Proximal Point in Topological Transformation Group
النقطة القريبة في زمرة التحويل التبولوجي

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Abstract

In this paper, we introduce the concept of proximal points in topological transformation group and it will be given a necessary and sufficient condition for a proximal points to be replete proximal points and almost periodic points also we show relation proximal points by syndetic set in side and locally almost periodic in other side .

الخلاصة

في هذا البحث قدمنا مفهوم النقطة الأقرب في زمرة التحويل التبولوجية كما سوف نعطي الشروط الضرورية والكافية لتكون النقطة الأقرب نقطة اقرب مفعمة ونقطة دورية تقريبا كذلك بيننا علاقة النقطة الأقرب بالمجموعة الرابطة من جهة و بالدورية تقريبا محلية من جهة أخرى .

Introduction

The proximal relation P(X,T) in transformation group (X,T) with compact hausdorff X has been introduced by R .Ellis and W.H. Gottschak (1.2). and studied in (3), (4) ,(5). The proximal relation is a reflexive, symmetric. The proximal relation plays an important role to characterize the transformation groups. The notions of replete proximal relation and replete regular relation has been strengthened and extended by consideration of replete set. In (6), the author introduced the r-homomorphism between two transformation groups The author(7) introduced regular relation R(X,T) on basis of the proximal relation P(X,T) This paper investigates the properties of there proximal relations and show some relations between proximal and replete proximal . We also give some necessary and sufficient conditions for the regular relation and proximal relation to be regular. We studied inverse image proximal points .

2.Preliminaries

Throughout this paper ,(X,T) will denote a transformation group with. compact hausdorff phasa spase X, A closed invariant non-empty subset A of X is called minimal set if dose not have any proper subset with these three properties. A point whose orbit closure is minimal set is called almost periodic point , (X,T) transformation group is said to be free effective if there exist $x \in X$ with $xt = x,$ then $t = e$ for each $t \in T$. A subset A of T is said to be {left}{right} syndetic in T provided that for some compact subset K of T. $\{T = AK\}\{T = KA\}$ and is said to be replete if for each compact set K there exist $g, g_1 \in T$ such that $gKg_1 \subseteq A$.A continuous map $\varphi : (X,T) \to (Y,T)$ with $\varphi(xt) = \varphi(x)t$ is called a homomorphism . if Y is minimal , φ is always onto .Especially , if φ is onto , φ is called an epimorphism . A homomorphism φ from (X,T) onto itself is called an endomorphism of (X,T) , and an isomorphism $\varphi : (X,T) \to (X,T)$ is called an automorphism of (X,T). A homomorphism $\varphi : (X,T) \to (Y,T)$ is called distal if $\varphi(x) = \varphi(y),$ x $\neq$ y.

Definition (2-1)(1،2).Let (X,T) be a transformation group a two points x and y of X are called proximal proved that for each index α in X ,there exist a $t \in T$ such that $(xt, yt) \in \alpha$.The set of all proximal pairs is called the proximal relation and is denoted by $P(X,T)$ or simply P.

Definition (2-2).Let (X,T) be a transformation group two points x and y of X are called replete proximal proved that for each index α in X ,there exist a replete subset $A \subset T$ such that $(xt, yt) \in \alpha$. The set of all replete proximal pairs is called the replete proximal relation and is denoted by $RP(X,T)$ or simply RP

Definition (2-3). (8)A transformation group (X,T) is called locally almost periodic if for each neighborhood U of x there is a neighborhood V of x and a syndetic subset $A \subset T$ with $VA \subset U$. (X,T) is said to be locally almost periodic if it is locally almost periodic at each point $x \in X$.

Definition (2-4). A minimal transformation group (X,T) is called regular minimal if (x, y) is an almost point of $(X \times X,T)$ and, there is an automomorphism h of (X,T) such that $h(x) = y$.

Definition (2-5) (5). The homomorphism φ: (X, T) $\to$ (Y, T) is said to be regular if for each $x, y \in \varphi^{-1}(y)$, there exists an automomorphism h of (X,T) such that $(hx, y) \in P(X,T)$ and $\varphi h = \varphi$ (y $\in$ Y).

Theorem (2-6). Let (X,T) be transformation group the fallowing hold:

1- If $(x, y) \in P(X,T)$ Then $P(X,T)$ be invariant set.
2- If $(x, y) \in P(X,T)$ Then $P(X,T)$ be closed set.
3- $(x, y) \in P(X,T)$ and (x,y) almost periodic Then $P(X,T)$ be minimal set.
4- If $(x, y) \in R(X,T)$ then $(h(x), y)T \subset P(X,T)$.

Proof: (3): Let $(x, y) \in P$, since $\overline{(x, y)T}$ be least closed invariant subset of $X \times X$ contain (x, y) such that $P \subset \overline{(x, y)T}$, since (x, y) almost periodic then $\overline{(x, y)T}$ be a minimal set by theorem (2-6 (1)(2)) we have obtain $P(X,T)$. be minimal set . Δ

Theorem (2-7). Let (X,T) be Transformation group and $(x, y) \in P$
Then the following hold:

1) $\overline{(x, y)T}$ be minimal set.
2) If $X \times X$ be a minimal set then P be dance set .

Proof.: We need only prove (1).Let $(x, y) \in P$,then $(x, y)T \subset PT$ and from theorem (2-6 number (1))we obtain $(x, y)T \subset P$,so $\overline{(x, y)T} \subset \overline{P}$ and P is closed set by theorem (2-6 number (2)),Then $\overline{(x, y)T} \subset P$,since $\overline{(x, y)T}$ be least closed invariant subset of $X \times X$ contain (x, y) therefore $P \subset \overline{(x, y)T}$.Thus we have $P = \overline{(x, y)T}$ therefore $\overline{(x, y)T}$ be minimal set . Δ

Theorem (2-8):Let (X,T) be transformation group and $(x, y) \in P$
Then α is invariant .

Proof: Let A be replete subset of T and $(x,y)\in P$ we obtain $(x,y)A\subset\alpha$
by hypothesis there exists $t\in T$ such that $(x,y)T\subset\alpha$. for all index α of X and $(x,y)TT\subset\alpha T$ since T be an invariant we have $(x,y)T\subset\alpha T$ $(x,y)T\cap(x,y)A\subset\alpha T\cap(x,y)A$ and $(x,y)T\cap(x,y)A\neq\varphi$.Thus $\alpha T\cap(x,y)A\neq\varphi$ so $\alpha T\subset(x,y)A\subset\alpha$ by hypotheses we obtain $\alpha\subset\alpha T$ then $\alpha=\alpha T$ therefore α is invariant. Δ

Theorem (2-9). Let (X,T) be Transformation group free effective and $(x,y)\in P$ If and only if $(x,y)\in RP$.

Proof: Let $(x,y)\in P$, then for each α index of X there exist $t\in T$ Such that $(x,y)T\subset\alpha$ .Since T be replete we obtain $(x,y)\in RP$ .Conversely Let $(x,y)\in RP$ then for each α index of X there exist A replete subset of T such that $(x,y)A\subset\alpha$,since A replete subset of T.then for each compact set K of T there exist $g_1,g_2\in T$ such that $g_1Kg_1\subset A$ and $(x,y)g_1Kg_2\subset\alpha$ and $(x,y)g_1Kg_2T\subset\alpha T$ by theorem (2-8) we get $(xg_1,yg_1)Kg_2T\subset\alpha$ since (X,T) be free effective then $(x,y)Kg_2T\subset\alpha$ and Kg_2 be compact set by hypothesis we have $(x,y)T\subset\alpha$ therefore $(x,y)\in P$ Δ

Theorem (2-10): If (X,T) be locally almost periodic and $U\subset X$ then fallowing hold:

1) U be invariant set.
2) If U be neighborhood of y then U be neighborhood of x.

Proof: We need only prove (1). Let (X,T) be locally almost periodic then for each neighborhood U of x there exist neighborhood V of x and syndetic subset A of T such that $VA\subset U$.$VAK\subset UK$,$VT\subset UK$ .Then for each $u\in U$ there exist $t\in T$ such that $xt=uk,\ xtk^{-1}=u$ so that $xT\subset U$, Since U neighborhood of x and xT orbit of x Then $U\subset xT$ thus we have $U=xT$ therefore U be invariant set. Δ

Theorem (2-11). Let (X,T) be Transformation group , $(x,y)\in P$ and U be neighborhood of x then (X,T) be locally almost periodic. Δ

Proof: Suppose that V be neighborhood of x ,and $(x,y)\in P$.In fact take h to be the identity therefore there exist syndetic set A of T such that $Ixa=ya$ for some $a\in A$,by hypotheses we have $xa\in xA\subset VA$
Since U be neighborhood of x then $xa\in UA\subset UT$ by theorem (2-10(1)) we get $xA\subset U$ and $AV\cap U\neq\varphi$ and $AV\subset U$ .Therefore (X,T) be locally almost periodic. Δ

Theorem (2-12):Let (X,T) be transformation group , T is abelain and $(x,y)\in P$, then A be syndetic set.

Proof: Let A be subset of T we show that A be syndetic set, since $(x,y)\in P$, Then for each α index in X there exist $t\in T$, such that $(x,y)T\subset\alpha$ by theorem (2-8) we have $\alpha T=\alpha$ and $\alpha TA=\alpha A$ since T be syndetic set there exist compact set K of T such that $\alpha TAK=\alpha AK$ and T be abelain group implies $\alpha TKA=\alpha AK$ and
$\alpha TA=\alpha AK$,by hypothesis , $\alpha TA\subset\alpha T$,thus $\alpha T\cap\alpha AK\neq\varphi$. $T\subset AK$. Therefore A be syndetic set. Δ

Theorem (2-13):If $\varphi : X \to Y$ be a distal homomorphism (X,T) be regular minimal if and only if φ be regular.

Proof: Let $\varphi(x) = \varphi(y)$.Since (X,T) be regular minimal and (x,y) almost periodic point of $(X \times X,T)$,then there exist an outomomorphim h in (X,T) such that $h(x) = y$ and therefore we have $(h(x),y) \in P(X,T)$, since φ be distal and $\varphi h(x) = \varphi(y) = \varphi(x)$.We have $\varphi h = \varphi$ .Thus we obtain φ is regular. Conversely suppose that (x,y) be almost periodic point in $(X \times X,T)$,we need to prove (X,T) be regular minimal ,because φ regular there exist automomorphism h of (X,T) such that $h(x)t = yt$ for some $t \in T$,by hypotheses there exist $t^{-1} \in T$ such that $h(x)tt^{-1} = ytt^{-1}$ implies $h(x) = y$ therefore (X,T) be regular minimal . Δ

Remark (2-14):Let (X,T) be regular and $\varphi : X \to Y$ be homomorphism then φ be distal.

Remark (2-15):

1. If (X,T) be regular then $(x,y) \in P$.
2. If (X,T) be regular minimal then (X,T) be locally almost period.

Theorem (2-16): If x and y be replete proximal then RP be invariant set

Proof: Let $(x,y) \in RP$,then for each index α in X ,there exist a replete subset $A \subset T$ such that $(xt,yt) \in \alpha$. since A be replete subset then for each compact sub set there exist $g_1, g_2 \in T$ such that $g_1Kg_2 \subset A$, that is $(x,y)g_1Kg_2 \subset \alpha$ and $(x,y)g_1Kg_2T \subset \alpha T$ by(2-8)we have $(x,y)g_1KT \subset \alpha$ since T syndetic set and g_1K compact set .Then $(x,y)T \subset \alpha$ and so $(x,y) \in P$ therefore $RP \subset P$ by theorem (2-9).Thus we have $RP = P$ and RP be invariant set. Δ

Remark (2-17): If $P(X,T)$ closed set and X be compact then P be compact set.

Theorem (2-18):Let (X,T) be transformation group . and $(x,y) \in RP$
A is replete, Then A is sub group.

Proof: Let $A = \{a \in A : g_1kg_2 = a\}$ is replete set and $(x,y) \in RP$ Then for each K compact subset of T there exist $g_1, g_2 \in T$ Such that $g_1Kg_2 \subset A$,Let $a,b \in A$ and $g_1kg_2 = a,\ g_1kg_2 = b$ for all $k \in K$ by hybothsis, there exist $b^{-1} \in T$ with $g_1kg_2b^{-1} = ab^{-1}$, since T be closed then $g_2b^{-1} \in T$ In fact take $g_2b^{-1} = d$ thus we get $g_1kd = ab^{-1}$, therefore $ab^{-1} \in A$,and A is replete group. Δ

Theorem (2-19):Let (X,T) be transformation group

<u>3-A Characterization of regular</u>

In this section, we give a characterization of regular

Definition (3-1). (7).Let (X,T) be a transformation group .The points x and y of X are said to be regular if h(x) and y are proximal for some automorphism h of (X,T).The set of all regular pairs is called the regular relation and is denoted by $R(X,T)$ or simply R .

Definition (3-2). Let (X,T) be a transformation group .Two points x and y are to replete regular if h(x) and y are replete proximal for some automorphism h of X . The set of all regular pairs is called the regular relation and is denoted by $R^*(X,T)$ or simply R^*

Definition (3-3).(9).Let (X,T) and (Y,T) be transformation groups .An epimorphism $\pi : X \to Y$ is called an r-homomorphism if a given h in (X,T) ,there exists a k in (Y,T) such that $\pi h = k\pi$.

Theorem (3-4) If $(x,y) \in R$, and $\pi : X \to Y$ be a r-homomorphism and $\theta : T \to T$ be onto homomorphism then $(\pi(x),\pi(y)) \in R(Y,T)$.

Proof: Let $\pi(x) = \pi(y)$. We need show that $(\pi(x),\pi(y)) \in R(Y,T)$,since $(x,y) \in R$,Then there exist automorphism h of (X,T) such that $(hx,y) \in P$.Therefore $(hx)t = yt$ $(t \in T)$, since $\pi : X \to Y$ is r-homomorphism then there exist K in (Y,T) such that $\pi h = k\pi$ Therefore we have $k\pi(x)\theta(t) = k\pi(xt) = \pi h(xt) = \pi(h(x)t) = \pi(yt) = \pi(y)\theta(t)$ .Huns we have $(k\pi(x),\pi(y)) \in P(Y,T)$ and therefore $(\pi(x),\pi(y)) \in R(Y,T)$. Δ

Theorem (3-5).Let $\pi : (X,T) \to (Y,T)$ be r-homomorphism distal (X,T) be regular minimal and $(x,y) \in P(Y,T)$ then $(\varphi^{-1}(x),\varphi^{-1}(y)) \in P(X,T)$.

Proof: Suppose that .We need only to show that $(\varphi^{-1}(x),\varphi^{-1}(y)) \in P(X,T)$.Since (X,T) is regular minimal and (x,y) is an almost periodic point of $(X \times X,T)$,there exists an automomorphism k in (Y,T) such that $k(x) = y$ and $\varphi^{-1}k(x) = \varphi^{-1}(y)$ since φ be r- homomorphism there exist an automomorphism h in (X,T) such that $\varphi h = k\varphi$ is equivalent $h\varphi^{-1} = \varphi^{-1}k$ implies $h(\varphi^{-1}(x)) = \varphi^{-1}(y)$ we have $(h(\varphi^{-1}(x)),\varphi^{-1}(y) \in P(X,T)$ and φ is distal then we get $(\varphi^{-1}(x),\varphi^{-1}(y) \in P(X,T)$.

Definition (3-6).Let $f : X \to X, g : X \to X$ .Then f and g are conjugate if there is a homomorphism $h : X \to X$ such that $h \circ g = f \circ h$ In this case, we write $f \approx_h g$.

Theorem (3-7):Let (X,T) be transformation group . and $(x,y) \in R$
$f : X \to X, g : X \to X$ If x be fixed under g and y fixed under f then $f \cong_h g$.
Proof: Let $(x,y) \in R$, then there exist automomorphism h in (X,T)
Such that $h(x) = y$ for some $e \in T$ and $f \circ h(x) = f(h(x)) = f(y)$ by hypothesis we have $f \circ h(x) = y$ so $h \circ g(x) = h(g(x)) = h(x) = y$ therefore
$f \circ h(x) = h \circ g(x)$. Δ

Theorem (3-8):Let (X,T) be transformation group , $f : X \to X$ bijective h(x) fixed under f Then $(x,y) \in R$.

Theorem (3-9): Let (X,T) be Transformation group free effective If $(x,y) \in R^*$ then $R^* = R$

Proof: Let $(x,y) \in R^*$ then there exist automomorphism h of (X,T)
Then $(hx,y) \in RP$ from theorem (2-9) we have $(hx,y) \in P$,then $(x,y) \in R$ therefore $R^* \subset R$ .similar method we have $R \subset R^*$ therefore
$R^* = R$. Δ

Remark (3-10): Let $\varphi : X \to Y$ be a distal homomorphism (X,T) be regular minimal then $(x,y) \in R(X,T)$.

Theorem (3-11).Let $\varphi: X \to Y$ be homomorphism , A be syndetic set and $(x, y) \in R$, then fallowing hold:

1) If y be fixed point.Then there exist an idempotent b in A .
2) A is subgroup of T.

Proof:: (1)Let $(x, y) \in R$.Then there exist automomorphism h in (X,T) Such that $h(x)a = ya$ for some $a \in A$ and A is syndetic set now have ϕ h(xa) $= \phi$ (h(x)a) $= \phi$(ya) since y be fixed point then
ϕ h(xa) $= \phi$ h(x) .Now have $xa = x$ .therefore $a = e$ .Thus A has contain an idempotent . Δ

(2) direct proof from (1).

Theorem (3-12):If $\varphi: X \to Y$ be a homomorphism Then following hold

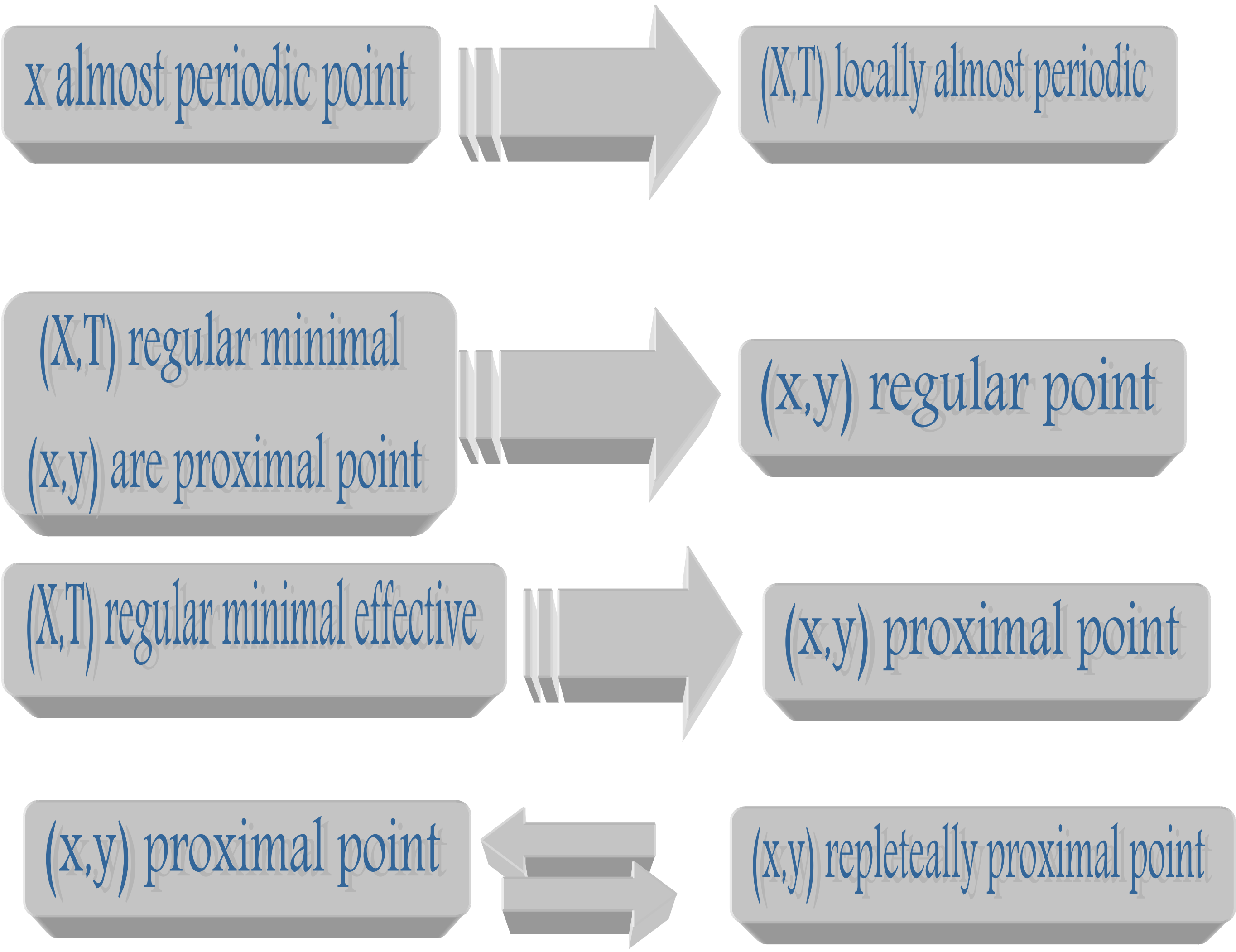
1) If $(x, y) \in R(X,T)$ then R is an invariant set.
2) If $(x, y) \in R(X,T)$ then R is an closed set.

Theorem (3-13):If $\varphi: X \to Y$ be a distal homomorphism and φ be regular if and only if $(x, y) \in R(X,T)$.

Proof: Let $(x, y) \in R$,then there exist automorphism h of (X,T) such that $(hx, y) \in P$ .From theorem (2-6(4)) we have $(hxt, yt) \in P$ for all $t \in T$, that is $(xt, yt) \in R$ and $(x, y)T \in R$ by hypotheses we obtain $R \subset (x, y)T$,therefore $R = (x, y)T$,from theorem (3-11(1),(2)) we get
$R = \overline{(x, y)T}$ and $\overline{(x, y)T}$ be minimal set then (x, y) be almost periodic by hypotheses there exist $e \in T$ such that $h(x) = y$ and $\varphi h(x) = \varphi(y)$ since φ be distal then $\varphi h(x) = \varphi(x)$ thus we get $\varphi h = \varphi$ .Therefore* φ be regular .Conversely direct from the definition of regular. Δ

References

(1) Ellis . R. Lectures on Topological Dynamics, W. A. Benjamin, New York, 1969.38-41
(2). Ellis. R. and Gottschalk .W. H., Homomorphisms of transformation groups, Trans. Amer. Math. Soc. 94 (1960), 258- 271.
(3). Auslander. J, On the proximal relation in topological dynamics, Proc. Amer. Math. Soc. II (1960), 88-86.
(4). Clay. J. P ,Proximity relations in transformation groups, Trans. Amer. Math. Soc. 108 (1963), 88-96.
(5)Woo.M.H ,Distal homomorphism ,j. of KMS,14(1978),235-2396
(6)Yu.J.O and Shin S.S.,The relative regionally regular relation,J.Chungcheong Math ,Soc.20(2007),no.4,495-501
(7) Yu. J. O. Regular relations in transformation group, J. Korean Math. Soc. 21 (1984), 41-48.
(8) Keynes .H.B.On the proximal relation being closed ,Proc. Amer,Soc.18(1967,518-522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(X,T) regular minimal
(x,y) are proximal point
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