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Abstract. This paper aims to study the geometrical properties of the conharmonic curvature tensor of a locally conformal almost cosymplectic manifold. The necessary and sufficient conditions for the conharmonic curvature tensor to be flat, the locally conformal almost cosymplectic manifold to be normal and an η-Einstein manifold were determined.

1. Introduction

The conformal transformation on the Riemannian manifold preserves the angle between two vectors. However, generally, this conformal transformation does not preserve the harmonicity of functions. A harmonic function is one with a vanishing Laplacian. Subsequently, Ishi [11] studied a conformal transformation that preserves the harmonicity of a certain function, referred to as the conharmonic transformation. In particular, he introduced a tensor of rank four that is invariant under conharmonic transformations for an n-dimensional Riemannian manifold, also known as conharmonic curvature tensor.

Many researchers studied the aforementioned tensor on certain classes of almost Hermatian and almost contact metric manifolds. Ghosh et al. [8] focused on conharmonically symmetric $N(K)$-manifolds, particularly to establish if an n-dimensional $N(K)$-manifold is conharmonically symmetric, then it is locally isometric to the product $E^{(n+1)}(0) \times S^n(4)$. De et al. [7] studied the properties of conharmonically semisymmetric and ξ-conharmonically flat generalised Sasakian space forms. Abood and Abdulameer [1] found the necessary and sufficient conditions required by the flat conharmonic Vaisman-Gray manifold to become an Einstein manifold. Further, Ignatochkina and Abood [10] investigated the geometric significance of the vanishing conharmonic curvature tensor of a Vaisman-Gray manifold and proved that the conharmonic flat Vaisman-Gray manifolds of dimensions greater than four are locally conformal Kähler

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manifolds with vanishing scalar curvature tensor. Prakasha and Hadimani [18] characterised locally Φ-conharmonically symmetric and flat Kenmotsu manifolds with respect to a generalised Tanaka-Webster connection $\widetilde{\nabla}$. Recently, Abood and Abdulameer [2] employed the G-adjoined structure space to study the geometry of the Vaisman-Gray manifold of pointwise constant holomorphic sectional conharmonic tensor.

2. Preliminaries

This section revisits the fundamental concepts in our work, particularly the structural equations of the locally conformal almost cosymplectic manifold.

Definition 2.1 ([4]). Let M^{2n+1} be a smooth manifold of odd dimension ≥ 3, η a differential contact 1-form, ξ *a characteristic* vector field and Φ a structure endomorphism of the module of the vector fields $\chi(M)$. The triplet of tensors (η, ξ, Φ) will be referred to as *an almost contact structure* if the following conditions hold:

(1) $\eta(\xi) = 1$;
(2) $\Phi(\xi) = 0$;
(3) $\eta \circ \Phi = 0$;
(4) $\Phi^2 = -id + \eta \otimes \xi$.

Moreover, if there is a Riemannian metric $g = \langle \cdot, \cdot \rangle$ on M such that $\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y)$, $X, Y \in \chi(M)$, then the quadruple (η, ξ, Φ, g) will be known as an almost contact metric structure. In this case, the manifold M equipped with the mentioned structure, is called an almost contact metric manifold.

Definition 2.2 ([13]). Let (M, η, Φ, g) be an almost contact metric manifold ($\mathcal{AC}$-manifold). On the module $\chi(M)$, there are two mutually complementary projections m and ℓ, where $m = \eta \otimes \xi$ and $\ell = -\Phi^2$; thus, $\chi(M) = L \oplus \aleph$, where $L = \text{Im}(\Phi) = \ker\eta$ and $\aleph = \text{Im}(m) = \ker\Phi$.

Definition 2.3 ([13]). In the module L^c (complexification of L) two mutually endomorphisms σ and $\bar{\sigma}$ are given as $\sigma = \frac{1}{2}(id - \sqrt{-1}\Phi)$ and $\bar{\sigma} = -\frac{1}{2}(id + \sqrt{-1}\Phi)$. Moreover, there are two projections given by the forms

$$\Pi = \sigma \circ \ell = -\frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi) \text{ and } \bar{\Pi} = \bar{\sigma} \circ \ell = \frac{1}{2}(-\Phi^2 + \sqrt{-1}\Phi),$$

where $\sigma \circ \Phi = \Phi \circ \sigma = i\sigma$ and $\bar{\sigma} \circ \Phi = \Phi \circ \bar{\sigma} = -i\bar{\sigma}$. Therefore, if we consider $\text{Im}\Pi = D_\Phi^{\sqrt{-1}}$ and $\text{Im}\bar{\Pi} = D_\Phi^{-\sqrt{-1}}$, then

$$\chi^c(M) = D_\Phi^{\sqrt{-1}} \oplus D_\Phi^{-\sqrt{-1}} \oplus D_\Phi^0,$$

where $D_\Phi^{\sqrt{-1}}$, $D_\Phi^{-\sqrt{-1}}$ and D_Φ^0 are proper submodules with values $\sqrt{-1}, -\sqrt{-1}$ and 0, respectively.

Definition 2.4 ([16])**.** The mappings $\sigma_p : L_p \longrightarrow D_\Phi^{\sqrt{-1}}$ and $\bar{\sigma}_p : L_p \longrightarrow D_\Phi^{-\sqrt{-1}}$ denote an isomorphism and an anti-isomorphism, respectively. Therefore, at each point $p \in M^{2n+1}$, there is a frame in $T_p^c(M)$ of the form $(p, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{\hat{1}}, \ldots, \varepsilon_{\hat{n}})$, where $\varepsilon_a = \sqrt{2}\sigma_p(e_p)$, $\varepsilon_{\hat{a}} = \sqrt{2}\bar{\sigma}(e_p)$, $\hat{a} = a + n$, $\varepsilon_0 = \xi_p$, and e_a are the bases of L_p. The frame $(p, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{\hat{1}}, \ldots, \varepsilon_{\hat{n}})$ is known as an A-frame.

Lemma 2.5 ([14])**.** *The components matrices of the tensors Φ_p and g_p in the A-frame are given as*:

$$(\Phi^i_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & o \\ 0 & 0 & -\sqrt{-1}I_n \end{pmatrix}, \ (g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -I_n \\ 0 & I_n & 0 \end{pmatrix},$$

where I_n is the identity matrix of order n.

Noteworthy is that the set of such frames defines a G-structure on M with structure group $1 \times U(n)$, which is represented by the matrices of the form $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{smallmatrix}\right)$, where $A \in U(n)$. The mentioned structure is known as a G-adjoined structure space.

Throughout this paper, indices $i, j, k, \ldots$ have been assumed to range from 0 to $2n$, while indices $a, b, c, d, f, g, \ldots$ from 1 to n; moreover, $\hat{a} = a + n$, $\hat{\hat{a}} = a$ and $\hat{0} = 0$ have been set.

Definition 2.6 ([4])**.** An antisymmetric tensor $\Omega(X, Y) = g(X, \Phi Y)$ is referred to as a fundamental form of the $\mathcal{AC}$-structure.

Lemma 2.7 ([16])**.** *An $\mathcal{AC}$-structure is normal if and only if the following is present on the G-adjoined structure space*:

$$\Phi^{\hat{a}}_{b,c} = \Phi^{a}_{\hat{b},\hat{c}} = \Phi^{\hat{a}}_{b,0} = \Phi^{a}_{\hat{b},0} = \Phi^{0}_{a,b} = \Phi^{0}_{\hat{a},\hat{b}} = \Phi^{0}_{a,0} = \Phi^{0}_{\hat{a},0} = 0.$$

Definition 2.8 ([9])**.** An almost contact metric structure $S = (\eta, \xi, \Phi, g)$ will be known as an almost cosymplectic structure ($\mathcal{AC}_f$-structure) if the following conditions hold.

(1) $d\eta = 0$;
(2) $d\Omega = 0$.

Definition 2.9 ([4])**.** A normal almost cosymplectic structure is said to be cosymplectic.

Definition 2.10 ([17])**.** A conformal transformation of an $\mathcal{AC}$-structure $S = (\eta, \xi, \Phi, g)$ on a manifold indicates the transformation of an S to an $\mathcal{AC}$-structure $\widetilde{S} = (\widetilde{\eta}, \widetilde{\xi}, \widetilde{\Phi}, \widetilde{g})$ such that

$$\widetilde{\eta} = e^{-\sigma}\eta, \qquad \widetilde{\xi} = e^{\sigma}\xi, \qquad \widetilde{\Phi} = \Phi, \qquad \widetilde{g} = e^{-2\sigma}g,$$

where σ is the determining function of the conformal transformation. If $\sigma =$ const, then the conformal transformation is said to be trivial.

Definition 2.11 ([17])**.** An $\mathcal{AC}$-structure S on a manifold M is said to be a locally conformal almost cosymplectic ($\mathcal{LCAC}$-structure) if the restriction of S on some neighbourhood U of a point $p \in M$ admits a conformal transformation of an almost cosymplectic structure. This transformation referred to as locally conformal. A manifold M equipped with a $\mathcal{LCAC}$-structure is known as a $\mathcal{LCAC}$-manifold.

Lemma 2.12 ([15])**.** *In the G-adjoined structure space, a $\mathcal{LCAC}$-manifold is said to be normal if and only if the following equalities hold*:

$$B^{abc} = B_{abc} = B^{ab} = B_{ab} = \sigma^a = \sigma_a = 0.$$

Lemma 2.13 ([12])**.** *In the G-adjoined structure space, the structural equations of a $\mathcal{LCAC}$-manifold hold the following form*:

(1) $d\omega^a = -\omega^a_b \wedge \omega^b + B^{ab}_c \omega^c \wedge \omega_b + B^{abc}\omega_b \wedge \omega_c + B^a_b \omega \wedge \omega^b + B^{ab}\omega \wedge \omega_b$;
(2) $d\omega_a = \omega^b_a \wedge \omega_b + B^c_{ab}\omega_c \wedge \omega^b + B_{abc}\omega^b \wedge \omega^c + B^b_a \omega \wedge \omega_b + B_{ab}\omega \wedge \omega^b$;
(3) $d\omega = C_b\omega \wedge \omega^b + C^b\omega \wedge \omega_b$;
(4) $d\omega^a_b = -\omega^a_c \wedge \omega^c_b + A^{acd}_b \omega_c \wedge \omega_d + A^a_{bcd}\omega^c \wedge \omega^d + A^{ac}_{bd}\omega^d \wedge \omega_c + A^a_{bc0}\omega \wedge \omega^c + A^{ac0}_b \omega \wedge \omega_c$;
(5) $dB^{abc} = -B^{dbc}\omega^a_d - B^{adc}\omega^b_d - B^{abd}\omega^c_d + B^{abcd}\omega_d + B^{abc}_d \omega^d + B^{abc0}\omega$;
(6) $dB_{abc} = B_{dbc}\omega^d_a + B_{adc}\omega^d_b + B_{abd}\omega^d_c + B_{abcd}\omega^d + B^d_{abc}\omega_d + B_{abc0}\omega$;
(7) $dB^{ab} = -B^{db}\omega^a_d - B^{ad}\omega^b_d + D^{abd}\omega_d + D^{ab}_d \omega^d + D^{ab0}\omega$;
(8) $dB_{ab} = B_{db}\omega^d_a + B_{ad}\omega^d_b + D_{abd}\omega^d + D^d_{ab}\omega_d + D_{ab0}\omega$;
(9) $d\sigma^b = -\sigma^c\omega^b_c + \sigma^{bc}\omega_c + \sigma^b_c\omega^c + \sigma^{b0}\omega$;
(10) $d\sigma_b = -\sigma_c\omega^c_b + \sigma_{bc}\omega^c + \sigma^c_b\omega_c + \sigma_{b0}\omega$;
(11) $d\sigma_0 = \sigma_{0b}\omega^b + \sigma^b_0\omega_b + \sigma_{00}\omega$.

Here B^{abc}, B_{abc}; B^{ab}, B_{ab}; B^a_b, B^b_a; C^{ab}, C_{ab}; C^b, C_b; A^{acd}_b, A^b_{acd}; A^{ac}_{bd}; A^{ac0}_b, A^b_{ac0}; B^{abci}, B_{abci}; D^{abi}, D_{abi} and σ_{ij} are smooth functions in the G-adjoined structure space.

Definition 2.14 ([6])**.** The Ricci tensor is a tensor of type $(2,0)$, which is defined by

$$r_{ij} = -R^k_{ijk}.$$

Lemma 2.15 ([3])**.** *In the G-adjoined structure space, all essential components of the Ricci tensor of a $\mathcal{LCAC}$-manifold are given by the following formulae*:

(1) $r_{ab} = 2(-2A^c_{(ab)c} - 4(\sigma^{[c}\delta^{h]}_{[b}B_{c]ha} + \sigma^{[c}\delta^{h]}_{[a}B_{c]hb}) + \sigma_0 B_{a[c}\delta^c_{b]} + \sigma_0 B_{b[c}\delta^c_{a]} + 2\sigma_0 B_{ab} - D_{ab0} - \sigma_{ab} - \sigma_a\sigma_b + 2B_{bah}\sigma^h$;
(2) $r_{\hat{a}b} = -4(\delta^{[a}_{[b}\sigma^{c]}_{c]} - \sigma_{[c}\delta^b_{h]}\sigma^{[h}\delta^{a]}_c - \frac{1}{2}\sigma^{[a}\delta^{h]}_b\sigma_h + B^{hca}B_{hcb} + B^{bch}B_{cha}) + (B^{cb}B_{ac} - B_{hb}B^{ah}) + A^{cb}_{ac} - \delta^a_b\sigma_{00} - 2n\sigma_0^2 - \sigma^a_b - \sigma^a\sigma_b$;
(3) $r_{a0} = -A^c_{ac0} - \sigma^c B_{ac} + n\sigma_0\sigma_a + 2(\sigma_{0[c}\delta^c_{a]} + B^{cb}B_{bca} - 2\sigma^{[c}\delta^{h]}_{[c}B_{a]h})$;
(4) $r_{oo} = -2n(\sigma_{00} + \sigma_0^2) - 2B_{hc}B^{ch} - 2(\sigma^c_c + \sigma^c\sigma_c) + 4\sigma^{[c}\delta^{h]}_c\sigma_h$.

The remaining components can be found by considering the complex conjugation operator of the above components.

Definition 2.16 ([3]). A $\mathcal{LCAC}_f$-manifold has a Φ-invariant property if $\Phi \circ r = r \circ \Phi$.

Lemma 2.17 ([3]). *A $\mathcal{LCAC}_f$-manifold has Φ-invariant property if and only if the following condition holds in the G-adjoined structure space*:

$$r_{b}^{\hat{a}} = r_{ab} = r_{0}^{\hat{a}} = r_{a0} = 0.$$

Definition 2.18 ([5]). A pseudo-Riemannian manifold M is known as an η-Einstein of type (α, β) if its Ricci tensor satisfies the following condition:

$$r = \alpha g + \beta \eta \otimes \eta, \tag{1}$$

where α and β are suitable smooth functions. If $\beta = 0$, then M is referred to as an Einstein manifold.

This section ends with the discussion of the conharmonic curvature tensor and its components.

Definition 2.19 ([11]). Let M be an $\mathcal{AC}$-manifold of dimension $2n+1$. A tensor T of rank $(4,0)$ is invariant under a conharmonic transformation and can be defined by the following:

$$T_{ijkl} = R_{ijkl} - \frac{1}{2n-1}(r_{jl}g_{ik} - r_{jk}g_{il} + r_{ik}g_{jl} - r_{il}g_{jk})$$

is called the conharmonic curvature tensor.

Lemma 2.20 ([3]). *In the G-adjoined structure space, the non-zero components of the conharmonic curvature tensor of a $\mathcal{LCAC}$-manifold are calculated using the following formulae*:

(1) $T_{abcd} = 2(2B_{[c|ab|d]} - 2\sigma_{[a}B_{b]cd} + B_{a[c}B_{d]b})$;
(2) $T_{\hat{a}bcd} = 2(A^{a}_{bcd} + 4\sigma^{[a}\delta^{h]}_{[c}B_{d]hb} - \sigma_0 B_{b[d}\delta^{a}_{c]}) - \frac{1}{2n-1}(r_{bd}\delta^{a}_{c} - r_{bc}\delta^{a}_{d})$;
(3) $T_{\hat{a}bc\hat{d}} = A^{ad}_{bc} + 4\sigma^{[a}\delta^{h]}_{c}\sigma_{[h}\delta^{d}_{b]} - 4B^{dah}B_{chb} + B^{ad}B_{bc} - \delta^{a}_{c}\delta^{d}_{b}\sigma_0^2$
$\qquad - \frac{1}{2n-1}(r^{d}_{b}\delta^{a}_{c} + r^{a}_{c}\delta^{d}_{b})$;
(4) $T_{\hat{a}\hat{b}cd} = 2(2\delta^{[b}_{[c}\sigma^{a]}_{d]} + 2B^{hab}B_{hdc} - \delta^{a}_{[c}\delta^{b}_{d]}\sigma_0^2) - \frac{4}{2n-1}(r^{[a}_{[c}\delta^{b]}_{d]})$;
(5) $T_{\hat{a}0cd} = 2(\sigma_{0[c}\delta^{a}_{d]} + B^{ab}B_{bcd} - 2\sigma^{[a}\delta^{h]}_{[c}B_{d]h}) - \frac{1}{2n-1}(r_{0d}\delta^{a}_{c} - r_{0c}\delta^{a}_{d})$;
(6) $T_{\hat{a}b\hat{c}0} = A^{ac0}_{b} + \sigma_b B^{ac} - \delta^{c}_{b}\sigma_0\sigma^{a} + \frac{1}{2n-1}(r^{a}_{0}\delta^{c}_{b})$;
(7) $T_{abc0} = 2B_{cab0} + 2B_{cab}\sigma_0$;
(8) $T_{\hat{a}0b0} = -\delta^{a}_{b}\sigma_{00} - \delta^{a}_{b}\sigma_0^2 - B_{cb}B^{ac} - \sigma^{a}_{b} - \sigma^{a}\sigma_{b} + 2\sigma^{[a}\delta^{c]}_{b}\sigma_c$
$\qquad - \frac{1}{2n-1}(r_{00}\delta^{a}_{b} + r^{a}_{b})$;
(9) $T_{\hat{a}0\hat{b}0} = 2\sigma_0 B^{ab} - D^{ab0} - \sigma^{ab} - \sigma^{a}\sigma^{b} + 2B^{bac}\sigma_c - \frac{1}{2n-1}(r_{\hat{a}\hat{b}})$.

The remaining components are conjugates to those given above or can be obtained using the symmetric properties for T or are identically equal to zero.

3. Geometry of conharmonic curvature tensor of a $\mathcal{LCAC}$-manifold

This section concerns the study of the flat conharmonic curvature tensor of a $\mathcal{LCAC}$-manifold. In particular, it deals with the necessary conditions for the locally conformal almost cosymplectic manifold to be an η-Einstein manifold.

Definition 3.1. A $\mathcal{LCAC}$-manifold is known to be conharmonically flat if its conharmonic curvature tensor vanishes.

Theorem 3.2. *Suppose M is a $\mathcal{LCAC}$-manifold of dimension > 3. Then the necessary and sufficient conditions for the conharmonic tensor to be flat are* $A^{ad}_{bc} = B^{abc} = B^{ab} = \sigma^a = 0$ *and* $\sigma_{00} = -(n + \frac{1}{2})\sigma_0^2$.

Proof. Let M be a conharmonically flat $\mathcal{LCAC}$-manifold. Considering Lemma 2.20(3), we have

$$A^{ad}_{bc} + 4\sigma^{[a}\delta^{h]}_{c}\sigma_{[h}\delta^{d}_{b]} - 4B^{dah}B_{chb} + B^{ad}B_{bc} - \delta^a_c\delta^d_b\sigma_0^2 - \frac{1}{2n-1}(r^b_d\delta^c_a + r^c_a\delta^b_d) = 0. \tag{2}$$

Symmetrising and then antisymmetrising (2) using indices (c, b), we get

$$4\sigma^{[a}\delta^{h]}_{c}\sigma_{[h}\delta^{d}_{b]} - 4B^{dah}B_{chb} = 0. \tag{3}$$

Symmetrising (3) by using indices (d, a), we have

$$B^{dah}B_{chb} = 0. \tag{4}$$

By contracting (4) using indices (a, b) and then (d, c), the following is obtained

$$\overline{B}_{dah}B_{dha} = 0 \Leftrightarrow \sum_{d,h,a} = |B_{dha}|^2 = 0 \Leftrightarrow B_{dha} = 0. \tag{5}$$

Consequently, we get

$$4\sigma^{[a}\delta^{h]}_{c}\sigma_{[h}\delta^{d}_{b]} = 0. \tag{6}$$

Contracting (6) with indices (h, c) and (d, b), we obtain

$$(n^2 - 2n + 1)(\sigma^a\sigma_h) = 0. \tag{7}$$

Once again, contracting (7) by using indices (a, h), we get

$$\sigma_a\overline{\sigma}_a = 0 \Leftrightarrow \sum_a |\sigma_a|^2 = 0 \Leftrightarrow \sigma_a = 0. \tag{8}$$

Moreover, from Lemma 2.20(1), we have

$$2(2B[c|ab|d] - 2\sigma_{[a}B_{b]cd}B_{a[c}B_{d]b}) = 0. \tag{9}$$

Symmetrising and then antisymmetrising (9) using indices (a, b), we deduce

$$B_{ac}B_{db} - B_{ad}B_{cb} = 0. \tag{10}$$

Antisymmetrising (10) by using indices (a, d), it follows that

$$B_{ac}B_{db} = 0. \tag{11}$$

Contracting (11) with indices (a, d) and (c, b), we get $B^2_{ac} = 0$, then

$$B_{ac} = 0. \tag{12}$$

Now, regarding (1) of Lemma 2.20 and taking into account the relations (5), (8) and (11), we obtain

$$-\delta^{ab}_{cd}\sigma^2_0 - \frac{1}{2n-1}(r^a_c\delta^d_b - r^a_d\delta^b_c - r^b_c\delta^a_d + r^b_d\delta^a_c) = 0,$$

where $\delta^{ab}_{cd} = \delta^a_c\delta^b_d - \delta^a_d\delta^b_c$.

By virtue of Lemma 2.15, we have

$$-\delta^{ab}_{cd}\sigma^2_0 - \frac{1}{2n-1}[-2\delta^{ab}_{cd}(\sigma_{00} + 2n\sigma^2_0) + \delta^b_d A^{hc}_{ah} - \delta^b_c A^{hd}_{ah} - \delta^a_d A^{hc}_{bh} + \delta^a_c A^{hd}_{bh}] = 0,$$

$$\frac{1}{2n-1}[2\delta^{ab}_{cd}((n+\frac{1}{2})\sigma^2_0 + \sigma_{00}) - \delta^b_d A^{hc}_{ah} + \delta^b_c A^{hd}_{ah} + \delta^a_d A^{hc}_{bh} - \delta^a_c A^{hd}_{bh}] = 0. \tag{13}$$

Once again, using the relations (5), (8) and (11), then equation (2) reduces to

$$A^{ad}_{bc} - \delta^a_c\delta^d_b\sigma^2_0 - \frac{1}{2n-1}(r^b_d\delta^c_a + r^c_a\delta^b_d) = 0.$$

According to Lemma 2.15, we have

$$A^{ad}_{bc} - \delta^a_c\delta^d_b\sigma^2_0 - \frac{1}{2n-1}[-2\delta^a_c\delta^d_b(\sigma_{00} + 2n\sigma^2_0) + \delta^a_c A^{hb}_{dh} + \delta^d_b A^{hc}_{ah}] = 0,$$

$$\frac{1}{2n-1}[2\delta^a_c\delta^d_b((n+\frac{1}{2})\sigma^2_0 + \sigma_{00}) + (2n-1)A^{ad}_{bc} - \delta^a_c A^{hb}_{dh} - \delta^d_b A^{hc}_{ah}] = 0. \tag{14}$$

Moreover, from Lemma 2.20(8), we have

$$-\delta^a_b\sigma_{00} - \delta^a_b\sigma^2_0 - \frac{1}{2n-1}(r_{00}\delta^a_b + r^a_b) = 0.$$

By substitution the component of the Ricci tensor, we get

$$-\delta^a_b\sigma_{00} - \delta^a_b\sigma^2_0 - \frac{1}{2n-1}[-2\delta^a_b(2n\sigma^2_0 + (n+\frac{1}{2})\sigma_{00}) + A^{hb}_{ah}] = 0,$$

$$\frac{1}{2n-1}[2\delta^a_b((n+\frac{1}{2})\sigma^2_0 + \sigma_{00}) - A^{hb}_{ah}] = 0. \tag{15}$$

Using the equations (13), (14) and (15), it follows that $A^{ad}_{bc} = 0$ and $\sigma_{00} = -(n+\frac{1}{2})\sigma^2_0$. Conversely, from Lemma 2.13, and according to the linear independence of the basic forms, we can get the requirement directly. □

As a consequence of Theorem 3.1, we can directly obtain the next result.

Corollary 3.3. *Suppose M is a conharmonically flat $\mathcal{LCAC}$-manifold. Then M is a conharmonically flat normal $\mathcal{LCAC}$-manifold.*

The next theorem gives the necessary condition for a $\mathcal{LCAC}$-manifold to be an η-Einstein manifold.

Theorem 3.4. *Let M be a $\mathcal{LCAC}$-manifold of dimension > 3 and conharmonically flat. Then M is an η-Einstein manifold of type (α, β), where $\alpha = -\frac{2n-1}{2}\sigma_0^2$ and $\beta = \frac{(n+2)(2n-1)}{2n}$.*

Proof. Suppose M is a conharmonically flat $\mathcal{LCAC}$-manifold.

According to Definition 3.1 and Lemma 2.7(3), we have

$$A^{ad}_{bc} + 4\sigma^{[a}\delta^{h]}_c \sigma_{[h}\delta^d_{b]} - 4B^{dah}B_{chb} + B^{ad}B_{bc} \\ - \delta^a_c\delta^d_b\sigma_0^2 - \frac{1}{2n-1}(r^b_d\delta^c_a + r^c_a\delta^b_d) = 0.$$

Taking into account Theorem 3.1, we have

$$-\delta^a_c\delta^d_b\sigma_0^2 - \frac{1}{2n-1}(r^d_b\delta^a_c + r^a_c\delta^d_b) = 0. \tag{16}$$

Contracting (16) with indices (a, b), we obtain

$$-\delta^c_d\sigma_0^2 = \frac{2r^c_d}{2n-1}, \tag{17}$$

$$r^c_d = \alpha\delta^c_d. \tag{18}$$

Using the Lemma 2.20(8), we immediately get

$$-\delta^a_b\sigma_{00} - \delta^a_b\sigma_0^2 - B_{cb}B^{ac} - \sigma^a_b - \sigma^a\sigma_b + 2\sigma^{[a}\delta^{c]}_b\sigma_c - \frac{1}{2n-1}(r_{00}\delta^a_b + r^a_b) = 0.$$

According to Theorem 3.1 and the equation (3.18), we have

$$(n - \frac{1}{2})\delta^a_b\sigma_0^2 - \frac{1}{2n-1}(r_{00}\delta^a_b - \frac{2n-1}{2}\sigma_0^2\delta^a_b) = 0. \tag{19}$$

Hence,

$$r_{00} = \frac{2n-1}{n}\sigma_0^2,$$

where $\beta = \frac{(n+2)(2n-1)}{2n}\sigma_0^2$.

Therefore, M is an η-Einstein manifold. □

Theorem 3.5. *If M is a $\mathcal{LCAC}$-manifold of $dim M < 5$ with Φ-invariance property and conharmonically flat, then M is an η-Einstein manifold of type (α, β), where $\alpha = \sigma_0^2 + \sigma_{00} + \sigma^1_1 + \sigma^1\sigma_1$ and $\beta = -3\sigma_0^2 - 3\sigma_{00} - 3(\sigma^1_1 + \sigma^1\sigma_1)$.*

Proof. Suppose M is a conharmonically flat $\mathcal{LCAC}$-manifold.

According to Definition 3.1 and Lemma 2.20(3), we have

$$A^{11}_{11} - 4B^{111}B_{111} + B^{11}B_{11} - \sigma_0^2 - 2r^1_1 = 0.$$

Making use of Theorem 3.1, we get

$$A^{11}_{11} - \sigma_0^2 - 2r^1_1 = 0.$$

By the virtue of Lemma 2.15, we obtain

$$A^{11}_{11} = 3\sigma_0^2 + 2\sigma_{00} + 2(\sigma^1_1 + \sigma^1\sigma_1). \tag{20}$$

Using relation (20), we have $r_1^1 = \alpha\delta_1^1$, where $\alpha = \sigma_0^2 + \sigma_{00} + \sigma_1^1 + \sigma^1\sigma_1$. Moreover, $r_{00} = \alpha + \beta$, where $\beta = -3\sigma_0^2 - 3\sigma_{00} - 3(\sigma_1^1 + \sigma^1\sigma_1)$. Using the Φ-invariance property, we obtain M as an η-Einstein manifold of type (α, β). $\square$

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