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GIBBS SAMPLING AND BAYESIAN ESTIMATORS FOR TIME CENSORING CONSTANT STRESS RELIABILITY/LIFE PREDICTION

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ABSTRACT

The main objective of this paper is to develop the Bayesian analysis for Constant Stress Accelerated Life Test (CSALT) under time censoring scheme of the Generalized Logistic (GL) Failure times. The power law function is used to represent the relationship between the stress and the scale parameters of a test unit. Bayes estimates are obtained using Markov Chain Monte Carlo (MCMC), simulation algorithm based on Gibbs sampling. Then, Monte Carlo error (MC error), credible intervals, and predicted values of the two scale parameters and the reliability function under design stress are obtained. Numerical illustration is addressed for illustrating the theoretical results. Win-Bugs software package is used for implementing Markov Chain Monte Carlo (MCMC) simulation and Gibbs sampling.

KeyWords: Accelerated Life Test; Constant Stress; Time Censoring; Power Law Function, Bayesian Method; Generalized Logistic Distribution; Markov Chain Monte Carlo; Gibbs Samples, Win-Bugs.

1. INTRODUCTION

In many industrial fields, the need for highly reliable components and materials are widely required for long-term performance. In particular, the extremely high reliability is essential in aviation and aerospace industries and also strongly required in automobile industry, electronic industry, semiconductor industry, and many others in the fields beyond engineering, such as medical science. The high reliability, however, brings about unacceptable length of time and cost of product life testing experiments

under use-condition since many such products are usually operated for years or more without failures. In Accelerated Life Testing (ALT) components and materials are tested under more severe conditions (or stresses) than use-condition and thus induce early failures; i.e., it provides significant reduction in time and cost of reliability testing.

There are two types of ALT: Constant-Stress ALT (CSALT) and Step-Stress ALT (SSALT). In constant-stress ALT the units are placed only under one higher than normal stress level, i.e, stress applied to the products is time-independent. Test units are subjected at a constant, higher-than-usual level of stress until either all units fail (without censoring) or the test is terminated, resulting in censored test data. Commonly all available test data obtained from ALT is used in analysis of life data. However, the obtained data may be incomplete or it may include uncertainty about the failure time. Therefore, life data could be separated into two categories: complete (all failure data are available) or censored (some of failure data are missing). Complete data consist of the exact failure time of test units, which means that the failure time of each sample unit is observed or known. In many cases when life data are analyzed, all units in the sample may not fail. This type of data is commonly called censored or incomplete data. Due to different types of censoring, censored data can be divided into time-censored data and failure-censored data. Time censored data is also known as type I censored. This type of data is usually obtained when censoring time is fixed, and then the number of failures in that fixed time is a random variable. Data are failure censored (or type II censored) if the test is

terminated after a specified number of failures, where, the time to the fixed number of failures is a random variable. Considering the censoring type, constant-stress ALT can also be divided into time censored CSALT and failure-censored CSALT. In this paper, time-censored CSALT is considered.

Generalized Logistic Distribution (GLD) is useful family of distributions in many practical situations. There are some who argue that the GLD is an inappropriate for modeling lifetime data because the left-hand limit of the distribution extends to negative infinity. This could conceivably result in modeling negative times-to-failure. However, provided that the distribution in question has a relatively high scale parameter α_1 and a relatively small scale parameter α_2 , the issue of negative failure times should not present itself as a problem, [Mathai and Provost (2004)]. The probability density function of the GLD of Molenberghs and Verbeke [3], has the following form

$$f(t) = \alpha_1 \alpha_2 e^{\alpha_1 t} \left(1 + \frac{\alpha_2 e^{\alpha_1 t}}{\alpha_3}\right)^{-(\alpha_3+1)}, \quad -\infty < t < \infty, \alpha_1, \alpha_2, \alpha_3 > 0 \dots (1)$$

where the parameters α_1, α_2 are referred to as the two scale parameters and α_3 the shape parameter of the GLD. The reliability function of Eq.(1) is given by

$$R(t) = \left(1 + \frac{\alpha_2 e^{\alpha_1 t}}{\alpha_3}\right)^{-\alpha_3}, \quad -\infty < t < \infty, \alpha_1, \alpha_2, \alpha_3 > 0$$

For the special case $\alpha_1 = \alpha_2 = \alpha_3 = 1$, the GLD is the standard logistic distribution. For $\alpha_1 = 1, \alpha_2 = \alpha_3$ is the Type-I GLD. Moreover, many distributions can be obtained by making some transformations on the pdf (1), for $y = e^t$ is the generalized log logistic distribution. For $y = e^{-t}, \alpha_2 = \alpha_3$ is the Burr III distribution. For $y = e^t, \alpha_2 = \alpha_3$ is the generalized Burr XII distribution.

The maximum likelihood method of estimation of distribution parameters assumes that the parameters are unknown, but fixed. The Bayesian approach, however, assumes that the parameters are random, and uncertainties on the parameters are described by a joint prior distribution, which is formulated before the failure data are collected, and is based on historical data, experience with similar products, design specifications, and experts' opinions. The capability of incorporating prior knowledge in the analysis makes the Bayesian approach very valuable in the reliability analysis because one of the major challenges associated with the reliability analysis is the limited availability of data. Inference on the model parameters is made in terms of probability statements, which are conditional on the observed data t .

The Bayesian approach has not been widely applied to analyze data from CSALT. Achcar [5] used Bayesian approach and assumed non-informative priors for the parameters of the exponential, Weibull, Birbaum-Saunders, and Inverse Gaussian distributions. He You [4] used Bayesian approach to estimate the parameters of the exponential distribution under different priors and different censoring schemes. Zhong and Meeker [9] estimated the

parameters of Weibull distribution assuming log-normal prior density. Liu and Tang [8] constructed a sequential CSALT scheme and its Bayesian inference using Weibull distribution and Arrhenius relationship. Aly and Bleed [2] developed Bayesian analysis for CSALT under Type-II censoring scheme with inverse power law model.

The inference on each parameter $(\alpha_1, \alpha_2, \alpha_3)$ is based on its marginal posterior density. To obtain the marginal posterior densities, multiple levels of integration are necessary. So, Markov chain Monte Carlo (MCMC) simulation is the easiest way to get reliable results without evaluating integrals, Gelman, *et al.* [1].

A MCMC algorithm that is particularly useful in high dimensional problems is the alternating conditional sampling called Gibbs sampling. Each iteration of the Gibbs sampling cycles through the unknown parameters, drawing a sample of one parameter conditioning on the latest values of all the other parameters. When the number of iterations is large enough, the samples drawn on one parameter can be regarded as simulated observations from its marginal posterior distribution. Functions of the model parameters, such as the p^{th} percentile of the lifetime distribution at the normal stress condition can be conveniently sampled. Posterior inference can be computed also using sample statistics.

In this paper, it is assumed that the test is done at high constant stresses $s_j, j = 1, 2, \dots, k$ where $s_0 < s_1 < s_2 < \dots < s_k$, and s_0 be the normal stress. A total of N units are divided into $n_j, j = 1, 2, \dots, k$ units

where $N = \sum_{j=1}^k n_j$, and the stress affected on the two scale parameters of the GLD. Then, the scale parameters α_{1j} and α_{2j} at the stress level s_j , $j = 1, 2, \dots, k$ of a test unit is a power law function of stress which is defined by Eq.(2), see Mann, Schafer and Singurwalla [6],

$$\alpha_l = c_l s_j^p, \quad l = 1, 2; \quad j = 1, 2, \dots, k \dots (2)$$

Where c_1 , c_2 and p constants to be estimated. Let t_i , $i = 1, 2, \dots, n$ be the life times of n units put to test under model (1). Then under time censored data, where the test is terminated at a pre-specified censored time T_j the likelihood function of this censored is given by

$$L = \prod_{j=1}^k \prod_{i=1}^{n_j} \left[\alpha_{1j} \alpha_{2j} e^{\alpha_{1j} t_i} \left(1 + \frac{\alpha_{2j}}{\alpha_3} e^{\alpha_{1j} t_i} \right)^{-(\alpha_3+1)} \right]^{\delta_i} \\ \cdot \left(1 + \frac{\alpha_{2j}}{\alpha_3} e^{\alpha_{1j} T_j} \right)^{-\alpha_3(1-\delta_i)} \dots (3)$$

$$\text{Such that, } \delta_i = \begin{cases} 1, & t_i \leq T_j \\ 0, & t_i > T_j \end{cases}$$

2. THE BAYES ESTIMATORS

The likelihood function of Eq.(3) under the power law function of Eq.(2) can be re-written as

$$L = \prod_{j=1}^k \prod_{i=1}^{n_j} \left[c_1 s_j^{c_1} c_2 s_j^{c_2} e^{c_1 s_j^{c_1} t_i} \left(1 + \frac{c_2 s_j^{c_2}}{\alpha_3} e^{c_1 s_j^{c_1} t_i} \right)^{-(\alpha_3+1)} \right]^{\delta_i} \cdot \left(1 + \frac{c_2 s_j^{c_2}}{\alpha_3} e^{c_1 s_j^{c_1} T_j} \right)^{-\alpha_3(1-\delta_i)} \dots (4)$$

and the log-likelihood function of Eq.(4) will be

$$\begin{aligned} l_T &= \ln c_1 \sum_{j=1}^k \sum_{i=1}^{n_j} \delta_i + \ln c_2 \sum_{j=1}^k \sum_{i=1}^{n_j} \delta_i + 2p \sum_{j=1}^k \sum_{i=1}^{n_j} \delta_i \ln s_j \\ &\quad + c_1 \sum_{j=1}^k \sum_{i=1}^{n_j} \delta_i s_j^{c_1} t_i - (\alpha_3 \\ &\quad + 1) \sum_{j=1}^k \sum_{i=1}^{n_j} \delta_i \ln \left(1 + \frac{c_2 s_j^{c_2}}{\alpha_3} e^{c_1 s_j^{c_1} t_i} \right) \\ &\quad - \alpha_3 \sum_{j=1}^k \sum_{i=1}^{n_j} (1 - \delta_i) \left(1 + \frac{c_2 s_j^{c_2}}{\alpha_3} e^{c_1 s_j^{c_1} T_j} \right) \end{aligned}$$

The Bayesian inference for two cases will be presented as follows:

Case I: If the parameters c_1 and p are unknown, then the natural family of the conjugate prior of c_1 is taken as the Gamma distribution with probability density function,

$$\pi(c_1) = c_1^{a-1} e^{-bc_1}, \quad a, b > 0, c_1 > 0$$

and the natural family of the conjugate prior of p given c_1 is taken as Gamma (d, gc_1) , then the prior density for c_1 and p is given by

$$\pi(c_1, p) = c_1^{a+d} p^d e^{-c_1(b+g)}, \quad a, b, d, g > 0, c_1, p > 0 \dots \quad (5)$$

Therefore, the posterior density of c_1 and p given t based on Eq.(4) and Eq.(5) is given by

$$\pi(c_1, p/t) = c_1^{a+d} p^d e^{-c_1(b+g - \sum_{j=1}^k \sum_{i=1}^{n_j} P_i)} \cdot \prod_{j=1}^k \prod_{i=1}^{n_j} [C_i \cdot C_i \cdot D_i \cdot D_{ij}] \dots \quad (6)$$

Where,

$$C_i = [c_1 s_j^p]^{\delta_i}, \quad C_i = [c_2 s_j^p]^{\delta_i}, \quad P_i = s_j^p \delta_i t_i,$$

$$D_i = \left(1 + \frac{c_2 s_j^p}{\alpha_3} e^{c_1 s_j^p t_i} \right)^{-(\alpha_3+1)\delta_{ij}},$$

$$D_i = \left(1 + \frac{c_2 s_j^p}{\alpha_3} e^{c_1 s_j^p t_j} \right)^{-\alpha_3(1-\delta_i)}.$$

Complicated integrations are often analytically intractable for Eq.(6). So, Markov Chain Monte Carlo (MCMC) simulation is the easiest way to get reliable results Gelman, *et al.* [1]. Through the MCMC approach, a sample of the posterior distribution can be used to obtain the Bayes estimators of c_1 and p . From the sample, approximations of moments and an approximation of the posterior distribution may be derived using Gibbs sampling. Gibbs sampling is used to draw a random sample of the parameters c_1 and p from their own marginal posterior distribution (c_1/t) , and (p/t) , respectively, then estimate the expected value of the parameters c_1 and p using the sample mean. Therefore Bayesian estimate of the scale parameter α_1 and the reliability function $R(t)$ at the mission time t_0 under the normal stress s_0 can be obtained.

Case II: If the parameters c_2 and p are unknown, then the natural family of conjugate prior of c_2 is taken as the Gamma distribution with probability density function,

$$\pi(c_2) = c_2^{h-1} e^{-vc_2}, \quad h, v > 0, c_2 > 0 \quad \dots \quad (7)$$

and the natural family of conjugate prior of p given c_2 is taken as Gamma (d, gc_2) , then the prior density for c_2 and p is given by

$$\pi(c_2, p) = c_2^{h+d} p^d e^{-c_2(v+g)}, \quad a, b, d, g > 0, c_2, p > 0 \quad \dots \quad (8)$$

Therefore, the posterior density of c_2 and p given t based on Eq.(4) and

Eq.(8) is given by

$$\pi(c_2, p/t) = c_2^{h+d} p^d e^{-c_2(v+g)} \cdot \prod_{j=1}^k \prod_{i=1}^{n_j} [C_i \cdot C_i \cdot D_i \cdot D_i] \dots \quad (9)$$

Where, $C_i \cdot C_i \cdot D_i \cdot D_i$ are defined in Eq.(6). Similarly, MCMC simulation is used to get reliable results of the Bayes estimator of the parameters c_2, p, α_2 under the normal stress s_0 and $R(t)$ at the mission time t_0 and under the normal stress s_0 . Win-Bugs is used, a specialized software package for implementing MCMC simulation and Gibbs sampling.

3. NUMERICAL ANALYSIS

1. Consider four accelerated stress levels $s_1 = 2, s_2 = 3, s_3 = 4, s_4 = 5$, and $s_0 = 1$.
2. Assume that the test is terminated at a specified failure times $T_j, j = 1, 2, 3, 4$ where $T_1 = 11, T_2 = 25, T_3 = 33, T_4 = 39$.
3. Accelerated life data from the GLD are generated using Math-Cad software.
4. The K-S test is used for assessing that the data set follows the GLD.
5. The posterior estimation of the parameters by applying Bayesian are gotten using generating data of the GLD under CSALT time censoring.
6. The parameters of interest are estimated as well as the scale parameters and the reliability function under normal stress are predicted.

The Case of unknown c_1 and p

1. Start with three Markov chains with different initial values ($c_1 = 0.05, p = 0.05$), ($c_1 = 0.05, p = 0.001$), ($c_1 = 0.05, p = 0.15$).

2. Assume that the values of (c_2, α_3) are known and set $(c_2 = 0.05, \alpha_3 = 0.5)$.
3. Assume the prior of c_1 is Gamma $(0.5, 0.75)$.
4. The conditional distribution of p given c_1 is Gamma $(0.5, 0.05)$.
5. Run 15,000 iterations for each Markov chain.
6. The trace plots of c_1 and p for 15,000 iterations are presented in Figure (1).

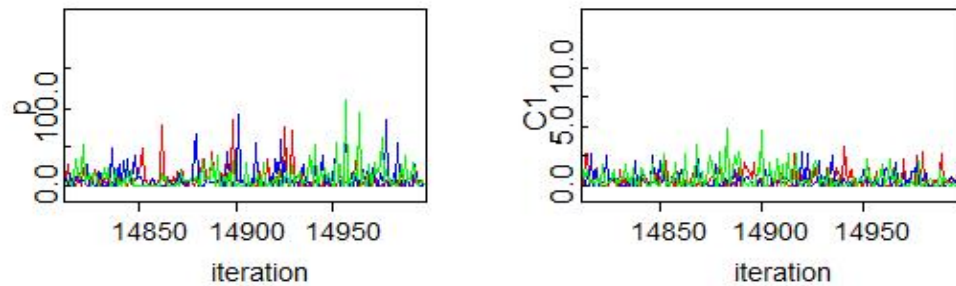


Figure 1: The trace plots of c_1 and p [case I]

7. To check convergence: Gelman-Rubin convergence statistic, R , is introduced, Figure (2) shows that the Gelman-Rubin statistic of (c_1, p) are believed to have converged, i.e., R is close to be one, Luo [7].

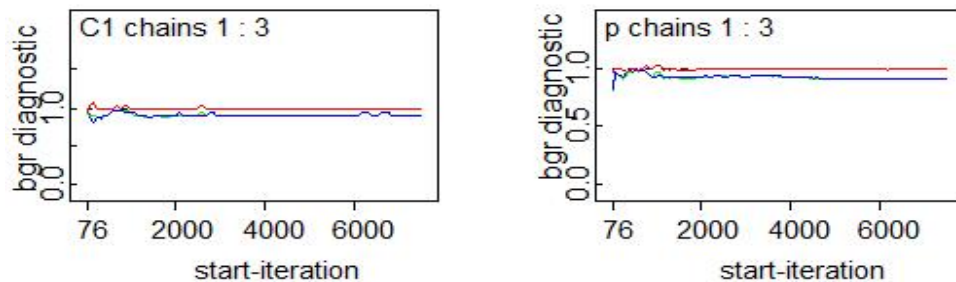


Figure 2: The Gelman-Rubin convergence statistic of c_1 and p [case I]

8. The sampling results of the unknown parameters c_1 and p : it can be generated showing posterior mean, median and standard deviation with a 95% posterior credible interval, see Table (1).

TABLE 1: THE SAMPLING RESULTS c_1 AND p [CASE I]

Parameter	Mean	S.D	MC error	2.50%	Median	97.50%
c_1	0.6631	0.9341	0.004168	6.804E-4	0.2979	3.333
p	10.12	14.47	0.07214	0.01057	4.542	51.22

9. The estimated values of α_1 and $R(t)$ under the normal stress s_0 : the estimated values of the scale parameter under the normal stress is 0.6631, and notes that the reliability function decreases when the mission time t_0 increases, see Table (2).

TABLE 2: THE SAMPLING RESULTS OF α_1 AND $R(t)$ [CASE I]

Parameter	α_1	$R(t_0)$				
		$t_0=3$	$t_0=2$	$t_0=1$	$t_0=0.05$	$t_0=0.005$
Mean	0.66310	0.744600	0.813500	0.894100	0.952000	0.953300
S.D	0.93410	0.286400	0.223700	0.116500	0.002205	0.000204
MC error	0.00417	0.001326	0.001014	0.000518	0.000010	0.000001
Median	0.29790	0.896400	0.920000	0.938800	0.952800	0.953400
2.50%	0.00068	0.021310	0.112100	0.512800	0.945700	0.952700
97.50%	3.33300	0.953400	0.953400	0.953400	0.953500	0.953500

10. The accuracy of the posterior estimate: The accuracy is calculated in terms of Monte Carlo standard error (MC error) of the mean according to Gelman, *et al.* [1]. The simulation is run until the MC error for each node is less than 0.05 of the sample standard deviation.

11. According to point (10), also the rule of MC error has been achieved in this paper. Table (1) and Table (2) show that , the MC error for each node is less than 0.05 of the sample standard deviation.

12. The shape of the posterior density of (c_1, p) can be shown from Figure (2).

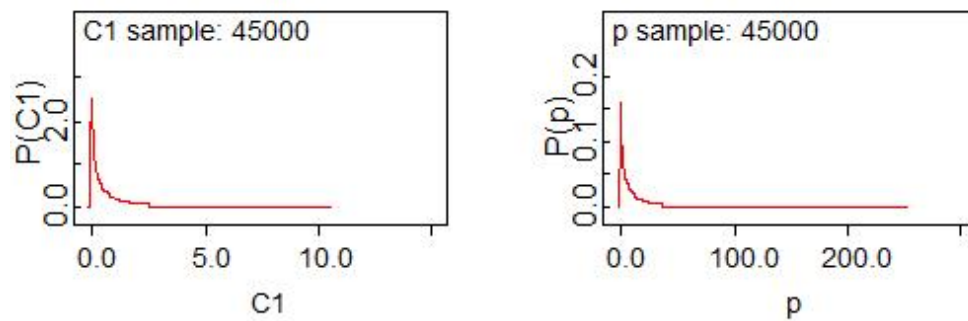


Figure 3: The posterior density of c_1 and p [case I]

The Case of unknown c_2 and p

1. Start with three Markov chains with different initial values $(c_2 = 0.15, p = 0.5), (c_2 = 0.15, p = 0.01), (c_2 = 0.15, p = 0.15)$.

2. Assume that the values of (c_1, α_3) are known and set $(c_1 = 0.05, \alpha_3 = 0.5)$.

3. Assume the prior of c_2 is Gamma $(0.15, 0.05)$.

4. Assume the conditional distribution of p given c_2 is Gamma $(0.05, 0.15)$.

5.Run 15,000 iterations for each Markov chain.

6.The trace plots of c_2 and p for 15,000 iterations are presented in Figure (4).

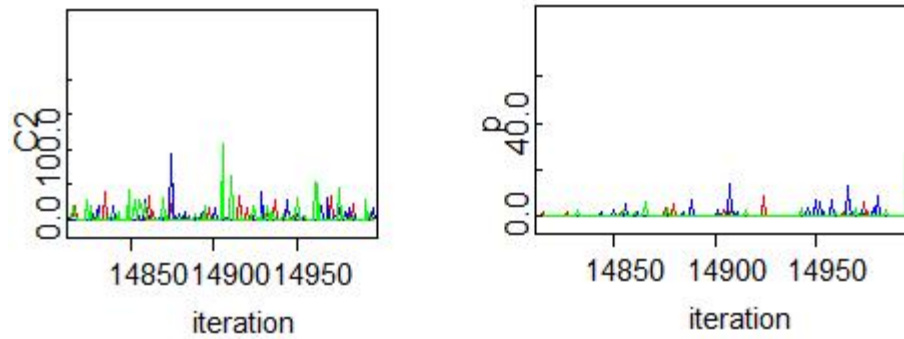


Figure 4: The trace plots of c_2 and p [case II]

7.To check convergence: Figure (5) shows that the Gelman-Rubin statistic of (c_2, p) are believed to have converged, i.e., R is close to be one, Luo [7].

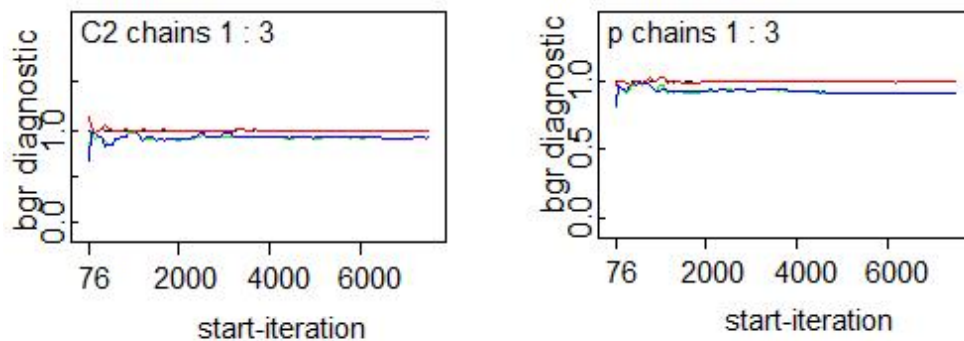


Figure 5: The Gelman-Rubin convergence statistic of c_2 and p [case II]

8.The sampling results: it can be generated showing posterior mean, median and standard deviation with a 95% posterior credible interval, see Table (3).

TABLE 3: THE SAMPLING RESULTS c_2 AND p [CASE II]

Parameter	Mean	S.D	MC error	2.50%	Median	97.50%
c_2	3.021	7.861	0.03564	2.73E-10	0.1198	25.73
p	0.3418	1.538	0.007162	5.176E-32	3.559E-6	3.875

9. The estimated values of α_2 and $R(t)$ under the normal stress s_0 : the estimated values of the scale parameters under the normal stress is 2.6830, and notes that the reliability function decreases when the mission time t_0 increases, see Table (4).

TABLE 4: THE SAMPLING RESULTS OF α_2 AND $R(t)$ [CASE II]

Parameter	α_2	$R(t_0)$				
		$t_0=3$	$t_0=2$	$t_0=1$	$t_0=0.05$	$t_0=0.005$
Mean	2.6830	0.82360	0.83020	0.83660	0.84270	0.84300
S.D	7.2710	0.19390	0.18940	0.18490	0.18060	0.18040
MC error	0.0335	0.00085	0.00083	0.00081	0.00079	0.00079
Median	0.0810	0.91530	0.92330	0.93060	0.93700	0.93730
2.50%	8.50E-11	0.43790	0.44770	0.45770	0.46710	0.46760
97.50%	23.6800	1.00000	1.00000	1.00000	1.00000	1.00000

10. The simulation is run until the MC error for each node is less than 0.05 of the sample standard deviation.

11. According to point (10), also the rule of MC error has been achieved in this paper. Table (3) and Table (4) show that, the MC error for each node is less than 0.05 of the sample standard deviation.

12. The shape of the posterior density of (c_2, p) can be shown from Figure (6).

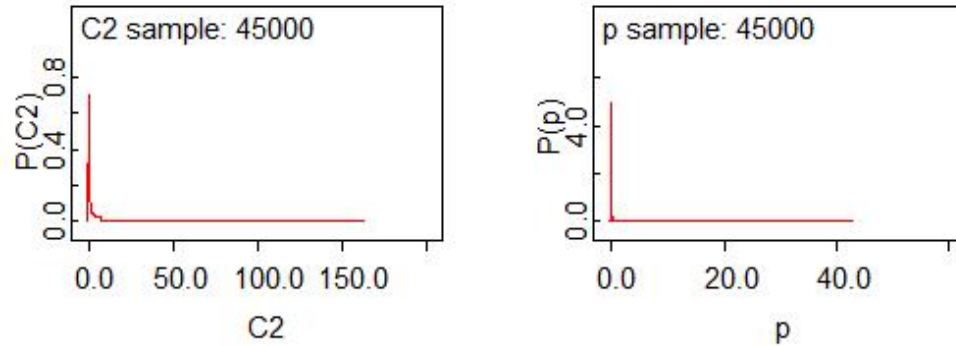


Figure 6: The posterior density of c_1 and p [case II]

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