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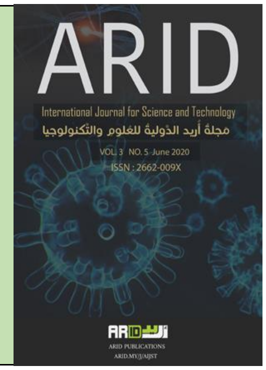

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Eigenvalues of the Laplacian Operator and Neighborhood Enlargement for Compact Manifolds

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القيم الذاتية لمؤثر لابلاس و نصف قطر تجسيم الجوار لمتعددات الطيات ذات التراص

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ABSRACT

In this investigation and under the conception of measure concentration phenomenon we found that the enlargement of the neighborhood for an ***n* – *dimensional*** compact Riemannian manifolds (M, g) relative to the eigenvalues λ of the Laplace operator Δ on (M, g). And we found that $r \sim \frac{1}{\sqrt{\lambda}}$.

Keywords: r-enlargement, Isoperimetric Inequalities, Concentration of Measure, Laplace Operator, Eigenvalues of the Laplace operator.

الملخص

تناول هذا البحث تجسيم الجوار لمتعدد الطيات المتراص في البعد n وذلك باستخدام مفهوم تركيز الحجم. كما وجدنا أن نصف القطر r لتجسيم الجوار لمتعدد الطيات يرتبط مع القيم الذاتية λ لمؤثر لابلاس Δ على متعدد الطيات.

الكلمات المفتاحية: نصف قطر تجسيم الجوار، متباينات متساوي المقاييس، تركيز الحجم، مؤثر لابلاس، القيم الذاتية لمؤثر لابلاس.

1. Introduction:

The conception of $r - enlargement$ was stood behind the concentration of measure phenomenon to estimate the local properties of the manifolds. Since it measure the basic best neighborhood U of points on the manifold. Let $A \subset \mathbb{R}^{n+1}$, we can estimate the local geometry of A with measure concentration phenomenon which is basically depends on the best neighborhood of A with best $r - enlargemnet$ and try to get the best topology of A. In this task we want get a relation between the $r - enlargement$ of the manifold and its related eigenvalues of the Laplace operator on it.

Berestycki and Nickl [1] have discussed the conception of ***concentration of measure*** phenomenon, they had stated that, and for $A \subseteq S^n \subset \mathbb{R}^{n+1}$, and if $\mu^n(\mathrm{A}) = \mu^n\big(B(x,r)\big)$, then $\mu^n(\mathrm{A_r}) \geq \mu^n\big(B(x, r+\varepsilon)\big)$, where B is an $n - ball$ and μ^n is standard measure on $\mathbb{R}^{n+1}$ and A_r is the $r - enlargement$ of A. This phenomenon affects at many fields such as learning theory, statistical learning, functional analysis, etc. The basic principle of this phenomenon is to study Benty [2] had gave more and analyze geometrical features at *n – dimensional* spaces for $n \to \infty$. historical information about this phenomenon beginning and its extension to an $n - dimensional$ manifolds.

F Chung and others had given the isoperimetric inequality related to eigenvalues on Riemannian Manifold and graphs [3].

To the best illustration on this task and for the purpose of the paper and [from [1] (Sec 1.3)] we had,

1.1 Theorem (Spherical Isoperimetric)[1]: Let $A \subseteq S^n \subset \mathbb{R}^{n+1}$ be a set, and let $A_\varepsilon = \{x \in S^n : d(x,y) < \varepsilon \; for \; some \; y \in A\}$. Let μ^n be the uniform probability measure on S^n. Let $\mu^n(A) \geq \frac{1}{2}$. Then

$$p(Z \in A_\varepsilon) = \mu^n(A_\varepsilon) \geq 1 - e^{-\frac{(n-1)\varepsilon^2}{2}}$$

The connections of isoperimetric inequalities and the eigenvalues of the Laplace operator had clarified at [4, 5, 6].

1.2 Theorem [4]: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\mathrm{B} \subset \mathbb{R}^n$ a ball with the same volume as Ω. Then, we have

$$\lambda_1(B) \leq \lambda_1(\Omega)$$

with equality hold if and only if Ω is equal to B up to a displacement.

Here λ_1 stands for the first eigenvalue of ***Laplace Operator***. Many topics discussed the analyzation of Laplace eigenvalues to the geometric of the $n - dimensional$ manifold [4, 6, 7]

Our 2nd section will describe the $r - enlargement$ of the neighborhood for the manifold and its relation to the notion of concentration of measure phenomenon explaining the basic theorems and principles. The 3rd section deals with the connection of the eigenvalues of the Laplacian to the isoperimetric inequalities in the sense of geometric features and we get some corollaries under the notion of measure concentration. The fourth section describes the *r – enlargement* as concentration of measure principle try to fix up the theorem of the number of covering ball on the manifold (M, g) and try to prove that it relates to the Weyl's Asymptotic formula [6] as conjecture counting method for the series of eigenvalues of the study and we seek our main theorem on the $\frac{1}{\lambda}$ – enlargement of the *n – dimensional* manifold. Lastly we had a brief discussion. Our theorems unfortunately are representing without a proof (not all of them).

Sometimes we get a proof in the context of the literature. The reader should refer for the standard proofs to the index reference front of it.

2. $\boldsymbol{r - enlargement}$ of the Manifold and Concentration of Measure Phenomenon

The isoperimetric inequalities states that whenever $\mu^n(A) = \mu^n(B)$ where $A \subseteq S \subset \mathbb{R}^{n+1}$and B is a ball on an S^n, then for every $r > 0$

$$\ldots\ldots(1)\mu^n(A_r) \geq \mu^n(B_r)$$

Where A_r is the r – *enlargement* of A. In the context we set $V = Vol$ and μ^n as the standard volume and standard measure respectively on an $n - dimenstional$ manifolds.

2.1 Theorem (Paul Levy's isoperimetric inequality)[2]: Let S_c denote the collection of all *Borel* sets in S^n with fixed normalized measure c, where $c \in (0,1)$. Then, for any $r > 0$ sufficiently small and any set $E_r \in S_c$, we have that $V(E_r) \geq V(C_r)$, where C denotes a spherical cap of measure c.

Theorem (2.1) concerns the geometric point of view of the isoperimetric inequality. Bates [7] had discussed the probability point of view of isoperimetric inequality.

2.2 Theorem[7]: (Weak Law of Large Number) For every $\varepsilon > 0$

$$\ldots\ldots(2)\lim_{n\to\infty} Vol(S_\varepsilon) = 1$$

Equivalently,

$$\ldots\ldots(3)\lim_{n\to\infty} Vol\left(\left\{x \in C^n : \left|\frac{S_n(x)}{n}\right| > \varepsilon\right\}\right) = 0$$

Where C^n is the n – *dimensional* Cube, and $S_\varepsilon = \left\{x \in C^n : \left|\frac{S_n(x)}{n}\right| < \varepsilon\right\}$.

Remark (1): If we imagine Theorem (2.2), we get Standard Normal distribution $N(X; \varepsilon)$ on an n – *dimensional* space. Also, the quantity ε stands for the $r - enlargement$ of the neighborhood of the point x. By the other hand this quantity appears obviously in the context of the concentration function.

2.3 Definition (Concentration function): Let (X, μ, d) be a metric space with metric d and $diam(X) \geq 1$, which is equipped with a *Borel* probability measure μ. Then the concentration function on X is (isometric constant)

$$\ldots\ldots(4)\alpha(X;\varepsilon) = 1 - \inf\left\{\mu(A_\varepsilon): A\ is\ a\ Borel\ subset\ of\ X, \mu(A) \geq \frac{1}{2}\right\}$$

Where $A_\varepsilon = \{x \in X: d(x, A) \leq \varepsilon\}$ is the $\varepsilon - extension$ of A. And there exists one value L_f such that:

$$\ldots\ldots(5)\mu(x \in X: \left|f(x) - L_f\right| \geq \varepsilon) \leq 2\alpha(X;\varepsilon)$$

Where f is $1 - Lipschitz$ function on X.

The conception of $r - enlargement$ appears also in the fields of convex geometry to get appropriate normal distribution of the data in the space.

2.4 Theorem: Let K be strictly convex body with a modulus of convexity $\delta(\varepsilon)$. Let $S = \partial K$ be the surface of K and let $A \subset S$ be a set such that $V(A) \geq \frac{1}{2}$. Then, for every $\varepsilon > 0$ such that $\delta(\varepsilon) \leq \frac{1}{2}$ we have:

$$\ldots\ldots(6)V\{x \in S: dist(x, A) \geq \varepsilon\} \leq 2\big(1 - \delta(\varepsilon)\big)^{2n} \leq 2e^{-2n\delta(\varepsilon)}$$

By the other hand, the relation which connect the $r - enlargement$ and concentration of measure phenomena appears in a sense of isoperimetric inequality. Its take the form $r - neighborhood$.

2.5 Proposition [2]: Let E_r be the $r-neighborhood$ of a great circle of S^n, $n \geq 2$. Then,

$$\ldots\ldots(7)\mu(E_r^c) \leq 2\exp\left(-\frac{(n-1)r^2}{2}\right)$$

Where E_r^c denotes the complement of E_r relative to S^n.

See [8] for more details and theorems.

Moreover, from Eq (7) we observe that the most important parameters in the concentration function (5) are dimension and the $r-enlargement$ of the neighborhood as radius or length of the interval. Moreover from the Embedding theorem,

2.6 Theorem: (Classical Devortzky's Theorem)[8] Let X be a normed space of dimension n. There exist a function $C(\varepsilon)$ such that, for all $k \geq C(\varepsilon)\log n$, $\ell_2^k \hookrightarrow \quad X$ which mean that $\|x\| \leq C\sqrt{k}|x|$, where $k \approx C(\varepsilon)\log n$. So

$$P\left(\|x\| \leq C\sqrt{k}|x|\right) \leq C_1 e^{-\frac{kC_2|x|^2}{\sigma^2}}$$

The quantity σ^2 here is related to the variance on X.

2.7 Theorem: For any $\varepsilon > 0$ and any positive integer k there exists a positive integer $N = N(k, \varepsilon)$ such that for any normed space W with $\dim W \geq N$ there exists a $k-dimensional$ subspace V of W which is $\varepsilon-close$ to Euclidean space $\mathbb{R}^n$.

So, every normed space admits a locally Euclidean space. And this concern the feature of $r-enlargement$ as

$$\|x\|_V \leq r|x|_{\mathbb{R}^n}$$

This implies that, and with $0 \leq r \leq 1$

$$\ldots\ldots(8)\left|\frac{\|x\|_V}{|x|_{\mathbb{R}^n}}\right| \leq r$$

So, $$\mu^n\left(\left|\frac{\|x\|_V}{|x|_{\mathbb{R}^n}}\right| \leq r\right) \leq 2e^{-\frac{nr^2}{2\sigma^2}}\ldots\ldots(9)$$

2.8 ***Corollary:*** *Under the assumption of Theorems (2.6) (2.7), Eq(9)and for $V \subset W$, then*

$$\ldots\ldots(10)\mu^n\left(\left|\frac{\|x\|_V}{|x|_{\mathbb{R}^n}}\right| \leq r\right) \leq 2\,exp\left(-\frac{N(n,r)}{2\sigma^2}\right)$$

Where $N(n,r)$ as in Theorem (2,7).

The quantity $|x|_{\mathbb{R}^n}$ takes the value $\sqrt{n}$ in many other papers [1, 8]. By the other hand, the exponential map which appears in the formula of concentration of measure function will represents the normal coordinates on the geodesic ball on manifold and that up to its ***Tylor*** expansion in the sense that $(x_0 - x) \equiv r - enlargement$.

2.9 **Definition (Injectivity Radius) [9]:** Let M be a Riemannian manifold, $p \in M$. Then the injective radius of p is

$$inj(p) = \sup\left\{\varepsilon > 0, exp_p\ is\ defined\ on\ \overline{B_{\mathbb{R}^n}(0,p)} and\ injective\right\}$$

2.10 ***Corollary:*** *The concentration function $\alpha(\varepsilon)$ define the normal coordinate on the geodesic ball*

$$\ldots\ldots(11)B_M(x) = \{x \in M : d(x,A) \leq inj(x)\}$$

Where, $\mu^n(A) \geq \frac{1}{2}$. And we had

$$\ldots\ldots(12)\alpha(\varepsilon) := e^{-\frac{(dim\,M-1)[inj(x)]^2}{2}}, for\ all\ A \subset B_M(x)$$

Now, we have the Levy – Gromove corollary which had treat the extension of the isoperimetric inequality to an $n - dimensional$ manifolds.

2.11 Corollary (Levy – Gromove): [2] Let M be an $n-dimensional$ manifold and suppose that its Ricci curvature is everywhere greater than that of S^n. Let $f: M \to \mathbb{R}$ be a $1-Lipschitz$ function. Then there exists $m \in \mathbb{R}$ such that, for any $t \geq 0$,

$$\ldots\ldots(13) V(\{x \in M, |f(x)-m| \geq t\}) \leq 2e^{-\frac{t^2}{2D^2}}$$

Where $D=\frac{1}{\sqrt{n-1}}$ and V is the natural measure on M, normalized so that $V(M)=1$.

2.12 Lemma:[9] Let (M,μ) be a Riemannian manifold with boundary of bounded geometry of dimension n. Then there exists $R_0>0$ and a constants $S_1, S_2>0$ such that for all $x \in M$ and $r \geq R_0$ one has

$$S_1 r^m \leq Vol\big(B_M(x,r)\big) \leq S_2 r^m$$

From (Lemma 52 of [6]) we deduce that

2.13 Corollary: *Let $A \subseteq S^n \subset M$ and let (M,μ) be an $n-dimensional$ manifold. Then there exists $\big(R_0 = inj(x)\big) > 0$ such that for any $r < R_0$ and all $x \in M$ we had*

$$\ldots\ldots(14) \mu^n\big(x: d(x,y) < r; y \in B_M(x,r)\big) \leq Ce^{-\frac{(n-1)r^2}{2}}$$

Where, C is universal constant, and μ^n is the normalized measure on M.

The curvatures of the $n-dimensional$ manifold also effects on the $r-enlargement$ of the points of M.

2.14 Lemma:[9] Let M has the curvature of radius $\rho > 0$, we further assume that

$$k \coloneqq inf_{x\in M}\, inf_{y\in M/B_M(x,\pi\rho)} \|x-y\|$$

is non – zero. Then $B_{\mathbb{R}^d}\left(x,\frac{k}{2}\right)\cap M\subset B_M(x,k)\subset B_M(x,\pi\rho)$. Particularly, if $x,y\in M$ and $\|x-y\|\le\frac{k}{2}$,

$$\frac{1}{2}d_M(x,y)\le\|x-y\|_{\mathbb{R}^d}\le d_M(x,y)\le k$$

Now, from Definition (2.9) and Corollary (2.13) and (Eq (3) & Assumption (1) of [10]) we deduce that $k\coloneqq inj(x)$

2.15 Corollary: *Let M have a finite radius of curvature $\rho>0$, and let k be as defined above, then we had*

$$\ldots\ldots(15)\ \alpha(\varepsilon)=e^{-\frac{C(\dim(M))k^2}{2}}$$

of Eqs(7,9, 10,12,13,14, 15) even **Remark (2):** The reader may be confused that denominator that of statement of Dvortzky Theorem are different? As we mentioned on Remark (1)The basic idea is stands behind the normal distribution $N(X;\mu,\sigma^2)$ where the density function of this distribution takes the formula $e^{\frac{|x-\mu|}{2\sigma^2}}$ with $\sigma^2=\frac{\sum_{i=1}^{n}(x-\mu)^2}{n-1}$ is the variance as on Eq (10) and Devortzky theorem. Concentration of measure behaves as the Standard normal distribution $N(X;0,1)$ where $\sigma^2=1$.

3. Eigenvalues of Δ on the Manifold & Concentration of Measure Phenomenon

The spectrum of Δ is not in general gets the geometry of the manifolds but we can give some geometric features from it. Canzani [6] had stated that ***Mark Kac*** in 1966 proved the formula

$$\sum_{i=1}^{\infty}e^{-\lambda_i t}\sim\frac{1}{4\pi t}\left(area(\Omega)-\sqrt{4\pi t}\,length(\partial\Omega)+\frac{2\pi t}{3}\left(1-\gamma(\Omega)\right)\right)$$

Where γ is the genus of Ω. And, C.Yu Xia on [11] stated that **“**on 1979 ***Li*** and ***Yau*** proved that $\lambda_1 \geq \frac{\pi^2}{2D^2(M)}$ and ***Hersch*** in 1974 states that $\lambda_1 \leq \frac{8\pi}{A(S^2)}$ “.

. In the feature of concentration phenomenon, we had

3.1 Theorem: (Faber Krahn Inequality)[6] Let $\Omega \subset \mathbb{R}^n$, and let B denotes a ball satisfies $V(\Omega) = V(B)$. Then,

$$\lambda_1(\Omega) \geq \lambda_1(B)$$

Where $\lambda_1(\Omega)$ and $\lambda_1(B)$ are the first eigenvalues for the Dirichlete eigenvalues on (Ω) and (B) respectively.

By the other hand and for the connection between the first eigenvalue of the Δ and the diameter of the manifold we had

3.2 Theorem: (Li and Yau(1979))[4] Let (M, g) be a compact n – dimensional Riemannian manifold without boundary. Suppose that $Ricci$ curvature satisfies $Ricci(M, g) \geq (n-1)k$ and that d denotes the diameter of (M, g). Then, if $k < 0$

$$\ldots\ldots(16)\lambda_1(M, g) \geq \frac{\exp -\left(1+\left(1-4(n-1)\right)^2 d^2 k\right)^{\frac{1}{2}}}{2(n-1)^2 d^2}$$

And if $k = 0$, then

$$\lambda_1(M, g) \geq \frac{\pi^2}{4d^2}$$

Where, d is the diameter of M. For simplicity the inequality (16) can be written as

$$\ldots\ldots(17)\lambda_1(M, g) \geq C_{(n)} \frac{e^{-(2(n-1)d\sqrt{k})}}{d^2}$$

3.3 Theorem: (Upper bound for λ_1)[2] If M is a compact $n-manifold$ with $Ricci \geq (n-1)(-k), k>0$, then

$$\lambda_1 \leq \frac{(n-1)^2}{4}k + \frac{c_n}{D^2(M)}$$

Where, c_n is a positive constant depending on n.

Now, and from (Proposition (1) and Corollary (3) of [6]) we deduce that

3.4 Corollary: *Let (M,g) be a compact $n-dimensional$ Riemannian manifold with $Ricci \geq (n-1)k, k<0$. Then*

$$C_n \frac{e^{-\left(2(n-1)d\sqrt{k}\right)}}{d^2} \leq \lambda_1(M,g) \leq \frac{(n-1)^2}{4}k + \frac{c_n}{d^2}$$

Where d is the dimeter of M. So $\lambda_1 \sim \frac{1}{d^2}$.

As a function the continuum of eigenvalues also affects the comparison inequalities in the feature of concentration of measure.

3.5 Theorem: (continuity in the $\mathbf{C^0}$ – topology of matrices)[6]Let M be a compact manifold and let g, g' be two Riemannian metrics on M that are close in the sense that there exists $\varepsilon > 0$, small making,

$$(1-\varepsilon)g' \leq g \leq (1+\varepsilon)g'$$

Then,

$$1-(n+1)\varepsilon + O(\varepsilon^2) \leq \frac{\lambda_k(g)}{\lambda_k(g')} \leq 1+(n+1)\varepsilon + O(\varepsilon^2)$$

3.6 Theorem: [6] Let λ_k be the kth eigenvalue of the Laplacian associated to g. λ_k is a continuous function on M with respect to the C^∞ topology. More precisely, $d(g,g') < \delta$ implies

$$\exp(-(n+1)\delta)\,\lambda_k(g') \le \lambda_k(g) \le \exp\big((n+1)\delta\big)\,\lambda_k(g')$$

3.7 ***Corollary:*** *Under the assumption of Theorems (3.5) & (3.6) we had*

$$\ldots\ldots(18)\mu^n\left(\left|\frac{\lambda_k(g)}{\lambda_k(g')}\right| \le (n+1)r\right) \le 2\,exp\left(-\frac{(n+1)r^2}{2\sigma^2}\right)$$

4. $r - enlargement$ and Eigenvalues of Δ on the Manifold

This section will describe our main results of this paper to get the relation between the $r - enlargement$ of the manifold and its Laplace eigenvalues.

4.1 **Lemma:[12]** Let (M, g) be a complete Riemannian manifold whose $Ricci$ curvature is bounded below, $Ricci \ge -(n-1)k$, where $k \ge 0$, and $\Omega \subset M$ be a pre compact domain that is star – shaped with respect to a point $x \in \Omega$. Then the quotient $\left(\frac{Vol(B(x,r)\cap M)}{Vol(B_k(r))}\right)$, where $B_k(r)$ is a ball in the space of constant curvature $(-k)$, is a non – increasing function in $r > 0$. In particular, for any $0 < r \le R$ we have

$$Vol(B(x,R)\cap M) \le e^{(n-1)R\sqrt{k}}\left(\frac{R}{r}\right)^n Vol(B(x,r)\cap M)$$

More over and according to Lemma (2.14) we had

4.2 ***Corollary:*** *Under the assumption of Lemmas (2.14) and (4.1) and Corollaries (2.15) and (37), and for*

$$\left(B_{\mathbb{R}^d}\left(x,\frac{k}{2}\right)\cap M\right) \subset B_M(x,k) \subset B_M(x,\pi\rho)$$

We had

$$\ldots\ldots(19)\mu^n\left(\frac{Vol\left(B_{\mathbb{R}^d}\left(x,\frac{k}{2}\right)\cap M\right)}{Vol\left(B_M(x,\pi\rho)\right)}\right) \le e^{\frac{(n-1)\sqrt{\pi\rho}k}{2}}\left(\frac{k}{2\pi\rho}\right)^n$$

Where $x \in M$*. Moreover,*

$$\ldots\ldots(20)\ \mu^n\left(\left|\frac{\lambda_{k_{B_{\mathbb{R}^d}}}}{\lambda_{k_M}}-1\right|\leq(n+1)\delta\right)\leq 2\exp-\left(\frac{(n+1)\delta^2}{2}\right)$$

Where, $\lambda_{k_{B_{\mathbb{R}^d}}}, \lambda_{k_M}$ *are the kth eigenvalues of the Laplace Operator on* $B_{\mathbb{R}^d}$ *a ball on* $\mathbb{R}^d$ *and on the manifold M respectively and* $d\left(\lambda_{k_{B_{\mathbb{R}^d}}}, \lambda_{k_M}\right) < \delta$*.And* μ^n *is the normalized measure.*

In [6] Y. Canzani had stated that "The mathematician ***Lorentz*** in 1910 stated his conjecture $N(\lambda) = \#\frac{area(\Omega)}{2\pi}\lambda$. And ***Hermann Weyl*** had Proved in 1911".

4.3 Theorem: (Weyl's Asymptotic formula)[6] Let M be a compact Riemannian manifold with eigenvalues $0 = \lambda_0 \leq \lambda_1 \leq \cdots$, each distinct eigenvalue repeated according to its multiplicity. Then for $N(\lambda) \coloneqq \#\{j: \lambda_j \leq \lambda\}$, we have

$$N(\lambda) \sim \frac{\omega_n}{(2\pi)^n} Vol_g(M)\lambda^{\frac{n}{2}}, \lambda \to \infty$$

In particular,

$$\lambda_j \sim \frac{\sqrt{2\pi}}{(\omega_n Vol(M))^{\frac{2}{n}}} j^{\frac{2}{n}}$$

Where, ω_n defined the volume of the unit $n - ball$.

Now, and from Polya Conjecture on [8] we found that

$$\ldots\ldots(21)\ \lambda_k \geq \frac{4\pi^2}{(\omega_n Vol(M))^{\frac{2}{n}}} k^{\frac{2}{n}}$$

Now we will discuss the eigenvalues of Δ relative to the $r - enlargement$ of the $n - dimensional$ geodesic ball $B_M(x, r)$. ***Jake Gipple*** in his paper [13] studied the volume of $n - Balls$.

4.4 Proposition:[13] For any natural number $n \geq 1$ and any real number $r > 0$

$$Vol\big(B^n(r)\big) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{1}{2}n+1\right)} r^n = Vol\left(B^n(1)\right) r^n$$

Where, $B^n(r)$ is the $n-dimensional$ ball of radius r and $B^n(1)$ is the unit $n-dimensional$ ball.

4.5 Theorem:[9] Let (X,d,μ) be a complete locally compact metric measured space, where μ is infinite measure. We assume that for all $r>0$, there exists an integer $N(r)$ such that each ball of radius r can be covered by $N(r)$ balls of radius $\left(\frac{r}{2}\right)$. If there exist an integer $k>0$ and radius $r>0$ such that for each $x \in M$,

$$\ldots\ldots(22)\mu\big(B(x,r)\big) \leq \frac{\mu(X)}{4N^2(r)k}$$

Then, there exists $k\ \mu-measurable$ subsets $A_1,\ldots,A_k$ of X such that $\mu(A_i) \geq \frac{\mu(X)}{2N(r)k}$ and for $i \neq j, d\big(A_i,A_j\big) \geq 3r$.

Now, from Eq (21) we had

$$N(r) \leq \left(\frac{\mu(X)}{4\mu(B(x,r)}\right)^{\frac{1}{2}}$$

If we replace $\mu(X)$ with $Vol(M)$ and as standard volume measure on an $n-dimensional$ manifold (M,g) we had

$$\ldots\ldots(23)N(r) \leq \left[\frac{Vol(M)}{4Vol\big(B^n(r)\big)}k\right]^{\frac{1}{2}}$$

And from Eq (21) we had

$$\ldots\ldots(24)Vol(M) \geq \frac{(4\pi)^n}{\lambda_k^{n/2}\omega_n}k$$

Substitute Eq (24) in Eq (23), we had

$$N(r) \leq \left[\frac{(4\pi)^n k}{4\lambda_k^{\frac{n}{2}}\omega_n Vol\big(B^n(r)\big)}\right]^{\frac{1}{2}}$$

And from Proposition (4.4) we get with $Vol\big(B^n(r)\big) = \omega_n r^n$ that

$$N(r) \leq \left[\frac{(4\pi)^n k}{4\lambda_k^{\frac{n}{2}} r_k^n [\omega_n]^2}\right]^{\frac{1}{2}}$$

Where $\omega_n = Vol\,\big(B^n(1)\big)$. This implies that

$$r_k^n \leq \frac{(4\pi)^n k}{4\lambda_k^{\frac{n}{2}}\big(N(r)\big)^2(\omega_n)^2}$$

So we can deduce that

$$r_k \leq \left[\frac{4\pi(k)^{\frac{1}{n}}}{4\sqrt{\lambda_k}(N(r)\omega_n)^{2/n}}\right] \Longrightarrow r_k \leq \left[\frac{C_{k,N(r),\omega_n}}{\sqrt{\lambda_k}}\right]$$

4.6 ***Corollary:*** *Let (M,g) be an $n-dimensional$ compact Riemannian manifold. Let $\{\lambda_i\}_{i=1}^n$ be its series of eigenvalues subjects to its Eigen function $\{\varphi_i\}_{i=1}^n$. Let $B^n(x,r_k) \subset M$ for $r_k > 0$ be $r_k - nieghborhood\ of\ M$. Then we have*

$$\ldots\ldots(25)\, r_k \leq \left[\frac{C}{\sqrt{\lambda_k}}\right]$$

Where $C = C(N(\lambda), N(r), \omega_n)$.

4.7 **Theorem: [11]** Let M be an $n-dimensional$ complete Riemannian manifold with Ricci curvature $Ricci \geq (n-1)$. If the first non – zero of the eigenvalues problem for M without boundaries $(\partial M = 0)$ is n, then M is isometric to a unite $n-sphere$.

4.8 Theorem: (The Maximal Diameter) [11] Let M be an $n-dimensional$ complete Riemannian manifold with Ricci curvature $Ricci \geq (n-1)$. If the diameter of M satisfies $d(M) \geq \pi$. Then M is isometric to an $n-dimensional$ unite sphere.

From Theorems (4.7) & (4.8) we deduce that $V(M) = V\big(B^n(1)\big)$, and with Eq (1), and Theorems (3.5) & (3.6) we deduce

4.9 Corollary: *Let (M,g) be an $n-dimensional$ Riemannian manifold with $Ricci \geq (n-1)$, and let $(\partial M = 0)$. Moreover, let $d(M) \geq \pi$. Then,*

$$\ldots\ldots(26)\mu(M) \geq \mu\big(B^n(1)\big)$$

Moreover, if

$$exp(-(n+1)r\,)\,\lambda_k\big(g_{B_1^n}\big) \leq \lambda_k(g_M) \leq exp\big((n+1)r\big)\,\lambda_k\big(g_{B_1^n}\big)$$

Then, and with Corollaries (2.10) and (4.6) and Eq (18) and for $r \sim \frac{1}{\sqrt{\lambda}}$ we had

$$\ldots\ldots(27)\mu^n\left(\left|\frac{\lambda_k(g_M)}{\lambda_k\big(g_{B_1^n}\big)}\right| \leq \frac{(n+1)}{\sqrt{\lambda}}\right) \leq exp\left(-\frac{(n+1)}{2\lambda}\right)$$

$$Where,\ d\big(g_M, g_{B_1^n}\big) \leq \frac{1}{\sqrt{\lambda}}.$$

Now, we get our main corollary in our context

4.10 Corollary: *Let (M,g) be an $n-dimensional$ compact Riemannian manifold with or without boundaries, and with $Ricci \geq (n-1)k$. Let μ be its probability measure. Moreover, let $\{\lambda_k\}_{k=1}^n$ be the eigenvalues of the Laplacian Δ on M. Let $A \subset M$ with $\mu(A) \geq \frac{1}{2}$. Let $A_{r_k} = \{x \in B_r^n : d(x,A) < r_k\}$, then we had according to Eq (27)*

$$\ldots\ldots(28)\mu\big(A_{r_k}\big) = \mu\left(A_{\frac{1}{\sqrt{\lambda_k}}}\right) \leq exp - \left[\frac{(n-1)}{2\lambda_k}\right]$$

$Where\ \lambda_k \sim \frac{1}{(r_k)^2}$.

5. **Conclusion:** From the previous investigation we conclude that we can use the series of eigenvalues of the Laplacian to describe the geometry of the structure under consideration, with the notion of concentration of measure. Moreover, this eigenvalues works more effectively on the area of concentration of measure upon the isoperimetric inequalities of this eigenvalues. It has been found that $r - enlargement$ relative to the eigenvalues of the Laplace operator and that is $\sim \frac{1}{\sqrt{\lambda}}$. The main result is equivalent to the frequency of the simple harmonic motion(SHM).

List of abbreviations

: Standard Volume on $\mathbb{R}^{n+1}$.V, Vol

: Standard measure on $\mathbb{R}^{n+1}$.μ^n, μ

: $r - enlargement$.r

: Eigenvalue of the Laplace operator.λ

: Laplace Operator on the ManifoldΔ

: Manifold on $\mathbb{R}^{n+1}$.M

: Standard metric on the Manifold$g$

: Simple Harmonic MotionSHM

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