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STUDY BROWNIAN MOTION AND SHOWING THE TRAJECTORIES OF PARTICLES USING FRACTAL GEOMETRY.

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ABSTRACT: In the present work, trajectories of particle in an infinite square well, finite square well, with simple harmonic oscillators and double well potentials are shown and fractal dimension of these trajectories is calculated. The fractal dimension to theses trajectories is found close to 1.5 which is fractal dimension of fractional Brownian motion. Here, to show that, one can modify Hermann and Al-Rashid approaches in direct applications of Scale Relativity Theory.

Keywords: Brownian Motion , Trajectories of Particles ,Fractal Geometry.

INTRODUCTION:

Brownian motion is the incessant random motion exhibited by microscopic particles immersed in a fluid. The effect is named after its discoverer, Robert Brown, a Scottish botanist who first noticed the effect in a suspension of pollen grains in 1827. It was determined quickly by Brown and others that the effect is exhibited by small particles of any material when they are suspended in a liquid or gas, and is not unique to pollen grains or other once-living matter[1,2].

The paths of particles exhibiting Brownian motion is of some mathematical interest in its own right. A motion picture of a Brownian particle's path will look pretty much the same if it is viewed at double speed but magnified four times or at triple speed and magnified nine times. The motion is thus self- similar and has no single characteristic length scale. Further the path taken by the Brownian particle, on all but the fastest time scales is highly irregular. The trajectory of the particle is constantly changing directions. It is, in fact, one of the objects characterized by the mathematician Benoit B. Mandelbrot as *fractal*, that is an object of fractional dimension. This is most easily seen if one thinks of drawing line segments connecting the locations of the particle at equal time intervals. For a particle moving along a smooth curve, the total length of the line segments over a given total time will begin to approach a clear limit as the time interval becomes smaller and smaller. For a Brownian particle, this distance appears to increase without limit, as long as the time intervals are long enough for several atomic collisions to occur. In this sense fractal geometry assigns to the Brownian paths a dimensionality greater than one but less than two[1,2,3,4].

Fractional Brownian motion is also known as the "Random Walk Process". It basically consists of steps in a random direction and with a step-length that has some characteristic value, hence the random walk process. A key feature to fBm is that if one zoom in on any part of the function one

will produce a similar random walk in the zoomed in part [2].
The two images in (Fig. 1) are examples of Brownian Motion. The first being a function over time. Where as t increases the function jumps up or down a varying degree. The second is the result of applying Brownian Motion to the xy-plane. It simply replaces the values in the function with x and y. It gives a nice random line that moves around the page[1,2].

E. E. Peter [5] in 1991 found that the fractal dimension of fractional Brownian motion was greater than 1.5 for the Nile Rivers levels. P.S. Addison [6] in 1997 showed a relationship between fractal dimension and Hurst exponent H aligns perfectly with the notion of fractal dimension as a measure of the roughness of an object. As H increases and the fractional Brownian motion has more persistence, the plot of the function becomes smoother and the box dimension DB decreases accordingly. Conversely, as H decreases and the fractional Brownian motion is more anti-persistent, the plot of the function becomes more jagged and D_B increases. He found that the fractal dimension of fractional Brownian motion paths lies between 1 and 2. Hermann [7] in 1997 showed the quantum behavior of particle in a box by using the principle of Scale Relativity Theory by Nottale [8] which is based on fractal space- time concept. He recommended further work to reveal that the trajectory of particle can be shown by numerical simulation of fractal position equation.

FRACTAL DIMENSIONS:

A geometric object whose dimension is fractional is called fractal. Fractals are self-similar or self-affine. For self-similar fractals, any small part of a fractal can be magnified to get the original fractal [9,10,11,13]. The common examples of this type of fractals are the Koch curve, the Koch snowflake and the Seirpinski triangle, as shown in Fig. (2) [10,14]. While, in self-affine fractals, a smaller piece of the whole appears to have undergone different scale reductions in the longitudinal and transverse directions. Examples of this latter type of fractals are shown in Fig. (3) [10,13,14].
Some other measures of dimension, which one collects under the term fractal dimension, are not limited to integers. Among objects classified as fractals, the fractal dimension of the object is usually (but not always) a non-integer dimension greater than its topological dimension [6,10]. Many types of fractal dimension exist, including the similarity dimension, the divider dimension, the Hausdorff dimension, the box counting dimension, the correlation dimension, the information dimension, the point wise dimension, the Lyapunov dimension, and others. As the names suggest, different dimension measures emphasize different aspects of the fractal object and may yield different results. What all of these dimension measures have in common is that they probe the fractal object to discover how much of it exists. How much space the object occupies is closely tied to the way that the object expresses its invariance under scaling [6].

Among the abundant of measures of fractal dimension, three of the most basic are the similarity dimension, the box counting dimension, and the Hausdorff dimension.
Similarity Dimension

All measures of dimension relate to the way that the object being measured scales, but the similarity dimension does so most literally. As a result, the similarity dimension is perhaps the easiest way to introduce the idea of measuring fractal dimensions. However, this measure of dimension only applies to exactly self-similar objects like the Sierpinski triangle (Fig.2), not to random fractals [6,10].
Imagine that one can be trying to measure how much space is filled by a line segment, a square, or a cube, in other words, the length of the line segment, the area of the square, or the volume of the cube. To do so, divide the object being measured into N equal-sized parts, each of which is a miniature copy of the object [6].
Let ε be the side length of each scaled-down copy, as Fig. (4) illustrates.

For the line segment, the square, and the cube, the amount of space that each miniature copy occupies is 1/ ε the length, 1/ ε2 the area, or 1/ε3 the volume, respectively, of the original object, one will refer to the amount of space that an object occupies as its hypervolume, abbreviated V. Thus, the hypervolume of a line segment is its length (V=Nε); a square, its area (V=Nε2); and a cube, its volume (V= Nε3). To generalize the measurement of hypervolume to objects with other dimensions, let the exponent of ε be the variable DS. Then the hypervolume is [6,10]:

$$N\varepsilon^{D_s} = V \qquad \text{----------------(1)}$$

Solving D_S, one can have

$$D_s = \log(N) - \log(V)/\log(\frac{1}{\varepsilon}) \qquad \text{--------(2)}$$

Although D_S, the similarity dimension, is an integer for line segments, squares,and cubes, it can also take on non-integer values (see ref.[6] for examples).

BOX COUNTING DIMENSION

Measuring the box counting dimension of an object is very much like measuring the similarity dimension, but instead of looking at how self-similar parts of the object itself scale, one can look at how boxes covering the object scale. Since one consider the self-similarity of the covering boxes and not of the object itself, the box counting dimension extends the basic idea of the similarity dimension to include fractal objects whose self-similarity is not exact. An advantage of the box counting dimension is that its calculation lends itself to computerization, as the covering boxes may be laid out in a grid over the object (Fig. 4).

Imagine again that one can be trying to measure how much space a line segment occupies, but now one does so by covering the line segment with N boxes of side length δ, as Fig. (5) shows [6].

To cover the line segment requires, at a minimum V/δ, one-dimensional boxes. Note that for a line segment, covering the object with boxes is identical to dividing the object into self-similar parts. The hypervolume of the line segment is Nδ. Now, generalizing this hypervolume measurement by letting the exponent of δ be the variable DB, one can have [6,10] :

$$N\delta^{D_B} = V \qquad \text{----------- (3)}$$

Solving DB, one can have

$$D_B = \log(N) - \log(V)/\log(\frac{1}{\delta}) \qquad \text{-----(4)}$$

Usually when one can be calculating DB, the box counting dimension, V is unknown. In order to determine the value of DB, one can rewrite the equation above in slope-intercept form [6]:

$$\log(N) = D_B \log(\frac{1}{\delta}) + \log(V) \qquad \text{-------(5)}$$

Thus, DB is the slope of the plot of log(N) against log(1/δ). Given a sufficient number of data points for log(N) versus log(1/δ), one can estimate DB without knowing V(see ref. for examples [6]).

HAUSDORFF DIMENSION

Of all the measures of dimension, the Hausdorff dimension is perhaps the most authoritative, since Mandelbrot once defined a fractal as an object whose Hausdorff dimension exceeds its topological dimension. However, the Hausdorff dimension is not especially useful in practice, as it is difficult to compute. Fortunately, the box counting dimension is often a passable alternative to the Hausdorff dimension [6,10].

Imagine once more that one can be trying to measure the hypervolume of an object by covering the object with N boxes, each of side length δ. Let Vm be the

hypervolume that one can measure using that set of boxes. The hypervolume measured when using boxes of dimension DE is [6,10]:

$$N\delta^{D_E} = V_m \quad \text{--------- (6)}$$

Suppose that the object in question is the smooth surface shown in Fig.(7) and one pick DE = 1. This is equivalent to trying to measure the area of the surface by covering it with N line segments, so Vm → ∞ as δ → 0. Suppose now that one pick DE = 3. In other words, one can be trying to measure the area of the surface by covering it with N cubes, so Vm → 0 as
δ → 0. Only if DE = 2 does Vm assume a value other than 0 or ∞.

In general, if the value selected for DE is too small, Vm → 0 as δ→ ∞. If DE is too large, Vm → 0 as δ→0. At the critical value DE = DH, where DH is called the Hausdorff dimension, limδ→∞ Vm switches from ∞ to 0 [6].

Results

Hermann [7] in his work said that the trajectory of particle can be shown by numerical simulations by plot between a number of boxes (a) on y-direction and a number of steps of time (c) on x-direction. He divided the region of particle in a box in 600 boxes and calculated probability density of this problem . He did that by making numerical simulations on fractal position equation which is [7]:

$$dx(t) = \frac{n\pi}{a}\tan\left(-\frac{n\pi}{a}x + \frac{\pi}{2}\right) + N(0,1) \quad \text{--------(7)}$$

where a is size of box(here, this region is divided into 600 pieces), n quantum number and N(0,1) is a normalized random variable.

Here if one uses this equation and plots between number of boxes(here 600 boxes)[7] and number of steps of time , the out put of our simulation gives trajectory of particle in a box[7].To do that, computer programmers were built using Matlab(ver.7) program, one following our modifying to Hermann approach [7] to reveal the shape of trajectory and second following random walk to calculate the fractal dimension (Fd) of these trajectories. Fig.(8) shows the trajectories of particle in a box for different values of quantum number(n=1,2,3and 9), from these shapes , the fractal dimension is around 1.5(which is fractal dimension of fractional Brownian motion when Horst exponent H=0.5 [1,2,6]). Here , we test our program to calculate Fd by calculating Fd to common fractal which is Koch curve so this program gives us exact results to this example. The above numerical simulations started with specific point which was x=100(no. of box is 100).

We were going to apply this program to other quantum systems solved by using scale relativity theory [8] and following Hermann approach [7] . Al-Rashid[15] in his work also solved three quantum problems which were finite square well, simple harmonic oscillators and double square well potentials. The fractal position equations of these problems are[15]:

$$dx(t) = \frac{1}{a}\begin{cases} 0.673 + N(0,1) & for \quad x < -a \\ -0.739\tan(\frac{0.739}{a}x) + N(0,1) & for \quad -a < x < a \\ -0.673 + N(0,1) & for \quad x > a \end{cases} \quad \text{---(8)}$$

and

$$dx(t) = \frac{1}{a}\begin{cases} 0.638 + N(0,1) & for \quad x < -a \\ 1.9\cot(\frac{1.9}{a}x) + N(0,1) & for \quad -a < x < a \\ -0.638 + N(0,1) & for \quad x > a \end{cases} \quad \text{--(9)}$$

for finite square well potential when α= 1(equivalent n=1) and 2 (equivalent n=2) respectively. Fig.(9) shows the trajectories of particle in this problem and their fractal dimension which is close to 1.5.

and for simple harmonic oscillators potential, the fractal position equation is [15]:

$$dx(t)=10^{-3}\left(-x+2n\left(H_{n-1}(x)/H_n(x)\right)\right)+\sqrt{10^{-3}}\,N(0,1) \quad \text{---}$$

(10)

where Hn is a Hermit polynomial of order n. Fig.(10) shows the trjectories of particle in a simple harmonic oscillators for n=1,2 and 3.The fractal dimensions of these trajectories are found between 1 and 2 or near 1.5 which is fractal dimension of fractional Brownian motion [6].

For this problem the numerical simulations were starting with x=100.Finally, for a double well potential problem, the fractal position equations are:

$$dx(t)=\frac{1}{a}\begin{cases} 1.35\cot\frac{1.35}{a}(x-a-b)+N(0,1) & for\ (3a)>x>(a) \\ 2.679\tanh(\frac{2.679}{a}x)+N(0,1) & for\ (a)>|x| \\ -1.35\cot\frac{1.35}{a}(x+a+b)+N(0,1) & for\ -(a)>x>-(3a) \end{cases} \quad -$$

(11)

for even parity when α =3, and

$$dx(t)=\frac{1}{a}\begin{cases} 1.35\cot\frac{1.35}{a}(x-a-b)+N(0,1) & for\ (3a)>x>(a) \\ 2.679\coth(\frac{2.679}{a}x)+N(0,1) & for\ (a)>|x| \\ 1.35\cot\frac{1.35}{a}(x+a+b)+N(0,1) & for\ -(a)>x>-(3a) \end{cases} \quad -$$

(12)

for odd parity when α= 3 too.

Fig.(11) shows the trajectories of particle in the current problem for even and odd parity solutions when α= 3 and the number of boxes is 1200.The fractal dimensions of these trajectories are found close to 1.5 .

Discussion and Conclusion :

Random walk and Brownian motion have been ubiquitous models in physical science. The theory of Brownian motion either focuses on the non stationary Gaussian process or treats it as the sum of a stationary pure random process[1]. The Brownian motion works on every particle. The trajectory in the theory of Brownian motion is continuous and non-differentiable, i.e. it is fractal [6,10].

The Scale Relativity Theory which is introduced by Nottale [15] is based on the concept of fractal space-time . By using the principles of this theory many quantum problems were solved. These solutions need to be completed by showing the shape of trajectories of particle in these quantum problems. In this work, one can be showing the trajectories of particle in problems discussed by Hermann[7] and Al-Rashid [15] but they didn't discuss trajectories of particle.

Here, one can show the trajectories of particle in a box, finite square well, simple harmonic oscillators and double well potentials. The fractal dimensions of these problems are calculated .The values of fractal dimension are closed to 1.5 which is the fractal dimension of fractional Brownian motion [1,6] when Hurst exponent $H \approx 0.5$. Then ,one can conclude that the trajectories of particle are fractal because they have fractional dimensions. This result gives more validity to scale relativity theory and Hermann approach to quantum problems in one-dimension.

REFERENCES

[1] E. Nelson ," Dynamical Theories of Brownian Motion" , Princeton, N.J.: Princeton University Press,1967.

[2] B. Mandelbrot and J. W. Van Ness ," Fractional Brownian motions, fractional noises and applications ", SIAM Review, 10, 422-437, 1968.

[3] H. P. McKean, Jr. and D. Sullivan ,"Brownian motion and harmonic functions on the class surface of the thrice-punctured sphere ", Adv. in Math., 51:203{211, 1984.

[4] S. J. Lin, " Stochastic analysis of fractional Brownian motions", Stochastics andStochastics Reports, 55, 121-140, 1995.

[5] E. E. Peters. Chaos and Order in the Capital Markets. John Wiley and Sons, Inc., New York, 1991.

[6] P. S. Addison ,"Fractals and Chaos: An Illustrated Course ", Institute of Physics Publishing, Bristol, 1997.

[7] R. P. Hermann, "Numerical Simulation of a Quantum Patrical in a Box", J. Phys. A; Math. Gen. 30, pp.3967-3975, 1997.

[8] L. Nottale, "Fractal Space-Time and Microphysics:To wards of a Theory of Scale Relativity", World Scientific, (First Reprint) 1998.

[9] B. Mandelbrot, "Fractal Geometry: What is it, and What Does It Do?", Proc. R. Soc., Vol. 423, No. 1864. London, pp. 3-16, 1989.

[10] B. Mandelbrot, "The Fractal Geometry of Nature", W. H. Freeman and Company, 1983.

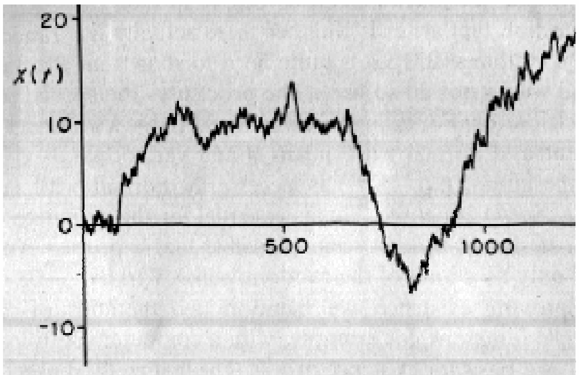
[11] F. Barnsley and A. D. Sloan. "A Better Way to Compress Images", Byte, pp. 215-223, 1988.

[12] P. P. Alex. "Fractal Based Description of Natural Scenes", Trans. on Pattern Analysis and Machine Intelligence, Vol. Pami-6, No.6, pp. 661-674, 1984.

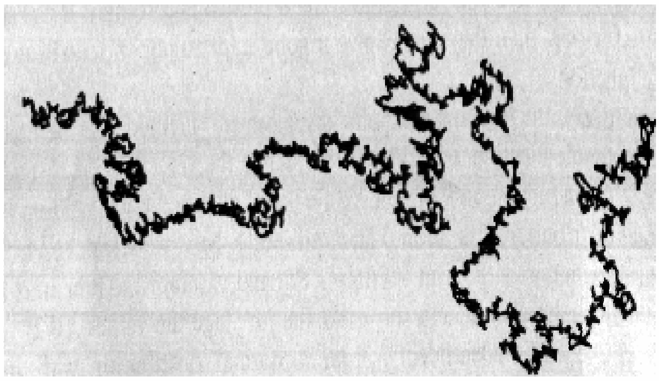
[13] M. F. Barnsley, "Fractals Everywhere", 2nd Ed., Acad. Press Professional, 1988

[14] D. M. Davis, "The Nature and Power of Mathematics", Princeton Univ. Press, 1993.

[15] S. N. T. Al-Rashid "Some Applications of Scale Relativity Theory in Quantum Physics", Ph.D Thesis, College of Science, Al-Mustansiriyah University,2006.



Graph of a Brownian Sample Function



A simulation of a Brownian Path in 2-Dimensions

Fig.(1) Examples of Brownian Motion [2].

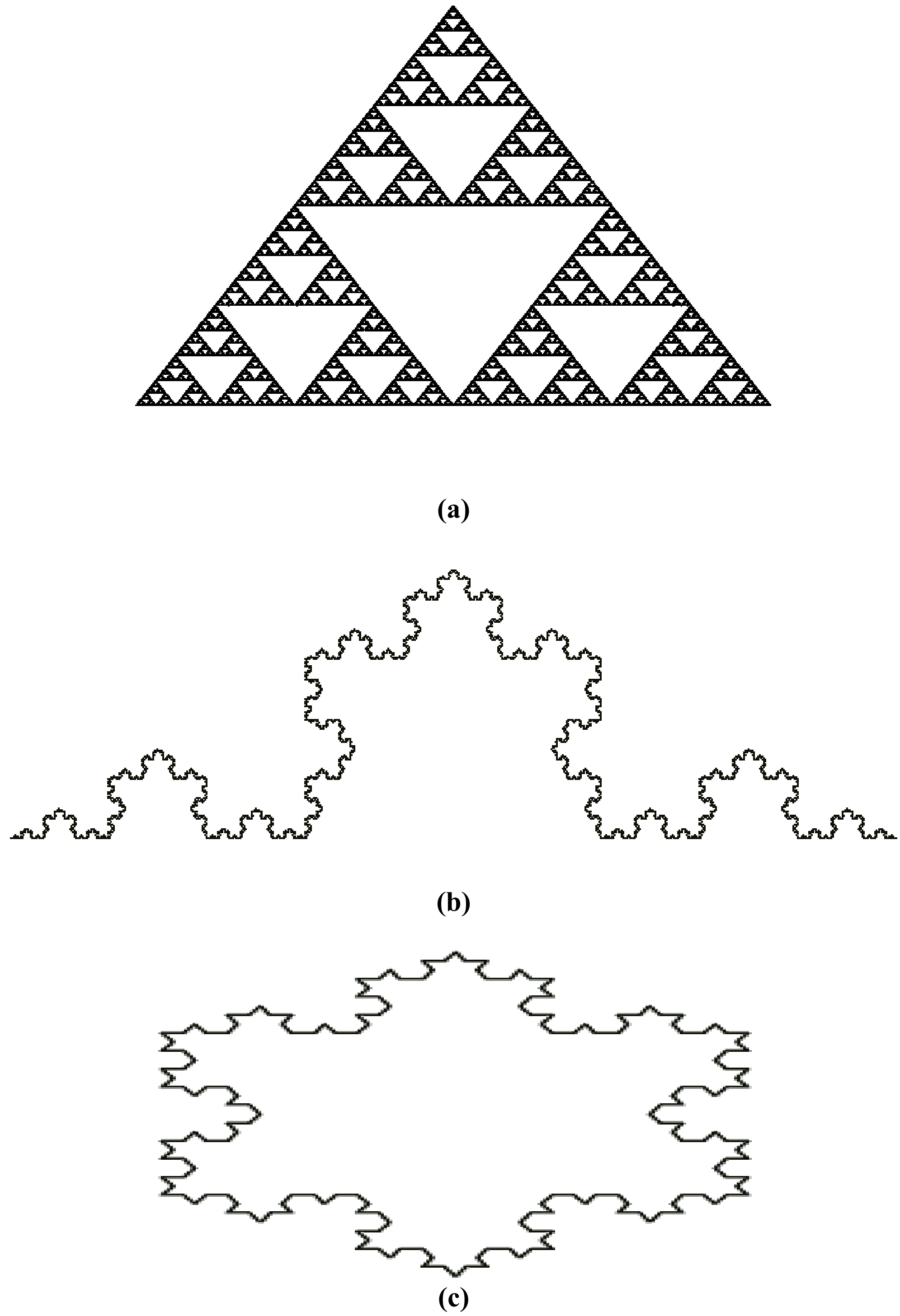


Fig. (2) Common self-similar fractals [10]: (a) Seirpinski triangle, (b) Koch curve and (c).Koch snowflake.

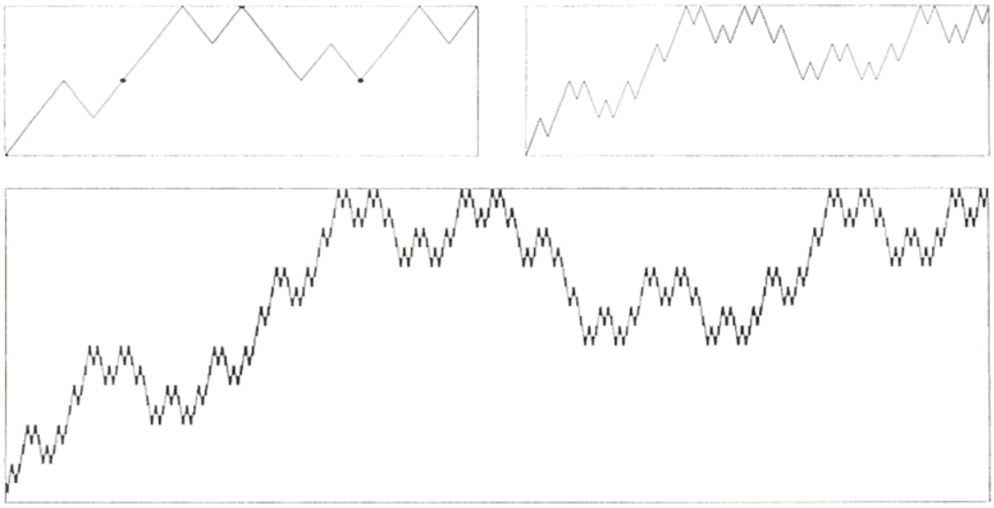


Fig (3) Common self-affine fractals(The Wiener Brownian motion WBM)[10,13].

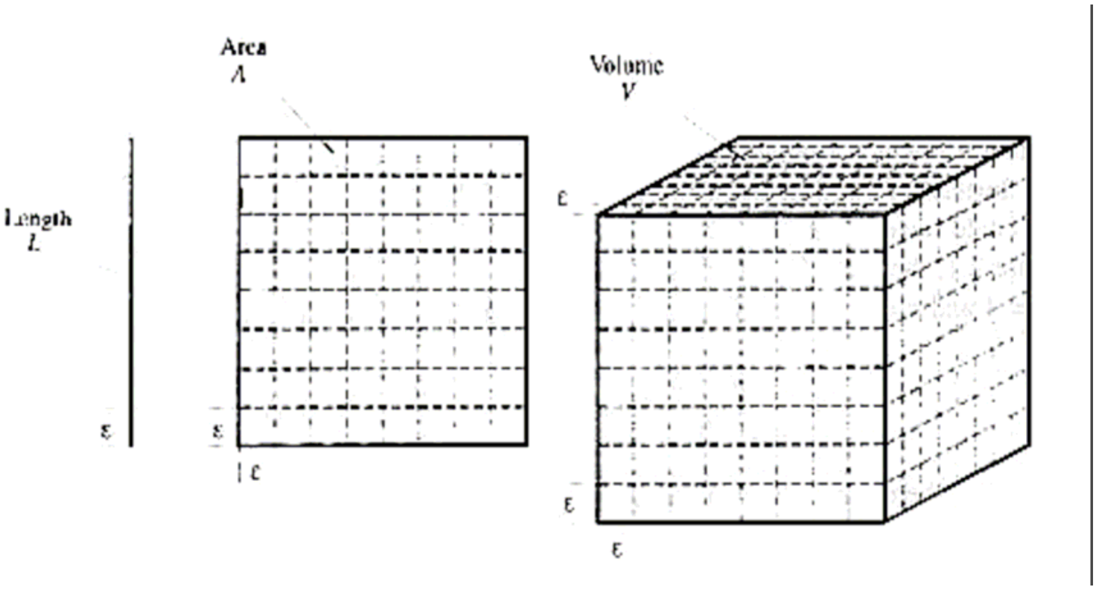


Fig.(4) Measuring the similarity dimension of a line segment, a square, and a cube [6].

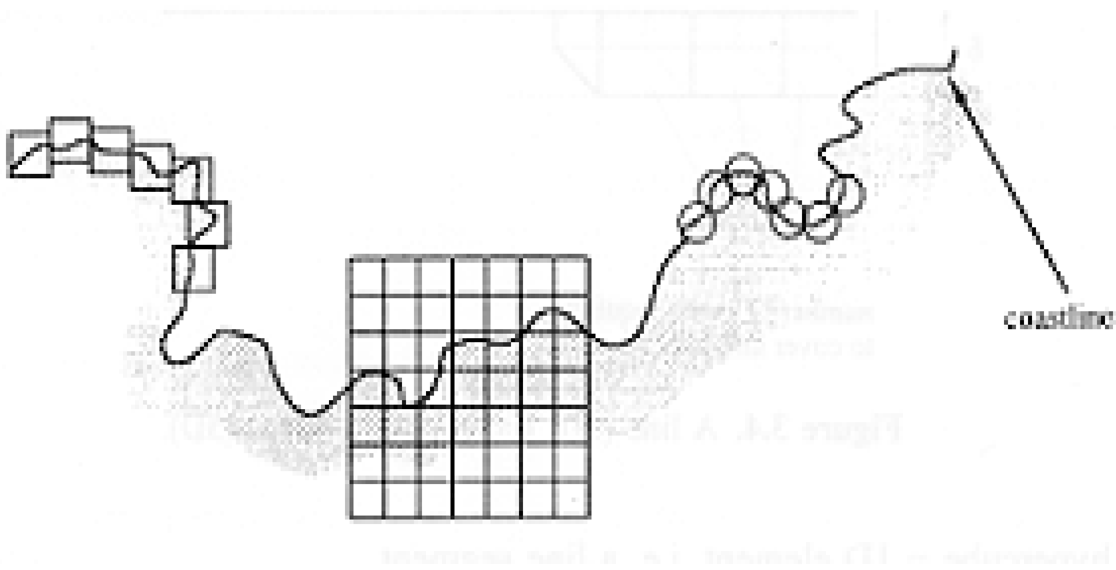


Fig.(5) In the box counting method, the covering boxes may be laid out in a grid over the object [6].

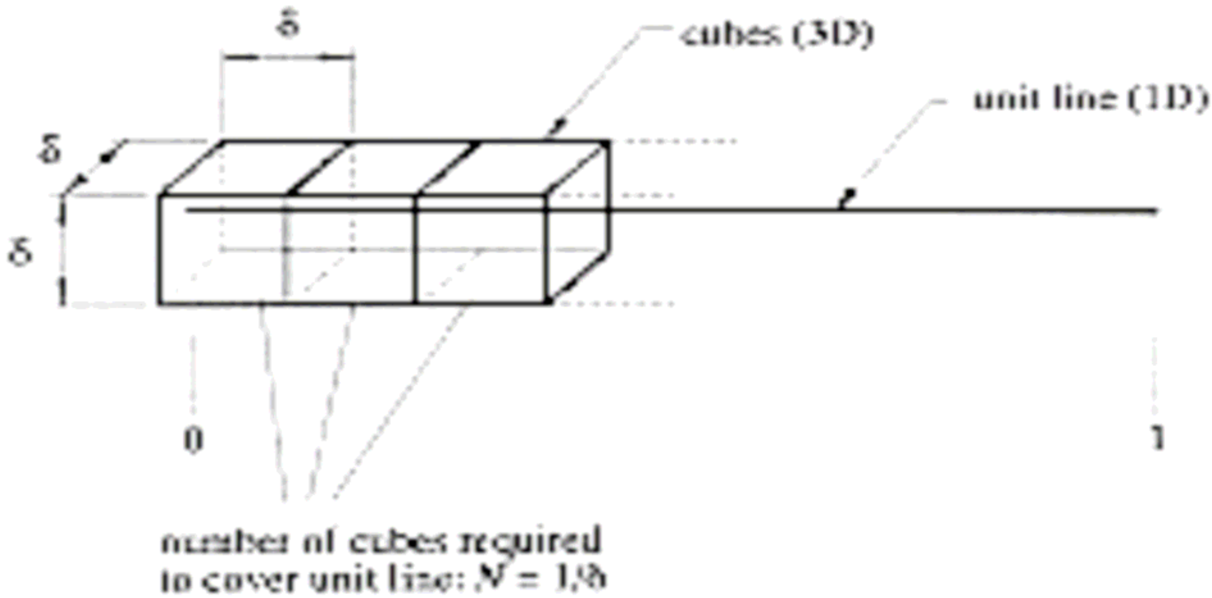


Fig.(6) Measuring the box counting dimension of a line segment [6].

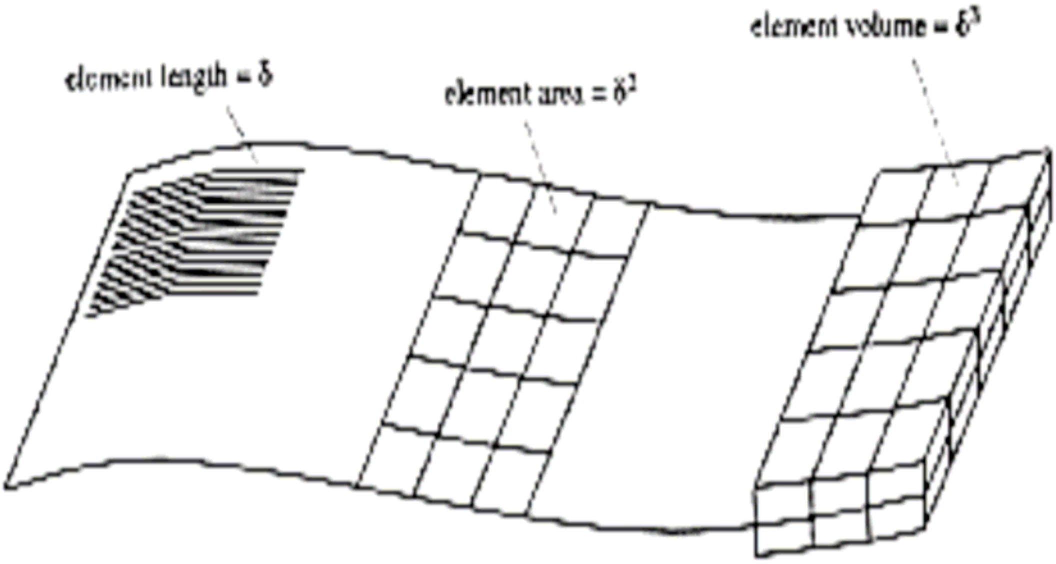


Fig.(7) Measuring the Hausdorff dimension of a smooth surface [6].

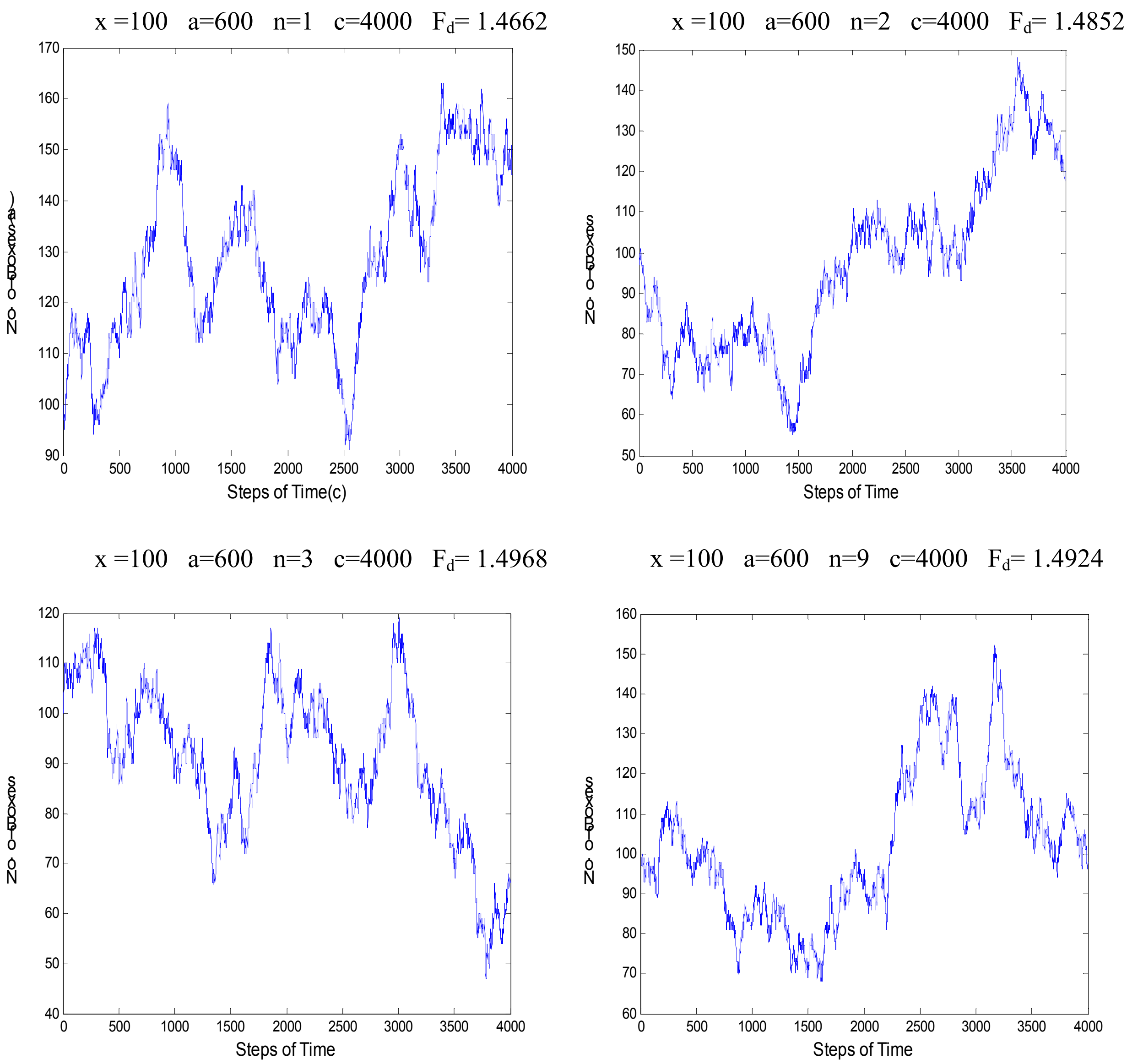


Fig.(8) Trajectories of particle in a box for n=1,2,3 and 9.

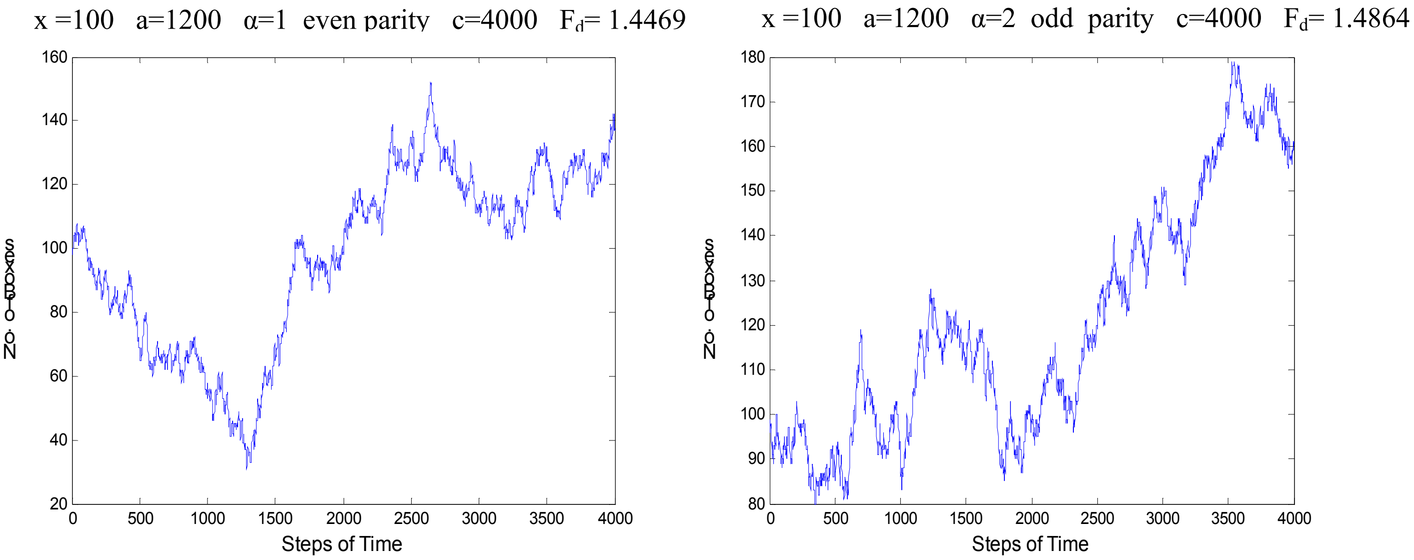


Fig.(9) Trajectories of particle in a finite square well potential for α=1 (even parity) and α=2 (odd parity).

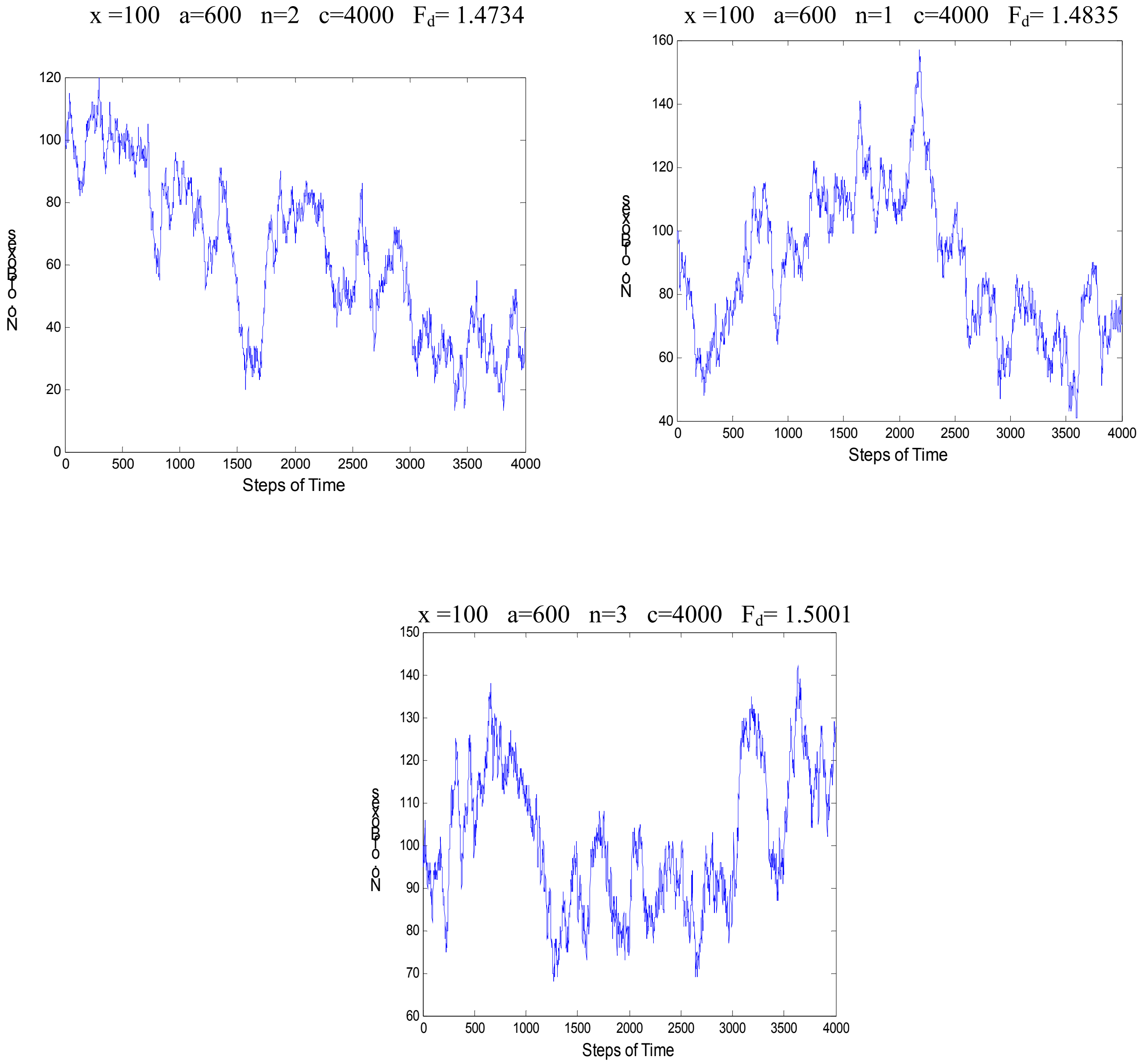


Fig.(10) Trajectories of particle in a simple harmonic oscillators for n=1,2 and 3.

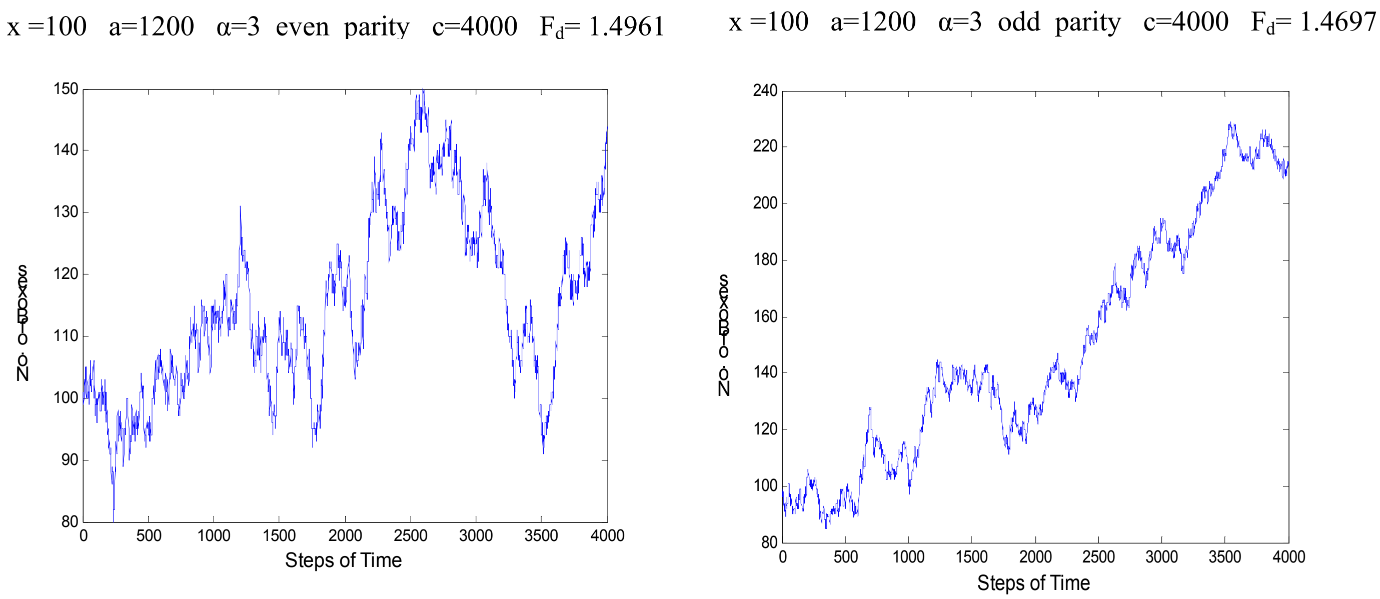


Fig.(11) Trajectories of particle in a double well potential when α=3 for even parity and odd parity.

دراسة الحركة البروانية وإظهار المسارات الكمية للجسيمات باستخدام الهندسة الكسورية

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الخلاصة:

تـم فـي هـذا البحث أظهار المسارات الكميه لجسيم يتحرك في الأنظمة الكمية التالية: صندوق ذي جـدران صلدة وصندوق ذي جدران غير صلدة و جهد توفقي متذبذب بسيط وأخيرا جهد ثنائي ،وقد حسب البعد ألكسوري لكلا من هذه المسارات ووجد أن البعد ألكسوري لتلك المسارات يقترب من 1.5 الذي هو الـبعد ألكسوري للحركة البر وانية الكسورية ، وأجراء ذلك تم عن طريق تعديل طريقتي هيرمان و الراشد التي استخدمتا في التطبيقات المباشرة لنظرية نسبية المقياس .