

PAPER • OPEN ACCESS

KT-E-invexity in E-differentiable vector optimization problems

To cite this article: Najeeb Abdulaleem 2021 *J. Phys.: Conf. Ser.* **1900** 012001

View the [article online](#) for updates and enhancements.

You may also like

- [Zwicky Transient Facility and Globular Clusters: the Period–Luminosity and Period–Luminosity–Color Relations for Late-type Contact Binaries](#)
Chow-Choong Ngeow, Szu-Han Liao, Eric C. Bellm et al.
- [COMPARING NUMERICAL METHODS FOR ISOTHERMAL MAGNETIZED SUPERSONIC TURBULENCE](#)
Alexei G. Kritsuk, Åke Nordlund, David Collins et al.
- [Kinetic Tomography. II. A Second Method for Mapping the Velocity Field of the Milky Way Interstellar Medium and a Comparison with Spiral Structure Models](#)
Kirill Tchernyshyov, J. E. G. Peek and Gail Zasowski



The Electrochemical Society
Advancing solid state & electrochemical science & technology

243rd ECS Meeting with SOFC–XVIII

More than 50 symposia are available!

Present your research and accelerate science

Boston, MA • May 28 – June 2, 2023

[Learn more and submit!](#)

KT-E-invexity in *E*-differentiable vector optimization problems

Najeeb Abdulaleem^{1,2}

¹ Department of Mathematics, Hadhramout University, P.O. BOX : (50511-50512), Al-Mahrah, Yemen.

² Faculty of Mathematics and Computer Science, University of Łódź, Banacha 22, 90-238 Łódź, Poland.

E-mail: nabbas985@gmail.com

Abstract. In this paper, a new concept of generalized convexity is introduced for *E*-differentiable vector optimization problems. Namely, the concept of *KT-E*-invexity is defined for (not necessarily) differentiable vector optimization problems in which the functions involved are *E*-differentiable. The sufficiency of the so-called *E*-Karush–Kuhn–Tucker optimality conditions is established for the considered *E*-differentiable multiobjective programming problem under assumption that is *KT-E*-invex at an *E*-Karush–Kuhn–Tucker point. Further, the examples of *KT-E*-invex optimization problems with *E*-differentiable functions are constructed to illustrate the aforesaid results. Moreover, the so-called vector Mond-Weir *E*-dual problem is also derived for the considered *E*-differentiable vector optimization problem and several *E*-duality theorems in the sense of Mond-Weir are derived under *KT-E*-invexity hypotheses.

Key words: *KT-E*-invex optimization problem; generalized convexity; *E*-differentiable function; *E*-optimality conditions; *E*-duality.

2020 Mathematics Subject Classification: 90C26, 90C29, 90C30, 90C46.

1. Introduction

The field of vector optimization, also known as multiobjective programming, has attracted a lot of attention since many real-world problems in engineering problems, physics, economics, management sciences, decision theory, game theory, optimal control can be modeled as nonlinear vector optimization problems. Therefore, considerable attention has been given recently to obtaining new optimality results for various classes of differentiable and non-differentiable nonlinear nonconvex multiobjective programming problems (see, for example, [1], [2], [3], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [22], [28], [31], [32], [34], and others). One of the most important of such generalizations of convexity is the invexity notion introduced by Hanson [19] for differentiable scalar optimization problems. Martin [24] defined the notion of a Kuhn-Tucker invex problem for differentiable scalar extremum problems. He also showed that an optimization problem is Kuhn-Tucker invex if and only if each point which satisfies the Kuhn-Tucker necessary optimality conditions is a global minimizer. In recent years, several generalizations of the *KT*-invexity have been introduced to optimization theory, also in a vectorial case (see, for example, [4], [26], [27], [28], [29], [30], and others).



The concept of E -convexity, introduced by Youness [34], is one of the notions of generalized convexity that weakens the convexity assumptions to prove the fundamental results in optimization theory for a new class of nonconvex differentiable optimization problems. Megahed et al. [21] presented a new concept of an E -differentiable convex function and they established optimality conditions for mathematical programming problems in which the functions involved are E -differentiable. Recently, Abdulaleem [7] introduced a new concept of generalized convexity. Namely, Abdulaleem defined the concept of E -differentiable E -invexity in the case of (not necessarily) differentiable vector optimization problems with E -differentiable functions.

In this paper, firstly, we characterize the class of E -differentiable E -invex functions by giving its new property. Namely, we show that every E -stationary point of any E -differentiable E -invex function is its global E -minimizer. Further, we consider a new class of E -differentiable vector optimization problems with inequality constraints. Namely, we define the class of so-called E -differentiable KT - E -invex multiobjective programming problems as a generalization of the concept of differentiable KT -invex vector optimization problem introduced by Osuna-Gómez et al. [26] and the definition of E -differentiable E -invex functions introduced by Abdulaleem [7]. Then, we prove the sufficient optimality conditions for this new class of E -differentiable vector optimization problems, that is, E -differentiable KT - E -invex ones. This result is illustrated by the examples of E -differentiable KT - E -invex optimization problems. Thus, we also show that the optimality results established in the paper are applicable for a larger class of E -differentiable vector optimization problems than under E -differentiable E -invexity hypotheses. Moreover, so-called vector Mond-Weir E -dual problem is defined for the considered (not necessarily) differentiable vector optimization problems with E -differentiable functions. Then, several E -duality theorems are established between the considered E -differentiable vector optimization problems and its vector E -duals under KT - E -invexity hypotheses.

2. Preliminaries

Let R^n be the n -dimensional Euclidean space and R_+^n be its nonnegative orthant. The following convention for equalities and inequalities will be used in the paper. For any vectors $x = (x_1, x_2, \dots, x_n)^T$ and $y = (y_1, y_2, \dots, y_n)^T$ in R^n , we define: $x > y \Leftrightarrow x_i > y_i, i = 1, 2, \dots, n$; $x \geq y \Leftrightarrow x_i \geq y_i, i = 1, 2, \dots, n$; $x \geq y \Leftrightarrow x \geq y, x \neq y$.

We now give the definition of an E -differentiable function introduced by Megahed et al. [21].

Definition 1 [21] Let $E : R^n \rightarrow R^n$ and $f : R^n \rightarrow R$ be a (not necessarily) differentiable function at a given point $u \in R^n$. It is said that f is an E -differentiable function at u if and only if $f \circ E$ is a differentiable function at u (in the usual sense), that is,

$$(f \circ E)(x) = (f \circ E)(u) + \nabla (f \circ E)(u)(x - u) + \theta(u, x - u) \|x - u\|, \quad (1)$$

where $\theta(u, x - u) \rightarrow 0$ as $x \rightarrow u$.

We now give the definitions of an E -invex set and an E -invex function introduced by Abdulaleem [7].

Definition 2 [7] Let $E : R^n \rightarrow R^n$. A set $M \subseteq R^n$ is said to be an E -invex set iff there exists a vector-valued function $\eta : M \times M \rightarrow R^n$ such that the relation

$$E(u) + \lambda \eta(E(x), E(u)) \in M$$

holds for all $x, u \in M$ and any $\lambda \in [0, 1]$.

Definition 3 [7] Let $E : R^n \rightarrow R^n$, $M \subseteq R^n$ be a nonempty open E -invex set with respect to the vector-valued function $\eta : R^n \times R^n \rightarrow R^n$ and $f : R^n \rightarrow R^k$ be an E -differentiable function on M . It is said that f is a vector-valued E -invex function with respect to η at u on M if, for all $x \in M$,

$$f_i(E(x)) - f_i(E(u)) \geq \nabla f_i(E(u))\eta(E(x), E(u)), \quad i = 1, \dots, k. \quad (2)$$

If inequalities (2) hold for any $u \in M$, then f is E -invex with respect to η on M .

In this paper, we consider the following (not necessarily differentiable) multiobjective optimization problem (VP):

$$\begin{aligned} & \text{minimize } f(x) = (f_1(x), \dots, f_p(x)) \\ & \text{subject to } g_j(x) \leq 0, \quad j \in J = \{1, \dots, m\}, \end{aligned} \quad (\text{VP})$$

where $f_i : R^n \rightarrow R$, $i \in I = \{1, \dots, p\}$, $g_j : R^n \rightarrow R$, $j \in J$, are real-valued functions defined on R^n . We shall write $g := (g_1, \dots, g_m) : R^n \rightarrow R^m$ for convenience. Let Ω denote the set of all feasible solutions in problem (VP), that is,

$$\Omega := \{x \in R^n : g_j(x) \leq 0, \quad j \in J\}.$$

Further, we denote by $J(x)$ the set of inequality constraint indices that are active at a feasible solution x , that is, $J(x) = \{j \in J : g_j(x) = 0\}$.

Definition 4 A point \bar{x} is said to be a weakly efficient solution (weak Pareto solution) of (VP) if there exists no x such that

$$f(x) < f(\bar{x}).$$

Definition 5 A point \bar{x} is said to be an efficient solution (a Pareto solution) of (VP) if there exists no x such that

$$f(x) \leq f(\bar{x}).$$

Let $E : R^n \rightarrow R^n$ be a given one-to-one and onto operator. Now, for the considered multiobjective programming problem (VP), we define its associated differentiable vector optimization problem (VP_E) as follows:

$$\begin{aligned} & \text{minimize } f(E(x)) = (f_1(E(x)), \dots, f_p(E(x))) \\ & \text{subject to } g_j(E(x)) \leq 0, \quad j \in J = \{1, \dots, m\}. \end{aligned} \quad (\text{VP}_E)$$

The problem (VP_E) is referred to as an E -vector optimization problem associated to (VP). Let Ω_E denote the set of all feasible solutions of (VP_E) , that is,

$$\Omega_E := \{x \in R^n : g_j(E(x)) \leq 0, \quad j \in J\}.$$

Since the functions constituting (VP) are assumed to be E -differentiable at any feasible solution of (VP), by Definition 1, the functions constituting (VP_E) are differentiable at any its feasible solution (in the usual sense). Further, we denote by $J_E(x)$ the set of inequality constraint indices that are active at a feasible solution $x \in \Omega_E$, that is, $J_E(x) = \{j \in J : (g_j \circ E)(x) = 0\}$.

Definition 6 A point $E(\bar{x})$ is said to be a weakly E -efficient solution (weak E -Pareto solution) of (VP) if there exists no $E(x)$ such that

$$f(E(x)) < f(E(\bar{x})).$$

Definition 7 A point $E(\bar{x})$ is said to be an E -efficient solution (an E -Pareto solution) of (VP) if there exists no $E(x)$ such that

$$f(E(x)) \leq f(E(\bar{x})).$$

Lemma 8 [5] Let $E : R^n \rightarrow R^n$ be a one-to-one and onto. Then $E(\Omega_E) = \Omega$.

Lemma 9 [5] Let $\bar{x} \in \Omega_E$ be a Pareto solution (a weak Pareto solution) of the problem (VP_E) . Then $E(\bar{x})$ is an E -Pareto solution (a weak E -Pareto solution) of the problem (VP).

Theorem 10 Let $E : R^n \rightarrow R^n$. $f : R^n \rightarrow R$ is an E -differentiable E -invex function with respect to η on R^n , where $\eta : R^n \times R^n \rightarrow R^n$ if and only if its every E -stationary point is a global E -minimum of f .

Proof. “ \Rightarrow ” Let $E : R^n \rightarrow R^n$. Clearly, if f is an E -differentiable E -invex vector-valued function with respect to η on R^n and $E(\bar{x})$ its E -stationary point, then $\nabla f(E(\bar{x})) = 0$ implies $f(E(\bar{x})) \leq f(E(x))$, $\forall x \in R^n$.

“ \Leftarrow ”

If $\nabla f(E(\bar{x})) = 0$, take $\eta(E(x), E(\bar{x})) = 0$.

If $\nabla f(E(\bar{x})) \neq 0$, take

$$\eta(E(x), E(\bar{x})) = \frac{f(E(x)) - f(E(\bar{x}))}{\nabla f(E(\bar{x})) \nabla f(E(\bar{x}))} \nabla f(E(\bar{x})).$$

■

Corollary 11 Let $E : R^n \rightarrow R^n$. If $f : R^n \rightarrow R^k$ has no E -stationary points, then f is an E -differentiable E -invex vector-valued function on R^n .

Example 12 Let $E : R \rightarrow R$, $f : R \rightarrow R$ be an E -differentiable function on R defined by

$$f(x) = \sqrt[3]{x}, \quad E(x) = x^9.$$

Note that f is not an E -differentiable E -invex function. Since the differentiable function $f(E(x)) = x^3$ has an E -stationary point at $E(x) = 0$, but it is not a global E -minimum.

Example 13 Let $E : R^2 \rightarrow R^2$, $f : R^2 \rightarrow R$ be an E -differentiable function on R^2 defined by

$$f(x) = \sqrt[3]{x_1} + \sqrt[3]{x_2} - 10\sqrt[3]{x_2} - \sqrt[3]{x_2}, \quad E(x_1, x_2) = (x_1^9, x_2^9).$$

Since the function f has no E -stationary points, then f is an E -differentiable E -invex function with respect to η .

3. KT - E -invexity and optimality

In this section, for the considered vector optimization problem (VP), we define a new concept of generalized convexity which is a generalization of a class of E -invexity defined by Abdalaleem [7] and the class of differentiable KT -invex multiobjective optimization problems introduced by Osuna-Gómez et al. [26].

Definition 14 Let $E : R^n \rightarrow R^n$ and the considered multiobjective optimization problem (VP) is said to be an E -differentiable KT - E -invex vector optimization problem at $E(\bar{x}) \in \Omega$ on Ω , if there exists a vector-valued function $\eta : \Omega \times \Omega \rightarrow R^n$ such that

$$\begin{cases} E(x) \in \Omega, \\ g_j(E(x)) \leq 0, \\ g_j(E(\bar{x})) \leq 0 \end{cases} \implies \begin{cases} f_i(E(x)) - f_i(E(\bar{x})) \geq \nabla f_i(E(\bar{x}))\eta(E(x), E(\bar{x})), \quad i \in I, \\ -\nabla g_j(E(\bar{x}))\eta(E(x), E(\bar{x})) \geq 0, \quad j \in J(E(\bar{x})). \end{cases} \quad (3)$$

If (3) is fulfilled at any point $E(\bar{x}) \in \Omega$ on Ω , then the considered multiobjective optimization problem (VP) is said to be an E -differentiable KT - E -invex vector optimization problem on Ω .

Definition 15 Let $E : R^n \rightarrow R^n$ and the considered multiobjective optimization problem (VP) is said to be E -differentiable strict KT - E -invex vector optimization problem at $E(\bar{x}) \in \Omega$ on Ω , if there exists a vector-valued function $\eta : \Omega \times \Omega \rightarrow R^n$ such that

$$\begin{cases} E(x) \in \Omega, \\ E(x) \neq E(\bar{x}), \\ g_j(E(x)) \leq 0, \\ g_j(E(\bar{x})) \leq 0 \end{cases} \implies \begin{cases} f_i(E(x)) - f_i(E(\bar{x})) > \nabla f_i(E(\bar{x}))\eta(E(x), E(\bar{x})), \quad i \in I, \\ -\nabla g_j(E(\bar{x}))\eta(E(x), E(\bar{x})) \geq 0, \quad j \in J(E(\bar{x})). \end{cases} \quad (4)$$

If (4) is fulfilled at any point $E(\bar{x}) \in \Omega$ on Ω , then the considered multiobjective optimization problem (VP) is said to be an E -differentiable strict KT - E -invex vector optimization problem on Ω .

Now, we also give the definitions of KT -invexity and strict KT -invexity for the differentiable E -vector optimization (VP_E) which is associated to the considered E -differentiable multiobjective programming problem (VP).

Definition 16 Let $E : R^n \rightarrow R^n$ and the E -vector optimization problem (VP_E) is said to be a differentiable KT -invex vector optimization problem at $\bar{x} \in \Omega_E$ on Ω_E , if there exists a vector-valued function $\eta_E : \Omega_E \times \Omega_E \rightarrow R^n$ such that

$$\begin{cases} x \in \Omega_E, \\ g_j(E(x)) \leq 0, \\ g_j(E(\bar{x})) \leq 0 \end{cases} \implies \begin{cases} f_i(E(x)) - f_i(E(\bar{x})) \geq \nabla f_i(E(\bar{x}))\eta(E(x), E(\bar{x})), \quad i \in I, \\ -\nabla g_j(E(\bar{x}))\eta(E(x), E(\bar{x})) \geq 0, \quad j \in J(E(\bar{x})). \end{cases} \quad (5)$$

If (5) is fulfilled at any point $\bar{x} \in \Omega_E$ on Ω_E , then the E -vector optimization problem (VP_E) is said to be a differentiable KT -invex vector optimization problem on Ω_E .

Definition 17 Let $E : R^n \rightarrow R^n$ and the E -vector optimization problem (VP_E) is said to be a differentiable strict KT -invex vector optimization problem at $\bar{x} \in \Omega_E$ on Ω_E , if there exists a vector-valued function $\eta_E : \Omega_E \times \Omega_E \rightarrow R^n$ such that

$$\begin{cases} x \in \Omega_E, \\ x \neq \bar{x}, \\ g_j(E(x)) \leq 0, \\ g_j(E(\bar{x})) \leq 0 \end{cases} \implies \begin{cases} f_i(E(x)) - f_i(E(\bar{x})) > \nabla f_i(E(\bar{x}))\eta(E(x), E(\bar{x})), \quad i \in I, \\ -\nabla g_j(E(\bar{x}))\eta(E(x), E(\bar{x})) \geq 0, \quad j \in J(E(\bar{x})). \end{cases} \quad (6)$$

If (6) is fulfilled at any point $\bar{x} \in \Omega_E$ on Ω_E , then the E -vector optimization problem (VP_E) is said to be a differentiable strict KT -invex vector optimization problem on Ω_E .

Remark 18 If (VP_E) is a differentiable KT -invex vector optimization problem on Ω_E , then (VP) is an E -differentiable KT - E -invex vector optimization problem on Ω and the converse is true.

We now recall the E -Karush-Kuhn-Tucker necessary optimality conditions established by Abdulaleem [7].

Theorem 19 [7] (E -Karush-Kuhn-Tucker necessary optimality conditions) Let $\bar{x} \in \Omega_E$ be a weak Pareto solution of the problem (VP_E) (and, thus, $E(\bar{x})$ be a weak E -Pareto solution of the problem (VP)). Further, let the objective functions f_i , $i \in I$, the constraint functions g_j , $j \in J$,

be E -differentiable at \bar{x} and the Guignard constraint qualification (GCQ) [7] be satisfied at \bar{x} . Then there exist Lagrange multipliers $\bar{\tau} \in R^p$, $\bar{\xi} \in R^m$ such that

$$\sum_{i=1}^p \bar{\tau}_i \nabla (f_i \circ E)(\bar{x}) + \sum_{j=1}^m \bar{\xi}_j \nabla (g_j \circ E)(\bar{x}) = 0, \quad (7)$$

$$\bar{\xi}_j (g_j \circ E)(\bar{x}) = 0, \quad j \in J, \quad (8)$$

$$\bar{\tau} \geq 0, \quad \bar{\xi} \geq 0. \quad (9)$$

Definition 20 $(E(\bar{x}), \bar{\tau}, \bar{\xi}) \in \Omega \times R^p \times R^m$ is said to be an E -Karush-Kuhn-Tucker point for the considered constrained vector optimization problem (VP) if the E -Karush-Kuhn-Tucker necessary optimality conditions (7)-(9) are satisfied at $E(\bar{x})$ with Lagrange multiplier $\bar{\tau}$, $\bar{\xi}$.

Definition 21 $(\bar{x}, \bar{\tau}, \bar{\xi}) \in \Omega_E \times R^p \times R^m$ is said to be a Karush-Kuhn-Tucker point for the E -vector optimization problem (VP_E) if the Karush-Kuhn-Tucker necessary optimality conditions (7)-(9) are satisfied at \bar{x} with Lagrange multiplier $\bar{\tau}$, $\bar{\xi}$.

Now, we prove the sufficiency of the E -Karush-Kuhn-Tucker optimality conditions for the E -differentiable multiobjective optimization problem (VP) under KT - E -invexity hypotheses.

Theorem 22 Let the considered multiobjective optimization problem (VP) be a vector KT - E -invex optimization problem on Ω . Then, every vector E -Karush-Kuhn-Tucker point of the multiobjective optimization problem (VP) is its weakly E -efficient solution.

Proof. Let the considered multiobjective optimization problem (VP) be a vector KT - E -invex optimization problem on Ω . Further, we assume that $E(\bar{x})$ is an E -Karush-Kuhn-Tucker point of the considered multiobjective optimization problem (VP). Then, by Definition 20, the E -Karush-Kuhn-Tucker necessary optimality conditions (7)-(9) are satisfied at $E(\bar{x})$ with Lagrange multipliers $\bar{\tau} \in R^p$ and $\bar{\xi} \in R^m$. We proceed by contradiction. Suppose, contrary to the result, that $E(\bar{x})$ is not a weakly E -efficient solution of the problem (VP). Hence, by Definition 6, there exists other $E(\tilde{x}) \in \Omega$ such that

$$f(E(\tilde{x})) < f(E(\bar{x})). \quad (10)$$

Since $\bar{\tau} \geq 0$, the above inequality yields

$$\sum_{i=1}^p \bar{\tau}_i f(E(\tilde{x})) < \sum_{i=1}^p \bar{\tau}_i f(E(\bar{x})). \quad (11)$$

Since the considered multiobjective optimization problem (VP) is an KT - E -invex vector optimization problem on Ω , by Definition 14, the following inequalities

$$\begin{aligned} f_i(E(\tilde{x})) - f_i(E(\bar{x})) &\geq \nabla f_i(E(\bar{x}))\eta(E(\tilde{x}), E(\bar{x})), \quad i \in I, \\ -\nabla g_j(E(\bar{x}))\eta(E(\tilde{x}), E(\bar{x})) &\geq 0, \quad j \in J(E(\bar{x})) \end{aligned} \quad (12)$$

hold. Multiplying the above inequalities by the corresponding Lagrange multipliers, respectively, we obtain

$$\begin{aligned} \bar{\tau}_i f_i(E(\tilde{x})) - \bar{\tau}_i f_i(E(\bar{x})) &\geq \bar{\tau}_i \nabla f_i(E(\bar{x}))\eta(E(\tilde{x}), E(\bar{x})), \quad i \in I, \\ -\bar{\xi}_j \nabla g_j(E(\bar{x}))\eta(E(\tilde{x}), E(\bar{x})) &\geq 0, \quad j \in J(E(\bar{x})). \end{aligned} \quad (13)$$

Adding both sides of the above inequalities, we obtain that the inequality

$$\sum_{i=1}^p \bar{\tau}_i f_i(E(\tilde{x})) - \sum_{i=1}^p \bar{\tau}_i f_i(E(\bar{x})) \geq \left[\sum_{i=1}^p \bar{\tau}_i \nabla (f_i \circ E)(\bar{x}) + \sum_{j=1}^m \bar{\xi}_j \nabla g_j(E(\bar{x})) \right] \eta(E(\tilde{x}), E(\bar{x})) \tag{14}$$

holds. Combining (11) and (14), we get that the inequality

$$\left[\sum_{i=1}^p \bar{\tau}_i \nabla (f_i \circ E)(\bar{x}) + \sum_{j=1}^m \bar{\xi}_j \nabla g_j(E(\bar{x})) \right] \eta(E(\tilde{x}), E(\bar{x})) < 0$$

holds, which is a contradiction to the E -Karush-Kuhn-Tucker necessary optimality condition (7). Thus, the proof of this theorem is completed. ■

Theorem 23 *Let the E -vector optimization problem (VP_E) be a vector KT -invex optimization problem on Ω_E . Then, every vector Karush-Kuhn-Tucker point of the E -vector optimization problem (VP_E) is its weakly efficient solution.*

Now, we present an example of an E -differentiable vector optimization problem in which the considered multiobjective optimization problem (VP) is KT - E -invex.

Example 24 *Consider the following nonconvex nondifferentiable vector optimization problem*

$$\begin{aligned} & \text{minimize } f(x) = (\sqrt[3]{x_1^2} + \sqrt[3]{x_2^2}, \sqrt[3]{x_1^2} + \sqrt[3]{x_2^2} + \sqrt[3]{x_1}) \\ & \text{subject to } g(x) = x_1 + \sqrt[3]{x_2^2} \leq 0. \end{aligned} \tag{VP1}$$

Note that $\Omega = \{(x_1, x_2) \in R^2 : x_1 + \sqrt[3]{x_2^2} \leq 0\}$. Let $E : R^2 \rightarrow R^2$ be a one-to-one and onto mapping defined as follows $E(x_1, x_2) = (x_1^3, x_2^3)$. Now, for the considered E -differentiable vector optimizations problem (VP1), we define its associated E -vector optimization problem $(VP1_E)$ as follows

$$\begin{aligned} & \text{minimize } f(E(x)) = (x_1^2 + x_2^2, x_1^2 + x_2^2 + x_1) \\ & \text{subject to } g(E(x)) = x_1^3 + x_2^2 \leq 0. \end{aligned} \tag{VP1_E}$$

Note that $\Omega_E = \{(x_1, x_2) \in R^2 : x_1^3 + x_2^2 \leq 0\}$ and $\bar{x} = (0, 0)$ is a Pareto solution in $(VP1_E)$, where minimize $x_1^2 + x_2^2 = 0 = f_1(\bar{x})$. However, $\tilde{x} = (-\frac{1}{2}, 0)$ is a Pareto solutions in $(VP1_E)$, where minimize $x_1^2 + x_2^2 + x_1 = -\frac{1}{4} = f_2(\tilde{x})$. Further, note that all functions constituting the considered multiobjective optimization problem (VP1) are E -differentiable at $\bar{x} = (0, 0)$. It can be shown, by Definition 16, that the E -vector optimization problem $(VP1_E)$ is KT -invex at Pareto solutions \bar{x} and \tilde{x} on Ω_E with respect to $\eta(E(x), E(u)) = (-x_1^2 - u_1^2, x_2^2 + u_2^2)$. Since all hypotheses of Theorem 23 are satisfied, therefore, \bar{x} and \tilde{x} are Pareto solutions to the E -vector optimization problem $(VP1_E)$ and, thus, all hypotheses of Theorem 22 are satisfied, therefore, $E(\bar{x}) = (0, 0)$ is an E -Pareto solution of the considered E -differentiable vector optimization problem (VP1) and, by Definition 14, the vector optimization problem (VP1) is KT - E -invex at $E(\bar{x})$ on Ω with respect to η given above. Further, note that the constraint function g is not E -invex on Ω_E . This follows from the fact that a stationary points of the constraint function g are not its global minimizer (see Theorem 10).

Now, we consider the example of a KT - E -invex optimization problem (VP), which is not KT -invex optimization problem given in [26].

Example 25 Consider the following nonconvex nondifferentiable vector optimization problem

$$\begin{aligned} \text{minimize } f(x) &= (\sqrt[3]{x_1^4} + (\sqrt[3]{x_2^2} + 1)^2, \sqrt[3]{x_1^2} + \sqrt[3]{x_2^4}) \\ \text{subject to } g(x) &= x_1 + x_2 - \sqrt[3]{x_1} \leq 0. \end{aligned} \quad (VP2)$$

Note that $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 - \sqrt[3]{x_1} \leq 0\}$. Let $E : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an one-to-one and onto mapping defined as follows $E(x_1, x_2) = (x_1^3, x_2^3)$. Now, for the considered E -differentiable vector optimization problem (VP2), we define its associated E -vector optimization problem (VP1 $_E$) as follows

$$\begin{aligned} \text{minimize } f(E(x)) &= (x_1^4 + (x_2^2 + 1)^2, x_1^2 + x_2^4) \\ \text{subject to } g(E(x)) &= x_1^3 + x_2^3 - x_1 \leq 0. \end{aligned} \quad (VP2_E)$$

Note that $\Omega_E = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^3 + x_2^3 - x_1 \leq 0\}$ and $\bar{x} = (0, 0)$ is a feasible solution of the problem (VP2 $_E$) at which the Karush-Kuhn-Tucker necessary optimality conditions are satisfied. Further, note that all functions constituting the considered vector optimization problem (VP2) are E -differentiable at $\bar{x} = (0, 0)$. It can be shown, by Definition 14, that the vector optimization problem (VP2) is KT - E -invex at $E(\bar{x})$ on Ω with respect to $\eta(E(x), E(u)) = (x_1^2 + u_1^2, x_2^2 + u_2^2 + 1)$. However, by the definition of a KT -invex optimization problem (see Osuna-Gómez et al. [26]), it follows that the multiobjective optimization problem (VP2) is not KT -invex at \bar{x} on Ω with respect to $\eta(x, u) = (\sqrt[3]{x_1^2} + \sqrt[3]{u_1^2}, \sqrt[3]{x_2^2} + \sqrt[3]{u_2^2} + 1)$.

4. Mond-Weir E -duality

In this section, for the considered E -differentiable multiobjective programming problem (VP), we define its vector dual problem (VD $_E$) in the sense of Mond-Weir [23].

Let $E : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a given one-to-one and onto operator. For the differentiable multicriteria E -optimization problem (VP $_E$), we define the following vector dual problem in the sense of Mond-Weir:

$$(VD_E) \quad f(E(y)) = (f_1(E(y)), \dots, f_p(E(y))) \rightarrow V - \max \quad (15)$$

$$\text{subject to} \quad \sum_{i=1}^p \tau_i \nabla f_i(E(y)) + \sum_{j=1}^m \xi_j \nabla g_j(E(y)) = 0, \quad (16)$$

$$\sum_{j=1}^m \xi_j g_j(E(y)) \geq 0, \quad (17)$$

$$\tau \in \mathbb{R}^p, \tau \geq 0, \xi \in \mathbb{R}^m, \xi \geq 0. \quad (18)$$

Further, let

$$W_E = \left\{ (y, \tau, \xi) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m : \sum_{i=1}^p \tau_i \nabla f_i(E(y)) + \sum_{j=1}^m \xi_j \nabla g_j(E(y)) = 0, \sum_{j=1}^m \xi_j g_j(E(y)) \geq 0, \tau \geq 0, \xi \geq 0 \right\}$$

be the set of all feasible solutions of the problem (VD $_E$). Let us denote $Y_E = \{y \in \mathbb{R}^n : (y, \tau, \xi) \in W_E\}$.

Theorem 26 (Mond-Weir weak duality between (VP_E) and (VD_E)). Let $x \in \Omega_E$ and $(y, \tau, \xi) \in W_E$. Further, assume that (VP_E) is KT -invex on $\Omega_E \cup Y_E$. Then

$$f(E(x)) \not\leq f(E(y)). \quad (19)$$

Proof. Let $x \in \Omega_E$ and $(y, \tau, \xi) \in W_E$. We proceed by contradiction. Suppose, contrary to the result, that the inequality

$$f(E(x)) < f(E(y)) \quad (20)$$

holds. By the feasibility of (y, τ, ξ) in the problem (VD_E) , the above inequality yields

$$\sum_{i=1}^p \tau_i f_i(E(x)) < \sum_{i=1}^p \tau_i f_i(E(y)). \quad (21)$$

By assumption, $x \in \Omega_E$ and $(y, \tau, \xi) \in W_E$. Since the problem (VP_E) is KT -invex on $\Omega_E \cup Y_E$, by Definition 16, the inequalities

$$f_i(E(x)) - f_i(E(y)) \geq \nabla f_i(E(y))\eta(E(x), E(y)), \quad i \in I \quad (22)$$

$$-\nabla g_j(E(y))\eta(E(x), E(y)) \geq 0. \quad j \in J(E(y)) \quad (23)$$

hold. Multiplying above inequalities by the corresponding Lagrange multipliers, respectively, and then summarizing the resulting inequalities, we obtain

$$\sum_{i=1}^p \tau_i f_i(E(x)) - \sum_{i=1}^p \tau_i f_i(E(y)) \geq \sum_{i=1}^p \tau_i \nabla f_i(E(y))\eta(E(x), E(y)), \quad (24)$$

$$- \sum_{j=1}^m \xi_j \nabla g_j(E(y))\eta(E(x), E(y)) \geq 0. \quad (25)$$

From (24) and (25), we obtain that the following inequality

$$\sum_{i=1}^p \tau_i f_i(E(x)) - \sum_{i=1}^p \tau_i f_i(E(y)) \geq \left[\sum_{i=1}^p \tau_i \nabla f_i(E(y)) + \sum_{j=1}^m \xi_j \nabla g_j(E(y)) \right] \eta(E(x), E(y)) \quad (26)$$

holds. Thus, by (16), it follows that the inequality

$$\sum_{i=1}^p \tau_i f_i(E(x)) \geq \sum_{i=1}^p \tau_i f_i(E(y)) \quad (27)$$

holds, contradicting (21). This means that the proof of the Mond-Weir weak duality theorem between the problems (VP_E) and (VD_E) is completed. ■

Theorem 27 (Mond-Weir weak E -duality between (VP) and (VD_E)). Let $E(x) \in \Omega$ and $(y, \tau, \xi) \in W_E$. Further, assume that (VP) is KT - E -invex on $\Omega \cup Y_E$. Then, the Mond-Weir weak E -duality between (VP) and (VD_E) holds, that is,

$$f(E(x)) \not\leq f(E(y)).$$

Proof. Let $E(x) \in \Omega$ and $(y, \tau, \xi) \in W_E$. We proceed by contradiction. Suppose, contrary to the result, that the inequality

$$f(E(x)) < f(E(y)) \quad (28)$$

holds. By the feasibility of (y, τ, ξ) in the problem (VD_E) , the above inequality yields

$$\sum_{i=1}^p \tau_i f_i(E(x)) < \sum_{i=1}^p \tau_i f_i(E(y)). \quad (29)$$

By assumption, $E(x)$ and (y, τ, ξ) are feasible solutions of the problems (VP) and (VD_E) , respectively. Since the problem (VP) is KT - E -invex on $\Omega \cup Y_E$, by Definition 14, the inequalities

$$f_i(E(x)) - f_i(E(y)) \geq \nabla f_i(E(y)) \eta(E(x), E(y)), \quad i \in I \quad (30)$$

$$-\nabla g_j(E(y)) \eta(E(x), E(y)) \geq 0. \quad j \in J(E(y)) \quad (31)$$

hold. Multiplying above inequalities by the corresponding Lagrange multipliers, respectively, and then summarizing the resulting inequalities, we obtain

$$\sum_{i=1}^p \tau_i f_i(E(x)) - \sum_{i=1}^p \tau_i f_i(E(y)) \geq \sum_{i=1}^p \tau_i \nabla f_i(E(y)) \eta(E(x), E(y)), \quad (32)$$

$$- \sum_{j=1}^m \xi_j \nabla g_j(E(y)) \eta(E(x), E(y)) \geq 0. \quad (33)$$

From (32) and (33), we obtain that the following inequality

$$\sum_{i=1}^p \tau_i f_i(E(x)) - \sum_{i=1}^p \tau_i f_i(E(y)) \geq \left[\sum_{i=1}^p \tau_i \nabla f_i(E(y)) + \sum_{j=1}^m \xi_j \nabla g_j(E(y)) \right] \eta(E(x), E(y)) \quad (34)$$

holds. Thus, by (16), it follows that the inequality

$$\sum_{i=1}^p \tau_i f_i(E(x)) \geq \sum_{i=1}^p \tau_i f_i(E(y)) \quad (35)$$

holds, contradicting (29). This means that the proof of the Mond-Weir weak E -duality theorem between the problems (VP) and (VD_E) is completed. ■

Theorem 28 (*Mond-Weir strong duality between (VP_E) and (VD_E) and also strong E -duality between (VP) and (VD_E)). Let $\bar{x} \in \Omega_E$ be an efficient solution (a weakly efficient solution) of the problem (VP_E) (and, thus, $E(\bar{x}) \in \Omega$ be an E -efficient solution (a weakly E -efficient solution) of the problem (VP)). Further, assume that the Guignard constraint qualification (GCQ) [7] be satisfied at \bar{x} . Then there exist $\bar{\tau} \in R^p$, $\bar{\xi} \in R^m$, $\bar{\xi} \geq 0$ such that $(\bar{x}, \bar{\tau}, \bar{\xi})$ is feasible for the problem (VD_E) . If all hypotheses of (Theorem 26) Theorem 27 are satisfied, then $(\bar{x}, \bar{\tau}, \bar{\xi})$ is an efficient solution (a weakly efficient solution) of a maximum type in the problem (VD_E) . In other words, if $E(\bar{x}) \in \Omega$ is an E -efficient solution (a weakly E -efficient solution) of the problem (VP) , then $(\bar{x}, \bar{\tau}, \bar{\xi})$ is an efficient solution (a weakly efficient solution) of a maximum type in the dual problem (VD_E) . This means that the Mond-Weir strong E -duality holds between the problems (VP) and (VD_E) .*

Proof. Since $\bar{x} \in \Omega_E$ is an efficient solution (a weakly efficient solution) of the problem (VP_E) and the Guignard constraint qualification (GCQ) is satisfied at \bar{x} , there exist $\bar{\tau} \in R^p$, $\bar{\xi} \in R^m$, $\bar{\xi} \geq 0$ such that

$$\sum_{i=1}^p \bar{\tau}_i \nabla (f_i \circ E)(\bar{x}) + \sum_{j=1}^m \bar{\xi}_j \nabla (g_j \circ E)(\bar{x}) = 0,$$

$$\bar{\xi}_j (g_j \circ E)(\bar{x}) = 0, \quad j \in J,$$

$$\bar{\tau} \geq 0, \quad \bar{\xi} \geq 0.$$

Thus, $(\bar{x}, \bar{\tau}, \bar{\xi})$ is a feasible solution for (VD_E) . If $(\bar{x}, \bar{\tau}, \bar{\xi})$ is not an efficient solution (a weakly efficient solution) for (VD_E) , then there exists a feasible solution $(\tilde{x}, \tilde{\tau}, \tilde{\xi})$ of (VD_E) such that $f(E(\tilde{x})) < f(E(\bar{x}))$, which contradicts Theorem 26. Hence, $(\bar{x}, \bar{\tau}, \bar{\xi})$ is an efficient solution (a weakly efficient solution) for (VD_E) . Moreover, we have, by Lemma 8, that $E(\bar{x}) \in \Omega$. Since $\bar{x} \in \Omega_E$ is an efficient solution (a weakly efficient solution) of the problem (VP_E) , by Lemma 9, it follows that $E(\bar{x})$ is an E -efficient solution (a weakly E -efficient solution) in the problem (VP) . Then, by the Mond-Weir strong duality between (VP_E) and (VD_E) , we conclude that also the Mond-Weir strong E -duality holds between the problems (VP) and (VD_E) . This means that if $E(\bar{x}) \in \Omega$ is a weak E -Pareto solution of the problem (VP) , there exist $\bar{\tau} \in R^p$, $\bar{\xi} \in R^m$, $\bar{\xi} \geq 0$ such that $(\bar{x}, \bar{\tau}, \bar{\xi})$ is a weakly efficient solution of a maximum type in the Mond-Weir dual problem (VD_E) . ■

Theorem 29 (Mond-Weir converse duality between (VP_E) and (VD_E)) Let $(\bar{x}, \bar{\tau}, \bar{\xi})$ be a (weak) efficient solution of a maximum type in the vector Mond-Weir dual problem (VD_E) such that $\bar{x} \in \Omega_E$. Further, assume that problem (VP_E) is KT -invex on $\Omega_E \cup Y_E$. Then \bar{x} is a (weak) Pareto solution of the problem (VP_E) .

Proof. Proof of this theorem follows directly from Theorem 26. ■

Theorem 30 (Mond-Weir converse E -duality between (VP) and (VD_E)) Let $(E(\bar{x}), \bar{\tau}, \bar{\xi})$ be a (weakly) efficient solution of a maximum type in the vector Mond-Weir E -dual problem (VD_E) such that $E(\bar{x}) \in \Omega$. Further, assume that problem (VP) is KT - E -invex on $\Omega \cup Y_E$. Then $E(\bar{x})$ is a (weak) E -Pareto solution of the problem (VP) .

Proof. Proof of this theorem follows directly from Theorem 27. ■

Theorem 31 (Mond-Weir restricted converse duality between (VP_E) and (VD_E)) Let $\bar{x} \in \Omega_E$ and $(\bar{y}, \bar{\tau}, \bar{\xi}) \in W_E$. Moreover, assume that problem (VP_E) is (strictly) KT -invex on $\Omega_E \cup Y_E$. Then $\bar{x} = \bar{y}$, that is, \bar{x} is a weak Pareto solution (a Pareto solution) of the problem (VP_E) and $(\bar{y}, \bar{\tau}, \bar{\xi})$ is a weakly efficient solution (an efficient solution) of a maximum type for the problem (VD_E) .

Proof. By means of contradiction, suppose that \bar{x} is not a weak Pareto solution of the problem (VP_E) . This means, by Definition 6, that there exists $\tilde{x} \in \Omega_E$ such that

$$f(E(\tilde{x})) < f(E(\bar{x})). \quad (36)$$

By assumption, $f(E(\bar{x})) = f(E(\bar{y}))$. Hence, (36) yields

$$f(E(\tilde{x})) < f(E(\bar{y})). \quad (37)$$

By assumption, $(\bar{y}, \bar{\tau}, \bar{\xi})$ is a feasible solution for (VD_E) . Then, it follows that $\bar{\tau} \geq 0$. Hence, the above inequality yields

$$\sum_{i=1}^p \bar{\tau}_i f_i(E(\tilde{x})) < \sum_{i=1}^p \bar{\tau}_i f_i(E(\bar{y})). \quad (38)$$

Since the problem (VP_E) is KT -invex on $\Omega_E \cup Y_E$ and by Definition 16, the inequalities

$$f_i(E(x)) - f_i(E(\bar{y})) \geq \nabla f_i(E(\bar{y}))\eta(E(x), E(\bar{y})), \quad i \in I \quad (39)$$

$$-\nabla g_j(E(\bar{y}))\eta(E(x), E(\bar{y})) \geq 0. \quad j \in J(E(\bar{y})) \quad (40)$$

hold. Multiplying above inequalities by the corresponding Lagrange multipliers, respectively, the inequalities

$$\sum_{i=1}^p \tau_i f_i(E(x)) - \sum_{i=1}^p \tau_i f_i(E(\bar{y})) \geq \sum_{i=1}^p \tau_i \nabla f_i(E(\bar{y}))\eta(E(x), E(\bar{y})), \quad (41)$$

$$- \sum_{j=1}^m \xi_j \nabla g_j(E(\bar{y}))\eta(E(x), E(\bar{y})) \geq 0. \quad (42)$$

hold for $x \in \Omega_E \cup Y_E$. Thus, they are also fulfilled for $x = \tilde{x} \in \Omega_E$. Hence, (41) and (42) yield, respectively,

$$\sum_{i=1}^p \tau_i f_i(E(\tilde{x})) - \sum_{i=1}^p \tau_i f_i(E(\bar{y})) \geq \sum_{i=1}^p \tau_i \nabla f_i(E(\bar{y}))\eta(E(\tilde{x}), E(\bar{y})), \quad (43)$$

$$- \sum_{j=1}^m \xi_j \nabla g_j(E(\bar{y}))\eta(E(\tilde{x}), E(\bar{y})) \geq 0. \quad (44)$$

Combining (43) and (44), the inequality

$$\sum_{i=1}^p \tau_i f_i(E(\tilde{x})) - \sum_{i=1}^p \tau_i f_i(E(\bar{y})) \geq \left[\sum_{i=1}^p \tau_i \nabla f_i(E(\bar{y})) + \sum_{j=1}^m \xi_j \nabla g_j(E(\bar{y})) \right] \eta(E(\tilde{x}), E(\bar{y})) \quad (45)$$

holds. Thus, by (16), it follows that the inequality

$$\sum_{i=1}^p \tau_i f_i(E(\tilde{x})) \geq \sum_{i=1}^p \tau_i f_i(E(\bar{y})) \quad (46)$$

holds, contradicting (38). Then, $\bar{x} = \bar{y}$ and this means by weak duality (Theorem 26) that \bar{x} is a weak Pareto solution of the problem (VP_E) and $(\bar{y}, \bar{\tau}, \bar{\xi})$ is a weakly efficient solution of a maximum type for the problem (VD_E) . Thus, the proof of this theorem is completed. ■

Theorem 32 (Mond-Weir restricted converse E -duality between (VP) and (VD_E)) Let $E(\bar{x})$ and $(\bar{y}, \bar{\tau}, \bar{\xi})$ be feasible solutions for the problems (VP) and (VD_E) , respectively. Moreover, assume that problem (VP) is KT - E -invex on $\Omega \cup Y_E$. Then $\bar{x} = \bar{y}$, that is, $E(\bar{x})$ is a weak E -Pareto solution (an E -Pareto solution) of the problem (VP) and $(\bar{y}, \bar{\tau}, \bar{\xi})$ is a weakly efficient solution (an efficient solution) of a maximum type for the problem (VD_E) .

Proof. Proof of this theorem is similar to that of Theorem 31. ■

5. Concluding remarks

In this paper, a new class of nonconvex (not necessarily) differentiable vector optimization problems has been defined. Namely, the concept of a KT - E -invex vector optimization problem in which all involved functions are E -differentiable has been introduced. The sufficiency of the so-called E -Karush-Kuhn-Tucker optimality conditions have been derived for such nonconvex nondifferentiable multicriteria optimization problems under assumptions that they are KT - E -invex at an E -Karush-Kuhn-Tucker point. In order to illustrate the results established in the paper, the suitable examples of KT - E -invex optimization problems have been presented. By the help of these examples, we have shown that the sufficient optimality conditions established in the paper are applicable for a significantly wider class of E -differentiable vector optimization problems in comparison to E -differentiable multicriteria optimization problems with E -invex functions introduced by Abdulaleem [7] and/or KT -invex functions introduced by Osuna-Gómez et al. [26]. Moreover, the so-called vector Mond-Weir E -dual problem has been defined for the considered (not necessarily) differentiable vector optimization problems. Then, several E -duality theorems have been established between the considered (not necessarily) differentiable vector optimization problems and its vector E -duals under KT - E -invexity hypotheses.

However, some interesting topics for further research remain. It would be of interest to investigate whether it is possible to prove similar results under KT - E -invexity hypotheses for other classes of E -differentiable vector optimization problems. We shall investigate these questions in subsequent papers.

Acknowledgements

This work was funded by Hadhramout University, Yemen. The author acknowledges with thanks to Hadhramout University for its financial support. The author is also thankful to the reviewers for their valuable suggestions, which certainly improved the presentation of the content of the paper.

References

- [1] Antczak, T 2001 On (p, r) -invexity-type nonlinear programming problems, *Journal of Mathematical Analysis and Applications*, 264(2), 382-397.
- [2] Antczak, T 2004 (p, r) -Invexity in multiobjective programming, *European Journal of Operational Research*, 152(1), 72-87.
- [3] Antczak, T 2001 (p, r) -invex sets and functions, *Journal of Mathematical Analysis and Applications*, 263(2), 355-379.
- [4] Antczak, T, Jiménez, M A 2016 KT - G -invexity in multiobjective programming, *International Journal of Mathematics and Computation*, 27, 23-39.
- [5] Antczak, T, Abdulaleem, N 2019 E -optimality conditions and Wolfe E -duality for E -differentiable vector optimization problems with inequality and equality constraints, *Journal of Nonlinear Sciences and Applications*, 12, 745-764 .
- [6] Antczak, T, Abdulaleem, N 2019 Optimality and duality results for E -differentiable multiobjective fractional programming problems under E -convexity, *Journal of Inequalities and Applications*, 2019 (1), 292.
- [7] Abdulaleem, N 2019 E -invexity and generalized E -invexity in E -differentiable multiobjective programming, *In ITM Web of Conferences*. (Vol. 24, p. 01002). EDP Sciences.
- [8] Abdulaleem, N 2019 E -duality results for E -differentiable vector optimization problems under (generalized) E -convexity. *6th International Conference on Recent Advances in Pure and Applied Mathematics*, Istanbul, Turkey, 2019, 9-26.
- [9] Abdulaleem, N 2018 Wolfe E -duality for E -differentiable E -invex vector optimization problems with inequality and equality constraints, *In 2018 International Conference on Applied Mathematics & Computational Science*, 156-1567, IEEE.
- [10] Abdulaleem, N 2019 E -optimality conditions for E -differentiable E -invex multiobjective programming problems, *WSEAS Transactions on Mathematics*, 18, 14-27.
- [11] Abdulaleem, N 2019 E -duality results for E -differentiable E -invex multiobjective programming problems, *In Journal of Physics: Conference Series*, (Vol. 1294, No. 3, p. 032027), IOP Publishing.

- [12] Aghezzaf, B, Hachimi, M 2000 Generalized invexity and duality in multiobjective programming problems, *Journal of Global Optimization*, 18(1), 91-101.
- [13] Ben-Israel, A, Mond, B 1986 What is invexity?, *The ANZIAM Journal*, 28(1), 1-9.
- [14] Craven, B D 1981 Invex functions and constrained local minima, *Bulletin of the Australian Mathematical Society*, 24(3), 357-366.
- [15] Craven, B D, Glover, B M 1985 Invex functions and duality, *Journal of the Australian Mathematical Society*, 39(1), 1-20.
- [16] Das, A K, Jana, R 2018 Invex programming problems with equality and inequality constraints, *Transactions of A. Razmadze Mathematical Institute*, 172(3), 361-371.
- [17] Goodrich, C.S., & Ragusa, M. A. (2019). Hölder continuity of weak solutions of p -Laplacian PDEs with VMO coefficients. *Nonlinear Analysis*, 185, 336-355.
- [18] Hanson, M A, Mond, B 1987 Necessary and sufficient conditions in constrained optimization, *Mathematical programming*, 37(1), 51-58.
- [19] Hanson, M A 1981 On sufficiency of the Kuhn-Tucker conditions, *Journal of Mathematical Analysis and Applications*, 80(2), 545-550.
- [20] Ivanov, V I 2011 Second-order Kuhn-Tucker invex constrained problems, *Journal of Global Optimization*, 50(3), 519-529.
- [21] Megahed, A E M A, Gomma, H G, Youness, E A, & El-Banna, A Z H 2013 Optimality conditions of E -convex programming for an E -differentiable function. *Journal of Inequalities and Applications*, 2013(1), 246.
- [22] Mohan, S R, Neogy, S K 1995 On invex sets and preinvex functions, *Journal of Mathematical Analysis and Applications*, 189(3), 901-908.
- [23] Mond, B, Weir, T 1981 Generalized concavity and duality, generalized concavity in optimization and economics. *Academic Press*, New York, 263-279.
- [24] Martin, D H 1985 The essence of invexity. *Journal of Optimization Theory and Applications*, 47(1), 65-76.
- [25] Mangasarian, O L 1994 Nonlinear programming. *Society for Industrial and Applied Mathematics (SIAM)*, Philadelphia.
- [26] Osuna-Gómez, R, Rufián-Lizana, A, & Ruiz-Canales, P 1998 Invex functions and generalized convexity in multiobjective programming, *Journal of Optimization Theory and Applications*, 98(3), 651-661.
- [27] Osuna-Gómez, R, Beato-Moreno, A, Rufián-Lizana, A 1999 Generalized convexity in multiobjective programming. *Journal of Mathematical Analysis and Applications*, 233(1), 205-220.
- [28] Pini, R, Singh, C 1997 A survey of recent [1985-1995] advances in generalized convexity with applications to duality theory and optimality conditions, *Optimization*, 39(4), 311-360.
- [29] Sach, P H, Lee, G M 2004 Efficiency and generalised convexity in vector optimisation problems, *The ANZIAM Journal*, 45(4), 523-546.
- [30] Sach, P H, Lee, G M (2005). Generalized convexity and nonsmooth problems of vector optimization, *Journal of Global Optimization*, 31(3), 383-403.
- [31] Treanță, S, Arana-Jiménez, M 2018 KT -pseudoinvex multidimensional control problem, *Optimal Control Applications and Methods*, 39(4), 1291-1300.
- [32] Treanță, S, Arana-Jiménez, M 2018 On generalized KT -pseudoinvex control problems involving multiple integral functionals, *European Journal of Control*, 43, 39-45.
- [33] Treanță, S 2020 Efficiency in generalised V - KT -pseudoinvex control problems, *International Journal of Control*, 93(3), 611-618.
- [34] Youness, E A 1999 E -convex sets, E -convex functions, and E -convex programming, *Journal of Optimization Theory and Applications*, 102(2), 439-450.