FIBREWISE COMPACTNESS IN FIBREWISE TOPOLOGICAL GROUP

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ABSTRACT

The aim of this paper is to study the fibrewise compactness and fibrewise locally compactness in fibrewise topological group. Also we give several results concerning it.

1. INTRODUCTION

The fibrewise viewpoint is standard in the theory of fibre bundles.However, it has been recognized relatively recently that the same viewpoint is also important in other areas such as general topology. A fibrewise topological space over B is just a topological space X together with a continuous function $p: X \to B$ is called projection. Most of the results obtained so far in this field can be found in James [3] (1984) and James [4] (1991). In [1] Radwan and others defined and studied a fibrewise group. In [11] Tantawy and others introduce the notion of fibrewise topological group and studied many properties.Our aim in this paper is to study the fibrewise compactness in fibrewise topological group. We study many properties and obtained some new results. Also we investigate some important properties of fibrewise compactness and fibrewise locally compactnessin fibrewise topological groupG using the fibre over the identity element e_B of B.

2. PRELIMINARIES

Throughout this section we will give the concepts and notations which we shall use in this paper:

2.1 Fibrewise topological space[4]

Definition 2.1.1: Let X be any set. Then a fibrewise set over B consists a set X together with a function $p: X \to B$, called the projection, where B is called a base set.

For each b of B, the fibre over b is the subset $X_b = p^{-1}(b)$ of X. Also for each subset $W$of b, we regard $X_W = p^{-1}(W)$ is a fibrewise set over $W$with the projection determined by p.

Proposition 2.1.2: Let X be a fibrewise set over B, with projection p. Then Y is fibrewise set over B with projection $p\alpha$ for each set Y and function $\alpha : Y \to X$.

In particular X' is fibrewise set over B with projection p/X' for each sub set X' of X. Also X is fibrewise set over B' with projection βp for each set B' and function $\beta : B \to B'$.

Definition 2.1.3: If X and Y are fibrewise sets over B, with projections p and q, respectively, a function $\varphi : X \to Y$ is said to be fibrewise if $q\varphi = p$. in other word $\varphi(X_b) \subseteq Y_b$ for each $b \in B$.

Definition 2.1.5: Let B be a topological space. Then a fibrewise topology on a fibrewise set X over B is any topology on X for which the projection p is continuous.

A fibrewise topological space over B is defined to be a fibrewise set over B with fibrewise topology.

Definition 2.1.6: The fibrewise topological space Xover B is fibrewise closed (fibrewise open) if the projection p is closed (open).

Definition 2.1.7: Let X and Y be fibrewise topological spaces over$B$. The fibrewise function $\varphi : X \to Y$ is perfect if φ is continuous, closed and for each $y \in Y_b$, whereb $\in B$, $\varphi^{-1}(y)$ is compact.

Definition 2.1.8: Let X be a fibrewise topological over B.Then$X$iscalled fibrewise compact if the projection $p: X \to B$ is perfect.

Proposition 2.1.9: Let X be a fibrewise topological spaces over B. Then X is fibrewise compact if and only if X is fibrewise closed and each fibre of X is compact (i.e. for everyb $\in B$, X_b is compact).

Proposition 2.1.10: Let $\varphi : X \to Y$ be a continuous fibrewise surjection, where X and $Y$are fibrewise topological spaces over$B$. If X is fibrewise compact then Y is so.

Definition 2.1.11: Let X be a fibrewise topological spaces over B. Then X is called fibrewise locally compact if for each $x \in X_b$, where b $\in B$, there exists a neighbourhood W of b in B and a neighbourhood$U \subset X_W$ of x such that $X_W \cap \overline{U}$ is fibrewise compact over W.

2.2 Fibrewise Group [1] and fibrewise topological group [11]

Definition 2.2.1: Let G be a group. A fibrewise group over B is a fibrewise set G with any binary operation makes G a group such that p : G $\to B$ is homomorphism.

Definition 2.2.2: Let G be a fibrewise group over$B$. Then any subgroup$H$ of G is a fibrewise group over B with projection $p_{/H}: H \to B$, we call this group a fibrewise subgroup of GoverB.

Definition2.2.3: Let G and K be two fibrewise groups over B. Then any homomorphism $\varphi : G \rightarrow K$ is called a fibrewise homomorphism if φ is a fibrewise map.

Definition2.2.4: A bijective fibrewise homomorphism is called a fibrewise isomorphism.

Theorem2.2.5: Let G be a fibrewise group over B and H be a fibrewise normal subgroup ofG. Then G/H is fibrewise group over B, with Projection $q : G/H \rightarrow B$ Such that $q\pi = p$.

Definition 2.2.8:A fibrewise topological group G is a fibrewise group endowed with fibrewise topology such that the mapping $g \rightarrow g^{-1}$ of G onto G and $(g, h) \rightarrow gh$ of $G \times G$ onto G are fibrewisecontinuousmaps.

Proposition 2.2.9: Let G be a fibrewise topological group over B. Then G_{B^*} is fibrewise topological group over $B^*$for each subgroup $B^*$of B.

Proposition2.2.10: Let G be a fibrewise topological group over B and H be a normal subgroup ofG. Then the quotient G/H is fibrewise topological space over B.

3. FIBREWISE COMPACTNESS IN TOPOLOGICAL GROUP

Definition3.1: Let G be a fibrewise topological group over B. Then G is fibrewise compact (fibrewise locally compact) if and only if the fibrewise topology on G is fibrewise compact (fibrewise locally compact).

Theorem 3.2: Let G be a fibrewise topological group over$B$ with aninjective projection, thenG is fibrewise compact if and only if the fibrewise topology on G is fibrewise closed and the fibre$G_{e_B}$is compact.

Proof:

Let G is fibrewise closed and $G_{e_B}$is compact and let $\{U_\alpha : \alpha \in \wedge\}$ be an open covering of $G_b$for any$b \in B$ then $\{{U_\alpha}^{-1} : \alpha \in \wedge\}$ is open covering of $G_{b^{-1}}$ then$\{U_\alpha {U_\alpha}^{-1} : \alpha \in \wedge\}$ is open cover of$G_b G_{b^{-1}} = G_{e_B}$, sinceG_{e_B} is compact then there is finite subset $\wedge_0$ of $\wedge$ such that $\{U_\alpha {U_\alpha}^{-1} : \alpha \in \wedge_0\}$ is cover of $G_{e_B}$and $\{U_\alpha : \alpha \in \wedge_0\}$ is a covering of G_b, hence G is fibrewise compact.

Conversely, direct from the fibrewise compact space.

Proposition 3.2: A closed fibrewise subgroup of fibrewise compact space is fibrewise compact.

Proof:

Let G be a fibrewise compact over$B$ and let H be a closed fibrewise subgroup ofG. Let H_b be a fibre of H, where b$\in B$and$H_b = G_b \cap H$, since H be a closed in G then H_b is closed inG_b for each b$\in B$. Since G is fibrewise compact then G_b is compact for each $b \in B$. This is implies H_b is compact for each $b \in B$ and H is closed fibrewise subgroup, then H is fibrewise compact over$B$.

Proposition 3.3: If G is fibrewise compact over$B$ and H is closed fibrewise normal subgroup ofG**,** thenG$/H$ is fibrewise compact.

Proof:

Let F be a closed set in G/H , then$\varphi^{-1}(F)$ is closed inG, since G is fibrewise closed thenp$\varphi^{-1}(F) = q(F)$ is closed in$B$. Hence $G/H$is fibrewise closed, sinceG is fibrewise compact and φ is continuous fibrewise surjection, then G/H is fibrewise compact from (Proposition2.1.10)

Proposition 3.4: For any fibrewise normal subgroup H of fibrewise locally compact G over$B$, the fibrewise topological group G/H is fibrewise locally compact.

Proof:

Let$x \in G_b$, where $b \in B$ then $xH \in (G/H)_b$ and let U be a neighbourhood of xH in G/H, then $\varphi^{-1}(U)$ is neighbourhood of x in G, since φ is continuous, G is fibrewise locally compact then there exist neighbourhood$W$ of b in $B$and neighbourhood$V$ of x in G_W such that $G_W \cap \bar{V}$ isfibrewise compact of Vover W in G. Since φ is open then$\varphi(V)$ is neighbourhoodof xH in $(G/H)_W$, and $(G/H)_W \cap \overline{\varphi(V)} = (G/H)_W \cap \varphi(\bar{V})$ is fibrewise compact of $\varphi(V)$over$W$.Hence G/H is fibrewise locally compact.

Theorem 3.5: G is fibrewise locally compact if and only ifthere exists a neighbourhood W of e_B in B and neighbourhood U of e_G in G_W such that $G_W \cap \bar{U}$ is fibrewise compact over W.

Proof:

Let G be a fibrewise locally compact, $e_G \in G_{e_B}$then there is neighbourhood$W$ of $e_B$in $B$and neighbourhood U of e_G such that $G_W \cap \bar{U}$ is compact.

Conversely, if there exists a neighbourhood W of e_{B} in B and neighbourhood U of e_{G} in G_W such that $\mathrm{G}_W \cap \overline{U}$ is fibrewise compact over W.

Now, for each$x \in \mathrm{G}_b$ where$b \in B$, xU is neighbourhood of x and $x\overline{U} = \overline{xU}$ also, $x(\overline{U} \cap \mathrm{G}_W) = x\overline{U} \cap x\mathrm{G}_W = \overline{xU} \cap x\mathrm{G}_W$ is fibrewise compact over bW.

Proposition 3.6: Let G and K be fibrewise topological groups over $B$and let $\varphi\colon \mathrm{G} \to \mathrm{K}$ be an open, closed continuous fibrewise surjection. If G is fibrewise locally compact then K is so.

Proof:

Let $y \in \mathrm{K}_b$, $b \in B$, then there exist $x \in \mathrm{G}_b$ such that $\varphi(x) = y$ since G is fibrewise locally compact, then there exist a neighbourhood W of b in B and neighbourhood U of x in G_W such that $\mathrm{G}_W \cap \overline{U}$ is fibrewise compact over W, then $\varphi(\mathrm{G}_W \cap \overline{U})$ is fibrewise compact over W in K since φ is open function then $\varphi(U)$ is neighbourhood of y hence $\varphi(\mathrm{G}_W \cap \overline{U}) = \varphi(\mathrm{G}_W) \cap \varphi(\overline{U}) = \mathrm{K}_W \cap \overline{\varphi(U)}$ is fibrewise compact over W.

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