SENSORS STRUCTURES AND REGIONAL EXPONENTIAL DETECTABILITY

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Abstract

In this paper, we deal with linear infinite dimensional systems in a Hilbert space where the dynamics of the system are governed by strongly continuous semi-groups. We study the concept of exponential regional detectability in connection with the structures of sensors for a class of parabolic distributed parameter systems. For different sensors structures, we give the characterization of the exponential regional detectability. Furthermore, we apply these results to a regional observer for a diffusion systems.

1 Introduction

The concept of regional detectability was introduced recently by Al-Saphory and El Jai [?], and was focused on the state exponential detection in a given part ω of the domain Ω. The purpose of this paper is to give some results related to the link between the regional detectability and sensors structures. We consider a class of distributed systems and we explore various results connected with different types of measurements, domains and boundary conditions. The main reason for introducing the concept of an exponential regional detectability is the possibility to construct a Luenberger regional observer for the considered system. The next section concerns the class of considered systems and the characterization of regional strategic sensors. The third section is devoted to the introduction of regional detectability problem. We discuss this notion with regional observability and structures of sensors. In the fourth section, we give an application to various situations of sensors locations and we present an application of regional observer to a diffusion system.

2 Considered systems

Let Ω be a regular bounded open set of $I\!R^n$ with boundary $\partial\Omega$ and ω be a nonempty given subregion of Ω. We denote $\mathcal{Q} = \Omega \times (0, \infty)$ and $\Theta = \partial\Omega \times (0, \infty)$. Let $X, U, \mathcal{O}$ be separable Hilbert spaces where X is the state space, U the control space and $\mathcal{O}$ the observation space. Usually, we consider $X = L^2(\Omega)$, $U = L^2(0, \infty; I\!R^p)$ and $\mathcal{O} = L^2(0, \infty; I\!R^q)$ where p and q hold for the number of actuators and sensors. The considered system is described by the following parabolic distributed parameter system

$$\begin{cases} \dfrac{\partial x}{\partial t}(\xi, t) & = Ax(\xi, t) + Bu(t) & \mathcal{Q} \\ x(\xi, 0) & = x_0(\xi) & \Omega \\ x(\eta, t) & = 0 & \Theta \end{cases} \tag{2.1}$$

augmented with the output function

$$z(., t) = Cx(., t) \tag{2.2}$$

where A is a second order linear differential operator which generates a strongly continuous semi-group $(S_A(t))_{t\geq 0}$ on the Hilbert space $X = L^2(\Omega)$ and is self-adjoint with compact resolvent. The operators $B \in \mathcal{L}(U, X)$ and $C \in \mathcal{L}(X, \mathcal{O})$ depend on the structure of actuators and sensors [?]. Under the given assumptions, the system (**??**) has a unique solution given by

$$x(\xi, t) = S_A(t)x_0(\xi) + \int_0^t S_A(t - \tau)Bu(\tau)d\tau. \tag{2.3}$$

The measurements can be obtained by the use of zone, pointwise or lines sensors which may be located in Ω (or on the boundary $\partial\Omega$).

• A sensor is defined by any couple (D, f) where D, a non-empty closed subset of Ω, is the spatial support of the sensor and $f \in L^2(D)$ defines the spatial distribution of the sensing measurements on D.

3 Regional detectability

The detectability is in some sense a dual notion of stabilizability. In El Jai and Pritchard [?] this notion was considered in the whole domain. In this section, we shall extend these results to the regional case by considering ω as subregion of Ω. Regional detectability characterization needs some assumptions which are related to the asymptotic behaviour of the system, i.e. stability.

3.1 Definitions and characterizations

• The semi-group $(S_A(t))_{t\geq 0}$ is said to be exponentially stable if for every initial state $x_0(.) \in L^2(\Omega)$ the solution $x(\xi, t)$ corresponding to the autonomous system of (??), converges exponentially to zero as t tends to ∞.
• The system (??) is said to be stable, if the operator A generates a semi-group which is exponentially stable, i.e. there exist positive constants M and α such that

$$\| S_A(t) \|_{L^2(\Omega)} \leq Me^{-\alpha t},\ \forall t \geq 0$$

If $(S_A(t))_{t\geq 0}$ is an exponentially stable semi-group, then for all $x_0(.) \in L^2(\Omega)$ the solution of the associated autonomous system satisfies

$$\lim_{t\to\infty} \| x(.,t) \|_{L^2(\Omega)} = \lim_{t\to\infty} \| S_A(t)x_0(.) \|_{L^2(\Omega)} = 0 \quad (3.1)$$

• The system (??) together with the output (??) is said to be exponentially detectable if there exists an operator $H : \mathcal{O} \longrightarrow X$ such that $(A - HC)$ generates a strongly continuous semi-group $(S_H(t))_{t\geq 0}$ which is exponentially stable.
• If a system is exponentially detectable then it is possible to construct a Luenberger observer type for the original system [?]. If we consider the system

$$\begin{cases} \dfrac{\partial y}{\partial t}(\xi, t) & = Ay(\xi, t) + Bu(t) \\ & \quad +H(z(.,t) - Cy(\xi, t)) \quad \mathcal{Q} \\ y(\xi, 0) & = y_0(\xi) \quad \Omega \\ y(\eta, 0) & = 0 \quad \Theta \end{cases} \quad (3.2)$$

then $y(\xi, t)$ estimates exponentially the state $x(\xi, t)$.

Remark 3.1. In this paper, we only need the relation (??) to be true on a given subdomain $\omega \subset \Omega$

$$\lim_{t\to\infty} \| x(.,t) \|_{L^2(\omega)} = 0 \quad (3.3)$$

we may refer to this as exponential ω-stability. If $(S_A(t))_{t\geq 0}$ is a semi-group generated by the operator A and defined by $S_A(t)x = \sum_{j=1}^{\infty} e^{\lambda_j t} < x, \varphi_j >_X \varphi_j$ where φ_j is an orthonormal base of eigenfunctions of A, associated to the eigenvalues λ_j. Then for a given region ω, we have

$$\begin{aligned} \| \chi_\omega S(t)x_0 \|^2_{L^2(\omega)} &= \sum_{j,k=1}^{\infty} e^{(\lambda_j+\lambda_k)t} < x_0, \varphi_j > \\ & \quad < x_0, \varphi_k >< \varphi_j, \varphi_k >_{L^2(\omega)} \end{aligned}$$

In particular if $x_0 = \varphi_{j_0}$, then we obtain

$$\| \chi_\omega S(t)x_0 \|^2_{L^2(\omega)} = e^{2\lambda_{j_0} t} \| \varphi_{j_0} \|^2_{L^2(\omega)}$$

and therefore $\lim_{t\to\infty} \| \chi_\omega S(t)x_0 \|_{L^2(\omega)} = 0 \Longleftrightarrow \lambda_{j_0} < 0$ or $\| \varphi_{j_0} \|_{L^2(\omega)} = 0$. If ω is such that $\| \varphi_j \|_{L^2(\omega)} \neq 0,\ \forall j \geq 1$. Consequently $\lim_{t\to\infty} \| \chi_\omega S(t)x_0 \| = 0 \Longrightarrow \lambda_{j_0} < 0$. Thus in this case, the exponential regional stability is equivalent to the exponential stability.

Definition 3.2. The system (??) is said to be exponentially regionally stable in ω (or ω-stable), if the operator A generates a semi-group which is exponentially regionally stable on the space $L^2(\omega)$ (or regionally stable on $L^2(\omega)$).

Definition 3.3. The system (??)-(??) is said to be exponentially regionally detectable in ω (or ω-detectable) if there exists an operator

$$H_\omega : \mathcal{O} \longrightarrow L^2(\omega)$$

such that $(A - H_\omega C)$ generates a strongly continuous semi-group $(S_{H_\omega}(t))_{t\geq 0}$ which is regionally stable on $L^2(\omega)$.

It is clear that, a system which is exponentially detectable, is ω-detectable. A system which is exponentially ω-detectable, is asymptotically ω-detectable. A system which is ω_1-detectable, is ω_2-detectable, for every subset ω_2 of ω_1.

3.2 Regional detectability and regional observability

It has been shown that a system which is exactly observable is detectable [?]. This interesting result remains non constructive because it is related to the choice of the operator H in (??). Define now the operator $K : x \in X \to Kx = CS_A(t)x \in \mathcal{O}$, then $z(.,t) = K(t)x_0(.)$. We denote by $K^* : \mathcal{O} \longrightarrow X$ the adjoint of K given by $K^* z^*(.,t) = \int_0^t S^*(s)C^* z^*(.,s)ds$. Consider a subdomain ω of Ω and let χ_ω be the function defined by

$$\begin{aligned} \chi_\omega : \quad & L^2(\Omega) \longrightarrow L^2(\omega) \\ & x \qquad \longrightarrow \chi_\omega = x_{|_\omega} \end{aligned} \quad (3.4)$$

where $x_{|_\omega}$ is the restriction of the state x to ω.
• The autonomous system associated to (??)-(??) is said to be exactly (respectively weakly) ω-observable if :

$$\mathrm{Im}\chi_\omega K^* = L^2(\omega) \text{ (respectively } \overline{\mathrm{Im}\chi_\omega K^*} = L^2(\omega)).$$

• The suite $(D_i, f_i)_{1\leq i\leq q}$ of sensors is said to be ω-strategic if the system (??) together with the output function (??) is weakly ω-observable [?, ?]. These definitions have been extended to regional boundary case by El Jai et al. [?, ?, ?]. However, one can easily deduce the following important results.

Corollary 3.4. *A system which is exactly observable, is asymptotically observable. Thus a system which is exactly ω-observable, is ω-detectable.*

From this result, we can easily deduce that there exists $\gamma > 0$ such that

$$\| CS_A(t)x(.,t) \|_{L^2(0,T,\mathcal{O})} \geq \gamma \| \chi_\omega S_A(t)x(.,t) \|_{L^2(\omega)}$$

$\forall x \in L^2(\omega)$. Thus the notion of ω-detectability is far less restrictive than that of exact ω-observability in ω.

3.3 Sensors and ω-detectability

As in Ref. El Jai and Pritchard [?], we shall develop a characterization result that links the regional detectability and sensors structures. For that purpose, let us consider the set (φ_i) of functions of $L^2(\Omega)$ orthonormal in $L^2(\omega)$ associated with the eigenvalues λ_i of multiplicity m_i and suppose that the system (**??**) has J unstable modes. Thus a sufficient condition of ω-detectability is given by the following theorem.

Theorem 3.5. *Suppose that there are q sensors $(D_i, f_i)_{1\leq i\leq q}$ and the spectrum of A contains J eigenvalues with non-negative real parts. The system (**??**)-(**??**) is ω-detectable if and only if : $q \geq m$ and rank $G_i = m_i$, $\forall i$, $i = 1,\ldots,J$ with*

$$G = (G_{ij}) = \left\{ \begin{array}{l} < \varphi_j(.), f_i(.) >_{L^2(D_i)} \\ \varphi_j(b_i) \\ < \varphi_j(.), f_i(.) >_{L^2(\Gamma_i)} \end{array} \right.$$

where sup $m_i = m < \infty$ and $j = 1,\ldots,\infty$.

Proof The proof is limited to the case of zone sensors. Under the assumptions of section 2, the system (**??**) can be decomposed by the projections P and $I - P$ on two parts, unstable and stable. The state vector may be given by $x(\xi,t) = [x_1(\xi,t) \;\; x_2(\xi,t)]^{tr}$ where $x_1(\xi,t)$ is the state component of the unstable part of the system (**??**), may be written in the form

$$\left\{ \begin{array}{lll} \dfrac{\partial x_1}{\partial t}(\xi,t) & = A_1x_1(\xi,t) + PBu(t) & \mathcal{Q} \\ x_1(\xi,0) & = x_{1_0}(\xi) & \Omega \\ x_1(\eta,t) & = 0 & \Theta \end{array} \right. \tag{3.5}$$

and $x_2(\xi,t)$ is the component state of the stable part of the system (**??**) given by

$$\left\{ \begin{array}{lll} \dfrac{\partial x_2}{\partial t}(\xi,t) & = A_2x_2(\xi,t) + (I-P)Bu(t) & \mathcal{Q} \\ x_2(\xi,0) & = x_{2_0}(\xi) & \Omega \\ x_2(\eta,t) & = 0 & \Theta \end{array} \right. \tag{3.6}$$

if the system (**??**)-(**??**) is ω-detectable, then the unstable subsystem (**??**) is weakly ω-observable and hence the suite $(D_i, f_i)_{1\leq i\leq q}$ of sensors is ω-strategic, i.e. $[K\chi^*_\omega x^*(.,t) = 0 \Longrightarrow x^*(.,t) = 0]$ ([?]). For $x^*(.,t) \in L^2(\omega)$, we have

$$\begin{array}{ll} K\chi^*_\omega x^*(.,t) & = (\sum_{j=1}^{J} e^{\lambda_j t} < \varphi_j(.), x^*(.,t) >_{L^2(\omega)} \\ & < \varphi_j(.), f_i(.) >_{L^2(\Omega)})_{1\leq i\leq q} \end{array}$$

If the unstable system (**??**) is not ω-strategic, there exists $(x^*(.,t) \neq 0) \in L^2(\omega)$, such that $K\chi^*_\omega x^*(.,t) = 0$, this leads $\sum_{j=1}^{J} < \varphi_j(.), x^*(.,t) >_{L^2(\omega)} < \varphi_j(.), f_i(.) >_{L^2(\Omega)} = 0$. The state vectors x_i may be given by

$$\begin{array}{ll} x_i(.,t) & = [< \varphi_1(.), x^*(.,t) >_{L^2(\omega)} \ldots \\ & < \varphi_J(.), x^*(.,t) >_{L^2(\omega)}]^{tr} \neq 0 \end{array}$$

we then obtain $G_ix_i = 0$ for all i, $i = 1,\cdots,J$ and therefore rank $G_i \neq m_i$ $\forall i$.

Reciprocally, by using the conditions of this theorem, we deduce that the suite $(D_i, f_i)_{1\leq i\leq q}$ of sensors is ω-strategic for the unstable part of the system (**??**), thus the subsystem (**??**) is ω-detectable, there exist M^1_ω, $\alpha^1_\omega > 0$ such that

$$\| x_1(.,t) \|_{L^2(\omega)} \leq M^1_\omega e^{-\alpha^1_\omega(t)} \| Px_0(.) \|_{L^2(\omega)} .$$

Since the semi-group generated by the operator A_2 is regionally stable on $L^2(\omega)$, then $\lim_{t\to\infty} \| x_2(.,t) \|_{L^2(\omega)} = 0$ and therefore $x(\xi,t) \longrightarrow 0$ when $t \longrightarrow \infty$. Finally, the system (**??**)-(**??**) is ω-detectable.■

4 Application to measurements structures

In this section we give the specific results related to some examples of sensors structures and we apply these results to different situations of the domain, which usually follow from symmetry considerations. We consider the two-dimensional system defined on $\Omega =]0,1[\times]0,1[$ by the form

$$\left\{ \begin{array}{ll} \dfrac{\partial x}{\partial t}(\xi_1,\xi_2,t) = \dfrac{\partial^2 x}{\partial \xi_1^2}(\xi_1,\xi_2,t) & \\ +\dfrac{\partial^2 x}{\partial \xi_2^2}(\xi_1,\xi_2,t) + x(\xi_1,\xi_2,t) & \mathcal{Q} \\ x(\eta_1,\eta_2,t) = 0 & t > 0 \\ x(\xi_1,\xi_2,0) = x_0(\xi_1,\xi_2) & \Omega \end{array} \right. \tag{4.1}$$

Let $\omega =]\alpha_1,\beta_1[\times]\alpha_2,\beta_2[$ be the considered region is subset of $]0,1[\times]0,1[$. In this case the eigenfunctions of the system (**??**) for Dirichlet boundary conditions are given by

$$\begin{array}{ll} \varphi_{ij}(\xi_1,\xi_2) & = [\dfrac{4}{(\beta_1-\alpha_1)(\beta_2-\alpha_2)}]^{1/2} \\ & \sin i\pi[\dfrac{\xi_1-\alpha_1}{\beta_1-\alpha_1}] \sin j\pi[\dfrac{\xi_2-\alpha_2}{\beta_2-\alpha_2}] \end{array} \tag{4.2}$$

associated with eigenvalues

$$\lambda_{ij} = -[\frac{i^2}{(\beta_1-\alpha_1)^2} + \frac{j^2}{(\beta_2-\alpha_2)^2}]\pi^2. \tag{4.3}$$

In the case of Neumann or mixed boundary conditions, we have different functions. We illustrate some practical examples of the linear parabolic system (**??**).

4.1 Case of a zone sensor

Consider the system (**??**)-(**??**) where the sensor supports D are located in Ω (or $\partial\Omega$).

4.1.1 Internal zone sensor

We discuss this case with different domains :

• **Rectangular domain.** The output function (**??**) can be written by the form

$$z(t) = \int_D x(\xi_1, \xi_2, t) f(\xi_1, \xi_2) d\xi_1 d\xi_2. \tag{4.4}$$

where $D \subset \Omega$ is the location of the zone sensor and $f \in L^2(D)$. In the case of (Fig. 1), the eigenfunctions and

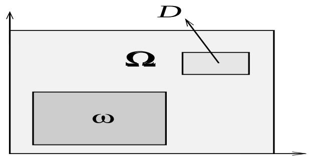

Figure 1: Domain Ω, subdomain ω and location D of internal zone sensor.

the eigenvalues are given in the equations (**??**) and (**??**). However, if we suppose that

$$(\beta_1 - \alpha_1)^2 / (\beta_2 - \alpha_1)^2 \notin I\!\!Q \tag{4.5}$$

then $m = 1$ and one sensor may be sufficient for ω-detectability. Let the measurement supports is rectangular with $D = [\xi_{0_1} - l_1, \xi_{0_1} + l_1] \times [\xi_{0_2} - l_2, \xi_{2_0} + l_2]$. We then have the result.

Corollary 4.6. *Suppose that f_1 is symmetric about $\xi_1 = \xi_{0_1}$ and f_2 is symmetric with respect to $\xi_2 = \xi_{2_0}$. The system (**??**)-(**??**) is not ω-detectable if $i(\xi_{0_1} - \alpha_1)/(\beta_1 - \alpha_1)$ and $i(\xi_{2_0} - \alpha_2)/(\beta_2 - \alpha_2) \in I\!\!N$ for some i, $i = 1, \ldots, J$.*

• **Disk domain.** Consider the system (**??**)-(**??**). So, this system may be given by the following form

$$\left\{ \begin{array}{lll} \dfrac{\partial x}{\partial t}(r, \theta, t) & = \dfrac{\partial^2 x}{\partial r^2}(r, \theta, t) & \\ & + \dfrac{\partial^2 x}{\partial \theta^2}(r, \theta, t) & \\ & + x(r, \theta, t) & \mathcal{Q} \\ x(r, \theta, 0) & = x_0(r, \theta) & \Omega \\ x(1, \theta, t) & = 0 & \theta \in [0, 2\pi], t > 0 \end{array} \right. \tag{4.6}$$

and with output function

$$z_i(t) = \int_{D_i} x(r_i, \theta_i, t) f(r_i, \theta_i) dr_i d\theta_i \quad 0 \le \theta_i \le 2\pi \tag{4.7}$$

where , $\frac{1}{2} r_{i_\omega} < r_i < \frac{1}{2}$, $2 \le i \le q$ $\Omega = D(0, 1)$ are defined as in (Fig. 2). So, the eigenfunctions and eigenvalues concerning the region $\omega = D(0, r_\omega) \subset \Omega$, $\forall r_\omega \in]0, 1[$ are defined by

$$\lambda_{ij} = -\beta_{ij}^2 \quad i \ge 0, \ j \ge 1 \tag{4.8}$$

where β_{ij} are the zeros of the Bessel functions J_i and

$$\begin{array}{lll} \varphi_{0j}(r, \theta) & = J_0(\beta_{0j} r) & j \ge 1 \\ \varphi_{ij_1}(r, \theta) & = J_i(\beta_{ij_1} r) \cos(i\theta) & i, j_1 \ge 1 \\ \varphi_{ij_2}(r, \theta) & = J_i(\beta_{ij_2} r) \sin(i\theta) & i, j_2 \ge 1 \end{array} \tag{4.9}$$

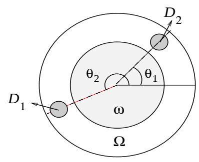

Figure 2: Domain Ω, subdomain ω and locations D_1, D_2 of internal zone sensors

with multiplicity $m_i = 2$ for all $i, j \neq 0$ and $m_i = 1$ for all $i, j = 0$. In this case, the regional detectability in ω is required at least two zone sensors $(D_i, f_i)_{2 \le i \le q}$ where $D_i = (r_i, \theta_i)_{2 \le i \le q}$ (see [**?**]). Thus we have the result.

Corollary 4.7. *Suppose that f_i and D_i, are symmetric with respect to $\theta = \theta_i$, for all $i, 2 \le i \le q$. Then the system (**??**)-(**??**) is not ω-detectable if $i_0(\theta_1 - \theta_2)/\pi \in I\!\!N$ for some i_0, $i_0 = 1, \ldots, J$.*

4.1.2 Boundary zone sensor

We consider the system (**??**)-(**??**) with the Dirichlet boundary conditions. We study this case with different geometrical domains :

• **The domain** $]0, 1[\times]0, 1[$**.** Now the output function (**??**) is given by

$$z(t) = \int_\Gamma \frac{\partial x}{\partial \nu}(\eta_1, \eta_2, t) f(\eta_1, \eta_2) d\eta_1 d\eta_2. \tag{4.10}$$

where $\Gamma \subset \partial\Omega$ is the support of the boundary sensor and $f \in L^2(\Gamma)$. The sensor (D, f) may be located on the boundary in $\Gamma = [\eta_{1_0} - l, \eta_{1_0} - l] \times \{1\}$, then we have.

Corollary 4.8. *If the function f is symmetric with respect to $\eta_1 = \eta_{1_0}$, then the system (**??**)-(**??**) is not ω-detectable if $i(\eta_{1_0} - \alpha_1)/(\beta_1 - \alpha_1) \in I\!\!N$ for some i, $1 \le i \le J$.*

When the sensor is located in $\bar{\Gamma} = [\bar{\eta}_{1_0} - l_1, 1] \times \{0\} \bigcup \{1\} \times [0, \bar{\eta}_{2_0} + l_2] = \Gamma_1 \bigcup \Gamma_2$ where $\bar{\Gamma} \subset \partial\Omega$ (Fig. 3), We obtain.

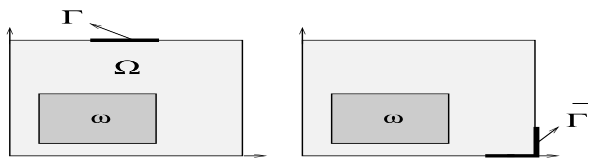

Figure 3: Rectangular domain and locations $\Gamma, \bar{\Gamma}$ of boundary zone sensors

Corollary 4.9. *Suppose that the function $f_{|\Gamma_1}$ is symmetric with respect to $\eta_1 = \bar{\eta}_{1_0}$, and the function $f_{|\Gamma_2}$ is symmetric about $\eta_2 = \bar{\eta}_{2_0}$. Then the system (**??**)-(**??**) is not ω-detectable if $(\bar{\eta}_{1_0} - \alpha_1)/(\beta_1 - \alpha_1)$ and $(\bar{\eta}_{2_0} - \alpha_2)/(\beta_2 - \alpha_2) \in I\!\!N$ for some i, $1 \le i \le J$.*

• **The domain** $D(0, 1)$. Here, we consider the system (**??**)

with output function

$$z_i(t) = \int_{\Gamma_i} \frac{\partial x}{\partial \nu}(1, \theta_i, t) f(1, \theta_i) d\theta_i \quad 0 \le \theta_i \le 2\pi, \ t > 0. \tag{4.11}$$

In this case, it is necessary to have at least two boundary zone sensors $(\Gamma_i, f_i)_{2 \le i \le q}$ with $\Gamma_i = (1, \theta_i)_{2 \le i \le q}$ and if the functions $f_{|\Gamma_i}$ are symmetric with respect to $\theta = (\theta i)_{2 \le i \le q}$ as in (Fig. 4). So, we have.

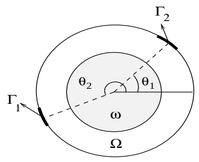

Figure 4: Circular domain and locations Γ_1, Γ_2 of boundary zone sensors

Corollary 4.10. *The system (??)-(??) is not ω-detectable if $i(\theta_1 - \theta_2)/\pi \in I\!N$ for some i, $1 \le i \le J$.*

4.2 Case of a pointwise sensor

Let us consider the case of pointwise sensor located inside of Ω or on the boundary of $\partial\Omega$.

4.2.1 Internal pointwise sensor

We can discuss the following :

• **Case of Fig. 5.** The system (**??**) is augmented with the following output

$$z(t) = \int_{\Omega} x(\xi_1, \xi_2, t) \delta(\xi_1 - b_1, \xi_2 - b_2) d\xi_1 d\xi_2. \tag{4.12}$$

where $b = (b_1, b_2)$ is the location of pointwise sensor in Ω is defined as in (Fig. 5). If $(\beta_1 - \alpha_1)/(\beta_2 - \alpha_1) \notin I\!\!Q$ then $m = 1$ and one sensor (b, δ_b) may be sufficient for ω-detectability. Then we obtain.

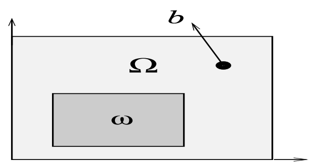

Figure 5: Rectangular domain and location b of internal pointwise sensor

Corollary 4.11. *The system (??)-(??) is ω-detectable if $i(b_1 - \alpha_1)/(\beta_1 - \alpha_1)$ and $i(b_2 - \alpha_2)/(\beta_2 - \alpha_2) \notin I\!N$, for every i, $i = 1, \ldots, J$.*

• **Case of Fig. 6.** Consider the system (**??**) with the function defined by the form

$$z_i(t) = \int_{\Omega} x(r_i, \theta_i, t) f_i(r_i, \theta_i) dr_i d\theta_i. \tag{4.13}$$

where $2 \le i \le q$, $0 \le \theta_i \le 2\pi$, $\frac{1}{2} r_{i_\omega} < r_i < \frac{1}{2}$, $t > 0$. The sensors may be located in $p_1 = (r_1, \theta_1)$ and $p_2 = (r_2, \theta_2) \in \Omega$ (see Fig. 6).

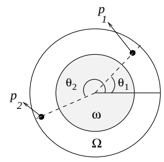

Figure 6: Circular domain and locations p_1, p_2 of internal pointwise sensors

Corollary 4.12.

1. *The system (??)-(??) is ω-detectable if $i(\theta_1 - \theta_2)/\pi \notin I\!N$ for all $i = 1, \ldots, J$.*
2. *If $r_1 = r_2$, the system (??)-(??) is ω-detectable if $i(\theta_1 - \theta_2)/\pi \notin I\!N$ for all $i = 1, \ldots, J$.*

Filament sensors. Consider the case where $\Omega =]0, 1[\times]0, 1[$ and $\omega =]\alpha_1, \beta_1[\times]\alpha_2, \beta_2[\subset \Omega$. Suppose that the observation on the curve $\sigma = \mathrm{Im}(\gamma)$ with $\gamma \in C^1(0, 1)$ (Fig. 7), then we have.

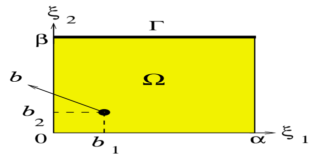

Figure 7: Rectangular domain and location σ of internal filament sensors

Corollary 4.13. *If the observation recovered by the filament sensor (σ, δ_σ) such that σ is symmetric with respect to the line $\xi = \xi_0$. The system (??)-(??) is ω-detectable if $i(\xi_{0_1} - \alpha_1)/(\beta_1 - \alpha_1)$ and $i(\xi_{0_2} - \alpha_2)/(\beta_2 - \alpha_2) \notin I\!N$, for every i, $i = 1, \ldots, J$.*

4.2.2 Boundary pointwise sensor

Let us consider the system (**??**) with Dirichlet boundary condition, so, we can study the following :

• **Case of Fig. 8** In this case the sensor (b, δ_b) is located on $\partial\Omega$. The output function is given by

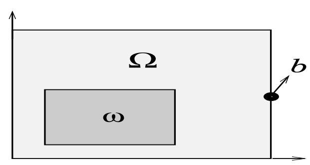

Figure 8: Rectangular domain Ω and location b of boundary pointwise sensor

$$z(t) = \int_{\partial\Omega} \frac{\partial x}{\partial \nu}(\eta_1, \eta_2, t) \delta(\eta_1 - b_1, \eta_2 - b_2) d\eta_1 d\eta_2. \tag{4.14}$$

Then we can obtain.

Corollary 4.14. *The system (**??**)-(**??**) is ω-detectable if $i(b_1-\alpha_1)/(\beta_1-\alpha_1)$ and $i(b_2-\alpha_2)/(\beta_2-\alpha_2) \notin I\!N$, for every i, $i=1,\ldots,J$.*

• **The case of Fig. 9** In this case we consider the system (**??**) with

$$z_i(t)=\int_{\partial\Omega}\frac{\partial x}{\partial\nu}(1,\theta_i,t)f(1,\theta_i)d\theta_i. \tag{4.15}$$

where $\theta_i\in[0,2\pi],\ t>0$. When the pointwise sensors at

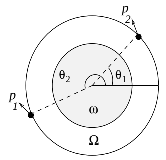

Figure 9: Circular domain and locations p_1,p_2 of boundary pointwise sensors

the polar coordinates $p_i=(1,\theta_i)$ where $\theta_i\in[0,2\pi]$ and $2\le i\le q$. We have the following result.

Corollary 4.15. *Then the system (**??**)-(**??**) is ω-detectable if $i(\theta_1-\theta_2)/\pi\notin I\!N$ for every i, $1\le i\le J$.*

These results can be extended with boundary Neumann conditions.

4.3 Application to a diffusion system

Consider the distributed parameter diffusion system described by the parabolic equations

$$\left\{\begin{array}{lll}\dfrac{\partial x}{\partial t}(\xi_1,\xi_2,t) & =\dfrac{\partial^2 x}{\partial\xi_1^2}(\xi_1,\xi_2,t)+\dfrac{\partial^2 x}{\partial\xi_2^2} & \\ & (\xi_1,\xi_2,t)+\beta x(\xi_1,\xi_2,t) & \mathcal{Q} \\ x(\eta_1,\eta_2,t) & =0 & t>0 \\ x(\xi_1,\xi_2,0) & =x_0(\xi_1,\xi_2) & \Omega\end{array}\right. \tag{4.16}$$

where the above-stated equations in discussing a heat-conduction problem [**?**]. Let ω be a subregion of $\Omega=]0,1[\times]0,1[$ and suppose that there exists a single sensor (D_1,f) with $D_1=[d_1-l_1,d_1+l_1]\times[d_2-l_2,d_2+l_2]\subset]0,1[\times]0,1[$. Using the corollary **??**, the system

$$\left\{\begin{array}{ll}\dfrac{\partial y}{\partial t}(\xi_1,\xi_2,t) & \\ =\dfrac{\partial^2 y}{\partial\xi_1^2}(\xi_1,\xi_2,t)+\dfrac{\partial^2 y}{\partial\xi_2^2}(\xi_1,\xi_2,t)+\beta y(\xi_1,\xi_2,t) & \\ +Bu(\xi_1,\xi_2,t)-H_\omega(Cy(\xi_1,\xi_2,t)-z(t)) & \mathcal{Q} \\ y(\eta_1,\eta_2,t)=0 & \Theta \\ y(\xi_1,\xi_2,0)=y_0(\xi_1,\xi_2) & \Omega\end{array}\right.$$

is a regional observer for the system (**??**) -(**??**) if $i(d_1-\alpha_1)/(\beta_1-\alpha_1)$ and $i(d_2-\alpha_2)/(\beta_2-\alpha_2)\notin I\!N$ [**?**].

5 Conclusion

In this paper we have presented sufficient condition of ω-detectability and we have discussed the regional problem for a class of distributed parameter systems. Many results concerning the choice of sensors structures are explored and illustrated in specific situations. We have shown that it is possible to construct a regional observer for distributed parameter diffusion system by using the concept of regional detectability. An extension of these results to the problem of regional stabilizability in connection to the actuator structures is under consideration.

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