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Article *in* European Journal of Pure and Applied Mathematics · July 2018

DOI: 10.29020/nybg.ejpam.v11i3.3261

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EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 11, No. 3, 2018, 671-681
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

Locally Conformal Almost Cosymplectic Manifold of Φ-holomorphic Sectional Conharmonic Curvature Tensor

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Abstract. The aim of the present paper is to study the geometry of locally conformal almost cosymplectic manifold of Φ-holomorphic sectional conharmonic curvature tensor. In particular, the necessary and sufficient conditions that locally conformal almost cosymplectic manifold is a manifold of point constant Φ-holomorphic sectional conharmonic curvature tensor have been found. The relation between the mentioned manifold and the Einstein manifold is determined.

2010 Mathematics Subject Classifications: 53C55, 53B35

Key Words and Phrases: Locally conformal almost cosymplectic manifold, conharmonic curvature tensor, Φ-holomorphic sectional conharmonic curvature tensor, Einstein manifold.

1. Introduction

Sectional curvature provides a lot of information with regard to substance geometry of Riemannian manifolds. Manifolds with constant sectional curvature are a great source of study. Morever, contact geometry plays important roles in Physics, optics, differential equations and phase spaces of a dynamical system. This stimulated the researchers to work in the domain of constancy holomorphic sectional curvatures of locally conformal almost cosymplectic manifold which is a motivating class of almost contact metric manifold..

The study of constant holomorphic sectional curvature of almost Hermitian manifolds was started by Tanno [19] in 1973. He obtained an algebraic characterization for an almost Hermitian manifold to constringe to a space of constant holomorphic sectional curvature, which he later extended for Sasakian manifold. In 1988, Kim [7] studied total spaces of constant Φ-holomorphic sectional curvature and in 1989, he studied [8] total spaces with flat contact Bochner curvature tensor for fibred Sasakian spaces with conformal fibres. In 1993, Takano [18] discuss fibred Sasakian spaces of constant Φ-holomorphic sectional

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DOI: https://doi.org/10.29020/nybg.ejpam.v11i3.3261

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and at the same time Nagaich [14] showed a generalized Tanno's results for indefinite almost Hermitian manifold. In 2009, Rani et al. [17] considered similar condition of [19] to another distinct class of almost contact manifold known as (ϵ)-Sasakian manifold. In 2012, Kirichenko and Kharitonova [12] studied the constancy of Φ-holomorphic sectional curvature of normal locally conformal almost cosymplectic Manifold.

2. Preliminaries

In this section, we will focus our efforts on the study of almost contact metric manifold. In particular, we dedicate our study on the construction of the class of locally conformal almost cosymplectic manifold in the G-adjoined structure space.

Definition 2.1. [1] *Let M be $2n+1$ dimensional smooth manifold , η be differential 1-form called a contact form, ξ be vector field called a characteristic, Φ be an endomorphism of the module of the vector fields $X(M)$ called a structure endomorphisim, then the triple (η, ξ, Φ) is called an almost contact structure if the following conditions hold*

(i) $\eta(\xi) = 1$;

(ii) $\Phi(\xi) = 0$;

(iii) $\eta \circ \Phi = 0$;

(iv) $\Phi^2 = -id + \eta \otimes \xi$.

Morover, if there is a Riemannian metric $g = \langle .,. \rangle$ on M such that $\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y)$, $X, Y \in X(M)$, then the tetrad of tensors (η, ξ, Φ, g) is called an almost contact metric structure. In this case the manifold M equipped with this structure is called an almost contact metric manifold.

Definition 2.2. [9] *Let (M, η, Φ, g) be almost contat metric manifold ($\mathcal{AC}$-manifold). In the module $X(M)$ we can determine two complementary projections m, ℓ, where $m = \eta \otimes \xi$ and $\ell = -\Phi^2$; thus $X(M) = L \oplus \aleph$, where $L = Im\Phi = ker\eta$ and $\aleph = Imm = ker\Phi$, where ℓ and m are the projections onto the submodules L and $\aleph$ respectively.*

Definition 2.3. [9] *In the complexification module L^c of the module L define two endomorphisms σ and $\bar{\sigma}$ as $\sigma = \frac{1}{2}(id - \sqrt{-1}\Phi)$ and $\bar{\sigma} = -\frac{1}{2}(id + \sqrt{-1}\Phi)$. We can define two projections by the forms:*

$$\Pi = \sigma \circ \ell = -\frac{1}{2}(\Phi^2 - \sqrt{-1}\Phi) \text{ and } \bar{\Pi} = \bar{\sigma} \circ \ell = \frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi),$$

where $\sigma \circ \Phi = \Phi \circ \sigma = i\sigma$ and $\bar{\sigma} \circ \Phi = \Phi \circ \bar{\sigma} = -i\bar{\sigma}$. Therefore, If we denote $Im\Pi = D_\Phi^{\sqrt{-1}}$ and $Im\bar{\Pi} = D_\Phi^{-\sqrt{-1}}$, then

$$X^c(M) = D_\Phi^{\sqrt{-1}} \oplus D_\Phi^{-\sqrt{-1}} \oplus D_\Phi^0,$$

where $D_\Phi^{\sqrt{-1}}$, $D_\Phi^{-\sqrt{-1}}$ and D_Φ^0 are proper submodules.

Definition 2.4. [11] *At each point* $p \in M^{2n+1}$, *there is a frame in* $T_p^c(M)$ *of the form* $(p, \varepsilon_0, \varepsilon_1, ..., \varepsilon_n, \varepsilon_{\hat{1}},$
$..., \varepsilon_{\hat{n}})$, *where* $\varepsilon_a = \sqrt{2}\sigma_p(e_p)$, $\varepsilon_{\hat{a}} = \sqrt{2}\bar{\sigma}(e_p)$, $\hat{a} = a + n$, $\varepsilon_0 = \xi_p$, *the mappings* $\sigma_p : L_p \longrightarrow D_\Phi^{\sqrt{-1}}$, $\bar{\sigma}_p : L_p \longrightarrow D_\Phi^{-\sqrt{-1}}$ *are isomorphism and anti-isomorphism respectively, and* e_a *are orthonormal bases of* L_p. *The frame* $(p, \varepsilon_0, \varepsilon_1, ..., \varepsilon_n, \varepsilon_{\hat{1}}, ..., \varepsilon_{\hat{n}})$ *is called an A-frame.*

Lemma 2.1. [13] *The matrices components of tensors* Φ_p *and* g_p *in A-frame have the following froms respectively:*

$$(\Phi_j^i) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & o \\ 0 & 0 & -\sqrt{-1}I_n \end{pmatrix}, \ (g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -I_n \\ 0 & I_n & 0 \end{pmatrix},$$

where I_n *is the identity matrix of order* n.

It is well known, that the set of such frames defines an G-structure on M with structure group $1 \times U(n)$, represented by matrix of the form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}$, where $A \in U(n)$. This structure is called an G-adjoined structure.

Definition 2.5. [1] *A skew-symmetric tensor* $\Omega(X, Y) = g(X, \Phi Y)$ *is called a fundamental form of the* $\mathcal{AC}$*-structure.*

Definition 2.6. [4] *An almost contact metric structure* $S = (\eta, \xi, \Phi, g)$ *is called an almost cosymplectic structure (* $\mathcal{AC}_\int$*-structure) if*

(i) $d\eta = 0$;

(ii) $d\Omega = 0$.

Definition 2.7. [15] *A conformal transformation of an* $\mathcal{AC}$*-structure* $S = (\eta, \xi, \Phi, g)$ *on a manifold is the passage from* S *to an* $\mathcal{AC}$*-structure* $\widetilde{S} = (\widetilde{\eta}, \widetilde{\xi}, \widetilde{\Phi}, \widetilde{g})$ *such that*

$$\widetilde{\eta} = e^{-\sigma}\eta, \qquad \widetilde{\xi} = e^{\sigma}\xi, \qquad \widetilde{\Phi} = \Phi, \qquad \widetilde{g} = e^{-2\sigma}g$$

where σ *is the determining function of the conformal transformation. If* σ =*const, then the conformal transformation is said to be trivial.*

Definition 2.8. [15] *An* $\mathcal{AC}$*-structure* S *on a manifold* M *is said to be locally conformal almost cosymplectic (*$\mathcal{LCAC}_\int$*-structure) if the restriction of this structure to some neighborhood* U *of an arbitrary point* $p \in M$ *admits a conformal transformation of an almost cosymplectic structure.This transformation is called a locally conformal. A manifold* M *equipped with an* $\mathcal{LCAC}_\int$*-structure is called an* $\mathcal{LCAC}_\int$*-manifold.*

Lemma 2.2. [6] *In the G-adjoined structure space, the collection of the structure equations of* $\mathcal{LCAC}_\int$*-manifold has the following forms:*

(i) $d\omega^a = -\omega^a_b \wedge \omega^b + B^{ab}_c \omega^c \wedge \omega_b + B^{abc}\omega_b \wedge \omega_c + B^a_b \omega \wedge \omega^b + B^{ab}\omega \wedge \omega_b;$

(ii) $d\omega_a = \omega^b_a \wedge \omega_b + B^c_{ab}\omega_c \wedge \omega^b + B_{abc}\omega^b \wedge \omega^c + B^b_a \omega \wedge \omega_b + B_{ab}\omega \wedge \omega^b;$

(iii) $d\omega = C_b \omega \wedge \omega^b + C^b \omega \wedge \omega_b;$

(iv) $d\omega^a_b = -\omega^a_c \wedge \omega^c_b + A^{acd}_b \omega_c \wedge \omega_d + A^a_{bcd}\omega^c \wedge \omega^d + A^{ac}_{bd}\omega^d \wedge \omega_c + A^a_{bc0}\omega \wedge \omega^c + A^{ac0}_b \omega \wedge \omega_c;$

where

(i) $B^{[abc]} = B_{[abc]} = 0;$

(ii) $B^{[ab]} = B_{[ab]} = 0;$

(iii) $B^a_b = B^b_a = \sigma_0 \delta^b_a;$

(iv) $C^{ab} = C_{ab} = 0;$

(v) $B^{ab}_c = 2\sigma^{[a}\delta^{b]}_c, B^c_{ab} = 2\sigma_{[a}\delta^c_{b]};$

(vi) $C^b = -\sigma^b, C_b = -\sigma_b;$

(vii) $A^{acd}_b = 2\delta^{[c}_b \sigma^{a]d} - 2\delta^{[d}_b \sigma^{a]c} + B^{acd}_b - 2\sigma^a \delta^{[d}_b \sigma^{c]} - 2\sigma_e B^{ae[d}\delta^{c]}_b + 2\sigma_b B^{abc};$

(viii) $A^{[acd]}_b = \sigma_e B^{e[da}\delta^{c]}_b;$

(ix) $A^{ac}_{[bd]} = -2\delta^{[c}_{[b}\sigma^{a]}_{d]} + 2\sigma^{[a}\delta^{e]}_b \sigma_{[e}\delta^c_{d]} - 2\sigma^{[a}\delta^{e]}_d \sigma_{[e}\delta^c_{b]} + \frac{1}{2}B^{aec}B_{ebd};$

(x) $A^{ac0}_b = -2\delta^{[c}_b \sigma^{a]0} + D^{ac}_b - \delta^a_b \sigma^c_0 - 2B^{aec}B_{eb} - \sigma^a \sigma_0 \delta^c_b + 2B^{ac}\sigma_b - B^{ae}\sigma_e \delta^c_b;$

(xi) $A^{[ac]0}_b = \sigma^{[c}_0 \delta^{a]}_b - \sigma_d \delta^{[a}_b B^{d]c} + \sigma_0 \sigma^{[c}\delta^{a]}_b + \frac{1}{2}B^{dca}B_{bd};$

(xii) $B^{a[bcd]} = -B^{a[db}\sigma^{C]};$

(xiii) $B^{abc0} = -2D^{a[bc]} - B^{adc}\sigma_0;$

(xiv) $\sigma^{[cd]} = \sigma_b B^{bcd}.$

Here B^{abc}, B_{abc}; B^{ab}, B_{ab}; B^a_b, B^b_a; C^{ab}, C_{ab}; C^b, C_b; A^{acd}_b, A^b_{acd}; A^{ac}_{bd}; A^{ac0}_b, A^b_{ac0}; B^{abci}, B_{abci}; D^{abi}, D_{abi} *and* σ_{ij} *are smooth functions in the G-adjoined structure space.*

The following lemma gives the expression for the nonzero components of Riemannian curvature tensor of $\mathcal{LCAC}_\int$-manifold in the G-adjoined structure space.

Lemma 2.3. [6] *In the G-adjoined structure space, the components of Riemannian curvature tensor of* $\mathcal{LCAC}_\int$*-manifold have the following forms:*

(i) $R^a_{bcd} = 2(A^a_{bcd} + 4\sigma^{[a}\delta^{h]}_{[c}B_{d]hb} - \sigma_0 B_{b[d}\delta^a_{c]});$

(ii) $R^a_{\hat{b}cd} = 2(2\delta^{[b}_{[c}\sigma^{a]}_{d]} + 2B^{hab}B_{hdc} - \delta^a_{[c}\delta^b_{d]}\sigma_0^2);$

(iii) $R^a_{bc\hat{d}} = A^{ad}_{bc} + 4\sigma^{[a}\delta^{h]}_c\sigma_{[h}\delta^d_{b]} - 4B^{dah}B_{chb} + B^{ad}B_{bc} - \delta^a_c\delta^d_b\sigma_0^2;$

(iv) $R^{\hat{a}}_{bcd} = 2(2B_{[c|ab|d]} - 2\sigma_{[a}B_{b]cd} + B_{a[c}B_{d]b});$

(v) $R^a_{0cd} = 2(\sigma_{0[c}\delta^a_{d]} + B^{ab}B_{bcd} - 2\sigma^{[a}\delta^{h]}_{[c}B_{d]h});$

(vi) $R^a_{b\hat{c}0} = A^{ac0}_b + \sigma_b B^{ac} - \delta^c_b\sigma_0\sigma^a;$

(vii) $R^{\hat{a}}_{bc0} = 2B_{cab0} + 2B_{cab}\sigma_0;$

(viii) $R^a_{0b0} = -\delta^a_b\sigma_{00} - \delta^a_b\sigma_0^2 - B_{cb}B^{ac} - \sigma^a_b - \sigma^a\sigma_b + 2\sigma^{[a}\delta^{c]}_b\sigma_c;$

(ix) $R^a_{0\hat{b}0} = 2\sigma_0B^{ab} - D^{ab0} - \sigma^{ab} - \sigma^a\sigma^b + 2B^{bac}\sigma_c.$

and the other components are conjugate to the above components or can be obtained by the property of symmetry for R or equal to zero.

Definition 2.9. [3] *A Ricci tensor is a tensor of type (2,0) which is defined by*

$$r_{ij} = -R^k_{ijk}$$

Lemma 2.4. *In the G-adjoined structure space, the components of the Ricci tensor of $\mathcal{LCAC}_f$-manifold are given by the following forms:*

(i) $r_{ab} = 2(-2A^c_{(ab)c} - 4(\sigma^{[c}\delta^{h]}_{[b}B_{c]ha} + \sigma^{[c}\delta^{h]}_{[a}B_{c]hb}) + \sigma_0B_{a[c}\delta^c_{b]} + \sigma_0B_{b[c}\delta^c_{a]} + 2\sigma_0B_{ab} - D_{ab0} - \sigma_{ab} - \sigma_a\sigma_b + 2B_{bah}\sigma^h;$

(ii) $r_{\hat{a}b} = -4(\delta^{[a}_{[b}\sigma^{c]}_{c]} - \sigma_{[c}\delta^b_{h]}\sigma^{[h}\delta^{a]}_c - \frac{1}{2}\sigma^{[a}\delta^{h]}_b\sigma_h + B^{hca}B_{hcb} + B^{bch}B_{cha}) + (B^{cb}B_{ac} - B_{hb}B^{ah}) + A^{cb}_{ac} - \delta^a_b\sigma_{00} - 2n\sigma_0^2 - \sigma^a_b - \sigma^a\sigma_b;$

(iii) $r_{a0} = -A^c_{ac0} - \sigma^cB_{ac} + n\sigma_0\sigma_a + 2(\sigma_{0[c}\delta^c_{a]} + B^{cb}B_{bca} - 2\sigma^{[c}\delta^{h]}_{[c}B_{a]h});$

(iv) $r_{oo} = -2n(\sigma_{00} + \sigma_0^2) - 2B_{hc}B^{ch} - 2(\sigma^c_c + \sigma^c\sigma_c) + 4\sigma^{[c}\delta^{h]}_c\sigma_h.$

and the other components can be found by taking the conjugate operator to the above components.

Proof. The above components can obtained directly from the Definition 2.10 and Lemma 2.5.

Definition 2.10. *An $\mathcal{LCAC}_f$-manifold has Φ-invariant Ricci tensor, if $\Phi \circ r = r \circ \Phi$.*

Lemma 2.5. *An $\mathcal{LCAC}_f$-manifold has Φ-invariant Ricci tensor if and only if, in the G-adjoined structure space, the following condition*

$$r_b^{\hat{a}} = r_{ab} = 0$$

holds.

We conclude this section by remembering the main concept of our study which is a conharmonic curvature tensor.

Definition 2.11. [5] *Let M be an $\mathcal{AC}$-manifold of dimension $2n+1$. A tensor T of type $(4,0)$ which is invariant under conharmonic transformation and defined by the form:*

$$T_{ijkl} = R_{ijkl} - \frac{1}{2n-1}(r_{il}g_{jk} - r_{jl}g_{ik} + r_{jk}g_{il} - r_{ik}g_{jl})$$

is called a conharmonic tensor, where $T_{ijkl} = -T_{jikl} = -T_{ijlk} = T_{klij}$.

Theorem 2.1. *In the G-adjoined structure space, the components of conharmonic curvature tensor of $\mathcal{LCAC}_f$-manifold are given by the following forms:*

(i) $T_{abcd} = 2(2B_{[c|ab|d]} - 2\sigma_{[a}B_{b]cd} + B_{a[c}B_{d]b})$;

(ii) $T_{\hat{a}bcd} = 2(A^a_{bcd} + 4\sigma^{[a}\delta^{h]}_{[c}B_{d]hb} - \sigma_0 B_{b[d}\delta^a_{c]}) - \frac{1}{2n-1}(r_{bc}\delta^a_d - r_{bd}\delta^a_c)$;

(iii) $T_{\hat{a}bc\hat{d}} = A^{ad}_{bc} + 4\sigma^{[a}\delta^{h]}_c\sigma_{[h}\delta^d_{b]} - 4B^{dah}B_{chb} + B^{ad}B_{bc} - \delta^a_c\delta^d_b\sigma^2_0 - \frac{1}{2n-1}(r^d_b\delta^a_c + r^a_c\delta^d_b)$;

(iv) $T_{\hat{a}\hat{b}cd} = 2(2\delta^{[b}_{[c}\sigma^{a]}_{d]} + 2B^{hab}B_{hdc} - \delta^a_{[c}\delta^b_{d]}\sigma^2_0) - \frac{4}{2n-1}(r^{[d}_{[a}\delta^{c]}_{b]})$;

(v) $T_{\hat{a}0cd} = 2(\sigma_{0[c}\delta^a_{d]} + B^{ab}B_{bcd} - 2\sigma^{[a}\delta^{h]}_{[c}B_{d]h}) + \frac{1}{2n-1}(r_{0d}\delta^a_c - r_{0c}\delta^a_d)$;

(vi) $T_{\hat{a}b\hat{c}0} = A^{ac0}_b + \sigma_b B^{ac} - \delta^c_b\sigma_0\sigma^a - \frac{1}{2n-1}(r^a_0\delta^c_b)$;

(vii) $T_{abc0} = 2B_{cab0} + 2B_{cab}\sigma_0$;

(viii) $T_{\hat{a}0b0} = -\delta^a_b\sigma_{00} - \delta^a_b\sigma^2_0 - B_{cb}B^{ac} - \sigma^a_b - \sigma^a\sigma_b + 2\sigma^{[a}\delta^{c]}_b\sigma_c + \frac{1}{2n-1}(r_{00}\delta^a_b + r^a_b)$;

(ix) $T_{\hat{a}0\hat{b}0} = 2\sigma_0 B^{ab} - D^{ab0} - \sigma^{ab} - \sigma^a\sigma^b + 2B^{bac}\sigma_c + \frac{1}{2n-1}(r_{\hat{a}\hat{b}})$.

and the other components are conjugate to the above or can be obtained by the property of symmetry for T or equal to zero.

Definition 2.12. [16] *A Riemannian manifold is called an Einstein manifold, if the Ricci tensor satisfies the equation $r_{ij} = eg_{ij}$.*

Definition 2.13. [10] *Let M be an $\mathcal{AC}$-manifold, an Φ-holomorphic sectional curvature (ΦHS-curvature) of a manifold M in the direction $X \in X(M)$; $X \neq 0$ is a function $H(X)$ which is defined as:*

$$H(X) = \langle R(X, \Phi X, X, \Phi X,)\rangle \|X\|^{-4}$$

Definition 2.14. [10] *An $\mathcal{AC}$-manifold is called a manifold of point constant ΦHS-curvature if*

$$\langle R(X, \Phi X, X, \Phi X,)\rangle = c\|X\|^4$$

where $c \in C^\infty(M)$; for all $X \in X(M)$

Lemma 2.6. [10] *An $\mathcal{AC}$-manifold is a manifold of point constant ΦHS-curvature c if and only if, on the G-adjoined structure,*

$$R^{(a}{}_{(bc)}{}^{d)} = \frac{c}{2}\delta^{\tilde{a}d}_{bc}$$

where $\delta^{\tilde{a}d}_{bc} = \delta^a_b\delta^d_c + \delta^a_c\delta^d_b$ is the symmetric second-order Kronecker delta.

Definition 2.15. *Let M be an $\mathcal{AC}$-manifold, an Φ-holomorphic sectional conharmonic curvature (ΦHTS-curvature) of a manifold M in the direction $X \in X(M)$; $X \neq 0$ is a function $H(X)$ which is defined as*

$$H(X) = \langle T(X, \Phi X, X, \Phi X,)\rangle \|X\|^{-4}$$

Definition 2.16. *An $\mathcal{AC}$-manifold is called a manifold of point constant ΦHST-curvature if*

$$\langle T(X, \Phi X, X, \Phi X,)\rangle = c\|X\|^4$$

where $c \in C^\infty(M)$; for all $X \in X(M)$.

3. The main results

This section is devoted to study the theoretical application of $\mathcal{LCAC}_\int$-manifold of point constant Φ-holomorphic sectional conharmonic curvature. In particular, we found the necessary and sufficient conditions in which the $\mathcal{LCAC}_\int$-manifold of point constant Φ-holomorphic sectional conharmonic curvature is an Eistein manifold.

The following theorems gives the necessary and sufficient condition in which an $\mathcal{LCAC}_\int$-manifold is a manifold of point constant ΦHS-curvature.

Theorem 3.1. *An $\mathcal{LCAC}_\int$-manifold is a manifold of point constant ΦHS-curvature c if and only if, the relation $A^{(ad)}_{(bc)} = \frac{1}{2}\delta^{\tilde{a}d}_{bc}(\sigma_0^2 + c) - 4\sigma^{[a}\delta^{h]}_c\sigma_{[h}\delta^d_{b]} + 4B^{(da)h}B_{chb} - B^{ad}B_{bc}$ holds on the G-adjoined structure space.*

Proof. According to the components of the Riemannian curvature tensor of $\mathcal{LCAC}_\int$-manifold, it follows that

$$R^a{}_{bc}{}^d = A^{ad}_{bc} + 4\sigma^{[a}\delta^{h]}_c\sigma_{[h}\delta^d_{b]} - 4B^{dah}B_{chb} + B^{ad}B_{bc} - \delta^a_c\delta^d_b\sigma_0^2$$

Symmetrizing with respect to the pair of upper and lower indices of the tensor $R^a{}_{bc}{}^d$, we get

$$R^{(a}{}_{(bc)}{}^{d)} = A^{(ad)}_{(bc)} + 4\sigma^{[a}\delta^{h]}_c\sigma_{[h}\delta^d_{b]} - 4B^{(da)h}B_{chb} + B^{ad}B_{bc} - \frac{1}{2}\delta^{\tilde{a}d}_{bc}\sigma_0^2$$

By Lemma 2.6, the constancy condition on the ΦHS-curvature c for a $\mathcal{LCAC}_f$-manifold, yields

$$A^{(ad)}_{(bc)} = \frac{1}{2}\delta^{\tilde{a}d}_{bc}(\sigma_0^2 + c) - 4\sigma^{[a}\delta^{h]}_c\sigma_{[h}\delta^d_{b]} + 4B^{(da)h}B_{chb} - B^{ad}B_{bc}$$

Theorem 3.2. *Suppose that M is $\mathcal{LCAC}_f$-manifold. Then the necessary and suffcient condition in which M is a manifold of point constant ΦHST-curvature C_0 is*

$$A^{ad}_{bc} = 4B^{dah}B_{chb} + B^{ad}B_{bc} - 4\sigma^{[a}\delta^{h]}_c\sigma_{[h}\delta^d_{b]} + \delta^a_c\delta^d_b\sigma_0^2 - C_0\delta^a_b\delta^d_c - \frac{1}{2n-1}(r^d_b\delta^a_c + r^a_c\delta^d_b)$$

Proof. Suppose that M is $\mathcal{LCAC}_f$-manifold of the point constant ΦHTS-curvature tensor.
According to the Definition 2.16, we get

$$\langle T(X, \Phi X, X, \Phi X,)\rangle = C_0\|X\|^4$$

In the G-adjoined structure space, we have

$$T_{ijkl}X^i(\Phi X)^jX^k(\Phi X)^l = C_0 g_{ij}g_{kl}X^iX^jX^kX^l$$

According to the property $(\Phi X)^a = \sqrt{-1}X^a, (\Phi X)^{\hat{a}} = -\sqrt{-1}X^{\hat{a}}$ and $(\Phi X)^0 = 0$ and then using the properties of conharmonic tensor, we get

$$-4T_{\hat{a}bc\hat{d}} = 4C_0\delta^a_b\delta^d_c$$

Hence

$$A^{ad}_{bc} = 4B^{dah}B_{chb} + B^{ad}B_{bc} - 4\sigma^{[a}\delta^{h]}_c\sigma_{[h}\delta^d_{b]} + \delta^a_c\delta^d_b\sigma_0^2 - C_0\delta^a_b\delta^d_c - \frac{1}{2n-1}(r^d_b\delta^a_c + r^a_c\delta^d_b)$$

Theorem 3.3. *If M is $\mathcal{LCAC}_f$-manifold of point constant ΦHST-curvature tensor with flat holomorphic sectional curvature tensor and Φ-invariant Ricci tensor. Then M is an Einstein manifold.*

Proof. Suppose that M is a manifold of point constant ΦHST-curvature tensor.
According to Theorem 3.2, we have

$$A^{ad}_{bc} - 4B^{dah}B_{chb} - B^{ad}B_{bc} + 4\sigma^{[a}\delta^{h]}_c\sigma_{[h}\delta^d_{b]} + \frac{1}{2n-1}(r^d_b\delta^a_c + r^a_c\delta^d_b) = \delta^a_c\delta^d_b\sigma_0^2 - C_0\delta^a_b\delta^d_c \quad (3.1)$$

Symmetrizing and then antisymmetrizing (3.1) by the indices (a, h) and (a, d) respectively and since M is a manifold with flat holomorphic sectional curvature tensor, then we have

$$\frac{1}{2n-1}(r^{[d}_b\delta^{a]}_c + r^{[a}_c\delta^{d]}_b) = \frac{1}{2}(\delta^d_b\delta^a_c - \delta^d_c\delta^a_b)(\sigma_0^2 + C_0) \quad (3.2)$$

Contracting (3.2) by the indices (d, c), we deduce

$$-\frac{(n-2)}{2(2n-1)}(r_b^a + r_d^d\delta_b^a) = -\frac{(n-1)}{2}\delta_b^a(\sigma_0^2 + C_0) \tag{3.3}$$

Symmetrizing and antisymmetrizing (3.3) by the indices (a, d), we obtain

$$r_b^a = e\delta_b^a$$

where $e = \dfrac{(2n-1)(n-1)}{(n-2)}(\sigma_0^2 + C_0)$
Since the Ricci tensor is Φ-invariant
Therefore M is Einstein manifold.

Theorem 3.4. *If M is $\mathcal{LCAC}_f$-manifold of point constant* Φ*HST-curvature tensor and* Φ*-invariant Ricci tensor, then M is an Einstein manifold if and only if* $A_{bc}^{ac} = B^{ac}B_{bc} + c_1\delta_b^a$.

Proof. Suppose that M is a manifold of point constant ΦHST-curvature tensor.
According to the Theorem 3.2, we have

$$A_{bc}^{ad} = 4B^{dah}B_{chb} + B^{ad}B_{bc} - 4\sigma^{[a}\delta_c^{h]}\sigma_{[h}\delta_{b]}^d + \delta_c^a\delta_b^d\sigma_0^2 - C_0\delta_b^a\delta_c^d - \frac{1}{(2n-1)}(r_b^d\delta_c^a + r_c^a\delta_b^d) \tag{3.4}$$

Symmetrizing (3.4) by the indices (a, h), we get

$$A_{bc}^{ad} = B^{ad}B_{bc} + \delta_c^a\delta_b^d\sigma_0^2 - C_0\delta_b^a\delta_c^d - \frac{1}{(2n-1)}(r_b^d\delta_c^a + r_c^a\delta_b^d) \tag{3.5}$$

Contracting (3.5) by the indices (c, d), we deduce

$$A_{bc}^{ac} = B^{ac}B_{bc} + (\sigma_0^2 - nC_0)\delta_b^a - \frac{2}{(2n-1)}r_b^a \tag{3.6}$$

Since M is an Einstein manifold, it follows that

$$A_{bc}^{ac} = B^{ac}B_{bc} + c_1\delta_b^a$$

where $c_1 = \sigma_0^2 - nC_0 - \frac{2e}{(2n-1)}$
Conversely, by substituted A_{bc}^{ac} in equation (3.6), we get

$$r_b^a = e\delta_b^a$$

According to Φ-invariant of Ricci tensor, it follows that M is Einstein manifold.

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