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A NEW SUBCLASS OF HARMONIC MEROMORPHIC FUNCTIONS INVOLVING QUANTUM CALCULUS

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Abstract. In this article, we introduce a new subclass of harmonic meromorphic functions which are defined by means of quantum calculus (q-calculus). With that, we study various interesting properties of this class. Further, q-integral operator is also defined and we show that the new class aforementioned is closed under this q-operator.

1. Introduction

Quantum calculus (q-calculus) has created many interests among the researchers due to its numerous applications in various branches of mathematics. Not to mention of its great influence in theoretical physics as well. The application of q-calculus was initiated by Jackson [14, 15], who was perhaps the first to develop q-integral and q-derivative in a systematic way. We also note that in [1, 2, 3], the q-analogue of Baskakov Durrmeyer operator has been proposed, which is based on q-analogue of beta function. Some other important generalizations of q-calculus of complex operators are the q-Picard and q-Gauss-Weierstrass singular integral operators discussed in [4],[5] and [6]. Very recently, other q-analogues of differential operators have been introduced in [16] and [8, 9, 10]. These q-operators are defined by using convolution of normalized analytic functions and q-hypergeometric functions, where several interesting results are obtained. We believe that deriving q-analogues of operators defined on the space of analytic functions, would be important in future. A comprehensive study on applications of q-analysis in operator theory may be found in [7].

For $z \in \mathbb{U}^* = \mathbb{U}\backslash\{0\}$, let M_H denote the class of functions:

$$f(z) = h(z) + \overline{g(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k z^k, \tag{1}$$

which are harmonic in the punctured unit disk $\mathbb{U}\backslash\{0\}$, where h and g are analytic in $\mathbb{U}^*$ and $\mathbb{U}$, respectively, and h has a simple pole at the origin with residue 1 here. The class M_H was studied in [13],[11] and [12]. We further denote by the subclass $M_{\overline{H}}$ of M_H consisting of functions f of the form

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty} |a_k| z^k,\ z \in \mathbb{U}\backslash\{0\} \text{ and } g(z) = -\sum_{k=1}^{\infty} |b_k| z^k,\ z \in \mathbb{U} \tag{2}$$

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which are univalent harmonic in the punctured unit disk $\mathbb{U}^*$.

We provide some notations and concepts of q-calculus used in this paper. All the results can be found in [7].

For $n \in \mathbb{N}$, the q-number is defined as follows:

$$[k]_q = \frac{1-q^k}{1-q}, \quad 0<q<1. \tag{3}$$

Hence, $[k]_q$ can be expressed as a geometric series $\sum_{i=0}^{k-1} q^i$, when $k \to \infty$ the series converges to $1\backslash 1-q$.

As $q \to 1$, $[n]_q \to n$, and this is the bookmark of a q-analogue: the limit as $q \to 1$ recovers the classical object.

The q-derivative of a function f is defined by

$$D_q(f(z)) = \frac{f(qz)-f(z)}{(q-1)z}, \quad q \neq 1, z \neq 0, \tag{4}$$

and $D_q(f(0)) = f'(0)$ provided $f'(0)$ exists. For a function $h(z) = z^k$ observe that

$$D_q(h(z)) = D_q(z^k) = \frac{1-q^k}{1-q} z^{k-1} = [k]_q z^{k-1},$$

then $\lim_{q\to 1} D_q(h(z)) = \lim_{q\to 1} [k]_q z^{k-1} = kz^{k-1} = h'(z)$, where h' is the ordinary derivative.

The q-Jackson definite integral of the function f is defined by

$$\int_0^z f(t)d_qt = (1-q)z\sum_{n=1}^{\infty} f(zq^n)q^n, \quad z \in \mathbb{C}.$$

DEFINITION 1. A function $f = h+\overline{g} \in M_H$ of the form (1) is said to be in the class $M_qS_H^*(\alpha)$ of meromorphically harmonic starlike functions of order α in $\mathbb{U}$ if it satisfies the condition

$$Re\left\{-\frac{qz\,D_q(h(z)) - qz\,\overline{D_q(g(z))}}{h(z)+\overline{g(z)}}\right\} > \alpha \ (z \in \mathbb{U},\ 0<q<1,\ 0 \leqslant \alpha < 1).$$

Also, denote $M_qS_{\overline{H}}^*(\alpha)$ the subclass of $M_qS_H^*(\alpha)$ consisting harmonic meromorphic functions $f = h+\overline{g}$ where h and g of the form (2).

In the first theorem we establish the sufficient coefficient condition for the class $M_qS_H^*(\alpha)$.

THEOREM 1. *If $f = h + \overline{g}$ is of the form (1)and satisfies the condition*

$$\sum_{k=1}^{\infty} [(q[k]_q + \alpha)|a_k| + (q[k]_q - \alpha)|b_k|] \leqslant 1 - \alpha, \tag{5}$$

where $(0 < q < 1)$, and $0 \leqslant \alpha < 1$ then f is harmonic univalent sense-preserving in $\mathbb{U}^*$ and $f \in M_q S_H^*(\alpha)$.

Proof. Let the function $f = h + \overline{g}$ given by (1), satisfying (5). For $0 < |z_1| \leqslant |z_2| < 1$, we have

$$\begin{aligned}
|f(z_1) - f(z_2)| &\geqslant |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)| \\
&\geqslant \frac{|z_1 - z_2|}{|z_1||z_2|} - |z_1 - z_2| \sum_{k=1}^{\infty} (|a_k| + |b_k|)|z_1^{k-1} + \cdots z_2^{k-1}| \\
&> \frac{|z_1 - z_2|}{|z_1||z_2|} \left[1 - |z_2|^2 \sum_{k=1}^{\infty} q[k]_q (|a_k| + |b_k|) \right] \\
&> \frac{|z_1 - z_2|}{|z_1||z_2|} \left[1 - \sum_{k=1}^{\infty} \left(\frac{q[k]_q + \alpha}{1 - \alpha} |a_k| + \frac{q[k]_q - \alpha}{1 - \alpha} |b_k| \right) \right].
\end{aligned}$$

This last expression is nonnegative by (5), and so f is univalent in $\mathbb{U}^*$. In order to show that f is sense-preserving in $\mathbb{U}^*$, it only needs to show that $|h'(z)| > |g'(z)|$ with ordinary derivative. For $0 < |z| = r < 1$, it follows that by using (5)

$$\begin{aligned}
|q\, D_q(h(z))| &\geqslant \frac{1}{|z|^2} - \sum_{k=1}^{\infty} q[k]_q |a_k||z|^{k-1} \\
&= \frac{1}{r^2} - \sum_{k=1}^{\infty} q[k]_q |a_k| r^{k-1} \\
&> 1 - \sum_{k=1}^{\infty} q[k]_q |a_k| \\
&\geqslant 1 - \sum_{k=1}^{\infty} \frac{q[k]_q + \alpha}{1 - \alpha} |a_k| \\
&\geqslant \sum_{k=1}^{\infty} \frac{q[k]_q - \alpha}{1 - \alpha} |b_k| \\
&> \sum_{k=1}^{\infty} q[k]_q |b_k| > \sum_{k=1}^{\infty} q[k]_q |b_k| r^{k-1} \\
&= \sum_{k=1}^{\infty} q[k]_q |b_k||z|^{k-1} > |q\, D_q(g(z))|.
\end{aligned}$$

Therefore,

$$h'(z) = \lim_{q \to 1} [|q\, D_q(h(z))|] > \lim_{q \to 1} [|q\, D_q(g(z))|] = g'(z)$$

which proves that f is sense-preserving in $\mathbb{U}^*$. In order to show $f \in M_q S_H^*(\alpha)$, it suffices to show

$$Re\left\{-\frac{qz\, D_q(h(z)) - qz\,\overline{D_q(g(z))}}{h(z)+\overline{g(z)}} - \alpha\right\} > 0\ , z \in \mathbb{U}.$$

It is known that $Re(p(z)) > 0$ if and only if $\left|\frac{p(z)-1}{p(z)+1}\right| < 1$ for an analytic function $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$.

Let

$$A(z) = -qz\, D_q(h(z)) + qz\,\overline{D_q(g(z))} - \alpha h(z) - \alpha\overline{g(z)} \tag{6}$$

and

$$B(z) = h(z) + \overline{g(z)}. \tag{7}$$

Then, we have to show that

$$|A(z)+B(z)| - |A(z)-B(z)| > 0.$$

Now from (6) and (7), it follows that

$$|A(z)+B(z)|$$

$$\begin{aligned}
&= \left|-qz\, D_q(h(z)) + qz\,\overline{D_q(g(z))} - \alpha h(z) - \alpha\overline{g(z)} + h(z) + \overline{g(z)}\right| \\
&= \left|\frac{2-\alpha}{z} - \sum_{k=1}^{\infty}(q[k]_q + \alpha - 1)a_k z^k + \sum_{k=1}^{\infty}(q[k]_q - \alpha + 1)\overline{b_k z^k}\right| \\
&\geqslant \frac{2-\alpha}{|z|} - \sum_{k=1}^{\infty}(q[k]_q + \alpha - 1)|a_k||z|^k - \sum_{k=1}^{\infty}(q[k]_q - \alpha + 1)|b_k||z|^k
\end{aligned}$$

and

$$|A(z)-B(z)|$$

$$\begin{aligned}
&= \left|-qz\, D_q(H(z)) + qz\,\overline{D_q(g(z))} - \alpha h(z) - \alpha\overline{g(z)} - h(z) - \overline{g(z)}\right| \\
&= \left|\frac{-\alpha}{z} - \sum_{k=1}^{\infty}(q[k]_q + \alpha + 1)a_k z^k + \sum_{k=1}^{\infty}(q[k]_q - \alpha - 1)\overline{b_k z^k}\right| \\
&\leqslant \frac{\alpha}{|z|} + \sum_{k=1}^{\infty}(q[k]_q + \alpha + 1)|a_k||z|^k + \sum_{k=1}^{\infty}(q[k]_q - \alpha - 1)|b_k||z|^k.
\end{aligned}$$

Therefore, we conclude

$$|A(z)+B(z)| - |A(z)-B(z)|$$

$$
\begin{aligned}
&\geqslant \frac{2(1-\alpha)}{|z|} - 2\sum_{k=1}^{\infty}(q[k]_q+\alpha)|a_k||z|^k - 2\sum_{k=1}^{\infty}(q[k]_q-\alpha)|b_k||z|^k \\
&\geqslant 2\left\{(1-\alpha) - \sum_{k=1}^{\infty}(q[k]_q+\alpha)|a_k||z|^{k+1} - \sum_{k=1}^{\infty}(q[k]_q-\alpha)|b_k||z|^{k+1}\right\} \\
&\geqslant 2\left\{(1-\alpha) - \left(\sum_{k=1}^{\infty}(q[k]_q+\alpha)|a_k| + \sum_{k=1}^{\infty}(q[k]_q-\alpha)|b_k|\right)\right\} \\
&> 0.
\end{aligned}
$$

Now, we prove that the condition (5) is necessary for functions in $M_qS^*_{\overline{H}}(\alpha)$.

THEOREM 2. *Let* $f = h + \overline{g} \in M_{\overline{H}}$ *where* h *and* g *of the form (2). Then* $f \in M_qS^*_{\overline{H}}(\alpha)$ *if and only if the inequality*

$$
\sum_{k=1}^{\infty}\left[(q[k]_q+\alpha)|a_k| + (q[k]_q-\alpha)|b_k|\right] \leqslant 1-\alpha, \tag{8}
$$

is satisfied.

Proof. In view of Theorem 1, it suffices to show that the “only if” part is true. Assuming that $f \in M_qS^*_{\overline{H}}(\alpha)$, then we have

$$
Re\left\{\frac{-qz\,D_q(h(z)) + qz\,\overline{D_q(g(z))} - \alpha h(z) - \alpha\overline{g(z)}}{h(z)+\overline{g(z)}}\right\} > 0
$$

$$
= Re\left\{\frac{\frac{1-\alpha}{z} - \sum_{k=1}^{\infty}(q[k]_q+\alpha)|a_k|z^k - \sum_{k=1}^{\infty}(q[k]_q-\alpha)|b_k|\overline{z}^k}{\frac{1}{z} + \sum_{k=1}^{\infty}|a_k|z^k - \sum_{k=1}^{\infty}|b_k|\overline{z}^k}\right\} > 0.
$$

The above condition must hold for all values of z in $\mathbb{U}^*$. Upon choosing the value of z on the positive real axis, where $0 < z = r < 1$, we conclude

$$
\frac{1-\alpha - \left\{\sum_{k=1}^{\infty}(q[k]_q+\alpha)|a_k|r^{k+1} + \sum_{k=1}^{\infty}(q[k]_q-\alpha)|b_k|r^{k+1}\right\}}{1 + \sum_{k=1}^{\infty}|a_k|r^{k+1} - \sum_{k=1}^{\infty}|b_k|r^{k+1}} > \alpha.
$$

If the condition (8) does not hold, then the numerator is negative for r sufficiently close to 1. Hence, there exist $z_0 = r$ in $(0,1)$ for which the quotient is negative. This contradicts the required condition for $f \in M_qS^*_{\overline{H}}(\alpha)$ and so the proof is complete.

A growth property for functions in the class $M_qS^*_{\overline{H}}([a_1])$ is contained in the following theorem:

THEOREM 3. *Let* $f = h + \overline{g} \in M_qS^*_{\overline{H}}(\alpha)$ *defined by (2). Then we have for* $|z| = r < 1$

$$
\frac{1}{r} - \frac{1-\alpha}{q(1+q)-\alpha}r \leqslant |f(z)| \leqslant \frac{1}{r} + \frac{1-\alpha}{q(1+q)-\alpha}r.
$$

Proof. Let $f \in M_qS^*_{\overline{H}}(\alpha)$, taking the absolute value of f defined by (2) and using Theorem 2, it follows that

$$\begin{aligned}
|f(z)| &= \left| \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k - \overline{\sum_{k=1}^{\infty} b_k z^k} \right| \\
&\leqslant \frac{1}{r} + \sum_{k=1}^{\infty} |a_k| r^k + \overline{\sum_{k=1}^{\infty} |b_k| r^k} \\
&\leqslant \frac{1}{r} + \sum_{k=1}^{\infty} (|a_k| + |b_k|) r \\
&= \frac{1}{r} + \frac{1-\alpha}{q(1+q)-\alpha} \sum_{k=1}^{\infty} \frac{q(1+q)-\alpha}{1-\alpha} [|a_k| + |b_k|]\, r \\
&\leqslant \frac{1}{r} + \frac{1-\alpha}{q(1+q)-\alpha} \sum_{k=1}^{\infty} \left[\frac{q[k]_q+\alpha}{1-\alpha} |a_k| + \frac{q[k]_q-\alpha}{1-\alpha} |b_k| \right] r \\
&\leqslant \frac{1}{r} + \frac{1-\alpha}{q(1+q)-\alpha} r.
\end{aligned}$$

The proof of the left inequality is similar to the proof of the right inequality.

THEOREM 4. *Let $f = h + \overline{g}$ where h and g are given by (2). Then $f \in M_qS^*_{\overline{H}}(\alpha)$ if and only if*

$$f(z) = \sum_{k=0}^{\infty} (\lambda_k h_k + \gamma_k g_k), \tag{9}$$

where

$$h_0(z) = \frac{1}{z}, \quad h_k(z) = \frac{1}{z} + \left(\frac{1-\alpha}{q[k]_q+\alpha} \right) z^k, k = 1, 2, \ldots, \tag{10}$$

and

$$g_0(z) = \frac{1}{z}, \quad g_k(z) = \frac{1}{z} - \left(\frac{1-\alpha}{q[k]_q-\alpha} \right) \overline{z^k}, k = 1, 2, \ldots, \tag{11}$$

where $1 \geqslant \lambda_k \geqslant 0, 1 \geqslant \gamma_k \geqslant 0$ *and* $\sum_{k=0}^{\infty} (\lambda_k + \gamma_k) = 1$.

Proof. Letting

$$\begin{aligned}
f(z) &= \sum_{k=0}^{\infty} (\lambda_k h_k + \gamma_k g_k) \\
&= \lambda_0 h_0(z) + \gamma_0 g_0(z) + \sum_{k=1}^{\infty} (\lambda_k h_k(z) + \gamma_k g_k(z)) \\
&= (\lambda_0 + \gamma_0) \frac{1}{z} + \sum_{k=1}^{\infty} \lambda_k \left(\frac{1}{z} + \frac{1-\alpha}{q[k]_q+\alpha} z^k \right) + \sum_{k=1}^{\infty} \gamma_k \left(\frac{1}{z} + \frac{1-\alpha}{q[k]_q-\alpha} \overline{z}^k \right),
\end{aligned}$$

then

$$\sum_{k=1}^{\infty}\left\{\frac{q[k]_q+\alpha}{1-\alpha}\left(\frac{1-\alpha}{q[k]_q+\alpha}\lambda_k\right)+\frac{q[k]_q-\alpha}{1-\alpha}\left(\frac{1-\alpha}{q[k]_q-\alpha}\gamma_k\right)\right\}$$
$$\sum_{k=1}^{\infty}(\lambda_k+\gamma_k)=1-\lambda_0-\gamma_0\leqslant 1,$$

so $f\in M_qS^*_{\overline{H}}(\alpha)$.

Conversely, suppose that $f\in M_qS^*_{\overline{H}}(\alpha)$. Set

$$\lambda_k=\frac{q[k]_q+\alpha}{1-\alpha}|a_k|, 0\leqslant\lambda_k\leqslant 1,$$

$$\gamma_k=\frac{q[k]_q-\alpha}{1-\alpha}|b_k|, 0\leqslant\gamma_k\leqslant 1,$$

$$\lambda_0=1-\sum_{k=1}^{\infty}\lambda_k-\sum_{k=1}^{\infty}\gamma_k.$$

Therefore, f can be written as

$$\begin{aligned}
f(z) &= \frac{1}{z}+\sum_{k=1}^{\infty}|a_k|z^k-\sum_{k=1}^{\infty}|b_k| \\
&= \frac{1}{z}+\sum_{k=1}^{\infty}\frac{1-\alpha}{q[k]_q+\alpha}\lambda_k z^k-\sum_{k=1}^{\infty}\frac{1-\alpha}{q[k]_q-\alpha}\gamma_k\overline{z}^k \\
&= (\lambda_0+\gamma_0)\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{1}{z}+\frac{1-\alpha}{q[k]_q+\alpha}z^k\right)\lambda_k+\sum_{k=1}^{\infty}\left(\frac{1}{z}-\frac{1-\alpha}{q[k]_q-\alpha}\overline{z}^k\right)\gamma_k \\
&= \sum_{k=0}^{\infty}(\lambda_k h_k+\gamma_k g_k), \text{as required.}
\end{aligned}$$

Next, we proceed three closure theorems which are convolution of the class $M_qS^*_{\overline{H}}(\alpha)$, convex linear combination of its members and finally we show that this class in closed under q-integral operator.

THEOREM 5. *Let* $f\in M_qS^*_{\overline{H}}(\alpha)$ *and* $F\in M_qS^*_{\overline{H}}(\alpha)$, *then the convolution function*

$$(f\tilde{*}F)(z)=\frac{1}{z}+\sum_{k=1}^{\infty}|a_k||A_k|z^k-\sum_{k=1}^{\infty}|b_k||A_k|\overline{z}^k$$

is in $M_qS^*_{\overline{H}}(\alpha)$.

Proof. Since $F\in M_qS^*_{\overline{H}}(\alpha)$, then by Theorem 2, $|A_k|\leqslant 1$ and $|B_k|\leqslant 1$, hence

$$\sum_{k=1}^{\infty}\left\{\frac{q[k]_q+\alpha}{1-\alpha}|A_k a_k|+\frac{q[k]_q-\alpha}{1-\alpha}|B_k b_k|\right\}$$

$$\leqslant \sum_{k=1}^{\infty}\left\{\frac{q[k]_q+\alpha}{1-\alpha}|a_k|+\frac{q[k]_q-\alpha}{1-\alpha}|b_k|\right\}\leqslant 1$$

by Theorem 2, as $f\in M_qS^*_{\overline{H}}(\alpha)$. Thus, $f\tilde{*}F\in M_qS^*_{\overline{H}}(\alpha)$.

We now examine the convex combination of $M_qS^*_{\overline{H}}(\alpha)$.

THEOREM 6. *Let the functions f_i defined as*

$$f_i(z)=\frac{1}{z}+\sum_{k=1}^{\infty}|a_{k,i}|z^k-\sum_{k=1}^{\infty}|b_{k,i}\,\overline{z}^k \tag{12}$$

be in the class $M_qS^*_{\overline{H}}([a_1])$ for every $i=1,2,\cdots,\ell.$, then the function

$$\xi(z)=\sum_{i=1}^{\ell}c_if_i(z)$$

is also in the class $M_qS^*_{\overline{H}}(\alpha)$, where $\sum_{i=1}^{\ell}c_i=1$.

Proof. According to the definition of ξ, we can write

$$\xi(z)=\frac{1}{z}+\sum_{k=1}^{\infty}\left(\sum_{i=1}^{\ell}c_i|a_{k,i}|\right)z^k-\sum_{k=1}^{\infty}\left(\sum_{i=1}^{\ell}c_i|b_{k,i}|\right)\overline{z}^k.$$

Further, since f_i are in $M_qS^*_{\overline{H}}(\alpha)$ for every $i=1,2,\cdots,\ell$. Then by (8), we have

$$\begin{aligned}&\sum_{k=1}^{\infty}\left\{q[k]_q+\alpha\left(\sum_{i=1}^{\ell}c_i|a_{k,i}|\right)+q[k]_q-\alpha\left(\sum_{i=1}^{\ell}c_i|b_{k,i}|\right)\right\}\\&=\sum_{i=1}^{\ell}c_i\left\{\sum_{k=1}^{\infty}\left(q[k]_q+\alpha|a_{k,i}|+q[k]_q-\alpha|b_{k,i}|\right)\right\}\\&\leqslant\sum_{i=1}^{\ell}c_i(1-\alpha)\leqslant 1-\alpha.\end{aligned}$$

Hence, the proof is complete.

COROLLARY 1. *The class $M_qS^*_{\overline{H}}(\alpha)$ is closed under convex combination.*

DEFINITION 2. Let $f=h+\overline{g}$ be defined by (2); then the q-integral operator $F_q: M_{\overline{H}}\to M_{\overline{H}}$ is defined by the relation

$$F_q(z)=\frac{[c]_q}{z^{c+1}}\int_0^z t^ch(t)d_qt+\overline{\frac{[c]_q}{z^{c+1}}\int_0^z t^cg(t)d_qt},(c>0),z\in\mathbb{U}^* \tag{13}$$

where $[a]_q$ is the q-number defined by (3).

From the Definition 2, we conclude that

$$F_q(z) = \frac{[c]_q}{z^{c+1}}\left[\int_0^z \left\{t^{c-1} + \sum_{k=1}^{\infty} |a_k| t^{k+c}\right\} d_q t - \overline{\int_0^z \left\{|b_k| t^{k+c}\right\} d_q t}\right]$$

$$= \frac{[c]_q}{z^{c+1}}\left[\left(1-q)z\sum_{n=0}^{\infty}(zq^n)^{c-1}q^n\right) + \sum_{k=1}^{\infty}|a_k|\left((1-q)z\sum_{n=0}^{\infty}(zq^n)^{k+c}q^n\right)\right.$$
$$\left. - \overline{\sum_{k=1}^{\infty}|b_k|\left((1-q)z\sum_{n=0}^{\infty}(zq^n)^{k+c}q^n\right)}\right]$$

$$= \frac{[c]_q}{z^{c+1}}\left[\frac{z^c}{[c]_q} + \sum_{k=1}^{\infty}\frac{1}{[k+c+1]_q}|a_k| z^{k+c+1} - \overline{\sum_{k=1}^{\infty}\frac{1}{[k+c+1]_q}|b_k| z^{k+c+1}}\right]$$

$$= \frac{1}{z} + \sum_{k=1}^{\infty}\frac{[c]_q}{[k+c+1]_q}|a_k| z^k - \sum_{k=1}^{\infty}\frac{[c]_q}{[k+c+1]_q}|b_k|\overline{z}^k, c>0, 0<q<1, |z|<1. \quad (14)$$

In the next theorem, we show that the class $M_qS^*_{\overline{H}}(\alpha)$ is closed under the q-integral operator defined by (13).

THEOREM 7. *Let $f = h + \overline{g}$ be given by (2) and $f \in M_qS^*_{\overline{H}}(\alpha)$, then F_q is defined by (13) also belongs to $M_qS^*_{\overline{H}}(\alpha)$.*

Proof. From the series representation of F_q defined by (14), we see that

$$[k+c+1]_q - [c]_q = \sum_{i=0}^{k+c} q^i - \sum_{i=0}^{c-1} q^i = \sum_{i=c}^{k+c} q^i > 0.$$

Therefore,

$$\sum_{k=1}^{\infty}\left\{q[k]_q + \alpha\left(\frac{[c]_q}{[k+c+1]_q}|a_k|\right) + q[k]_q - \alpha\left(\frac{[c]_q}{[k+c+1]_q}|b_k|\right)\right\}$$

$$\leqslant \sum_{k=1}^{\infty}\left\{(q[k]_q + \alpha)|a_k| + (q[k]_q - \alpha)|b_k|\right\} \leqslant 1-\alpha,$$

hence, $F_q \in M_qS^*_{\overline{H}}(\alpha)$.

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