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Najeeb Abdulaleem

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E-duality results for *E*-differentiable vector optimization problems involving *E*-type I functions

Najeeb Abdulaleem
Department of Mathematics
Hadhramout University
P. O. Box (50511-50512)
Al-Mahrah
Yemen

and

Faculty of Mathematics and Computer Science
University of Łódź
Banacha 22, 90–238
Łódź
Poland

Abstract

In this paper, the class of *E*-differentiable vector optimization problems is considered. The so-called vector Wolfe and mixed *E*-duality problems are defined for the considered *E*-differentiable multiobjective programming problem and several Wolfe and mixed *E*-duality theorems are derived under appropriate *E*-type I functions.

Subject Classification: *90C26, 90C30, 26B25.*

Keywords: *E-type I functions, E-differentiable function, E-optimality conditions, Wolfe E-duality, Mixed E-duality.*

1. Introduction

Consider the following multiobjective programming problem with inequality constraints:

E-mail: `nabbas985@gmail.com`

$$\begin{aligned} \text{minimize } f(x) &= (f_1(x),\dots,f_p(x)) \\ \text{subject to } g_t(x) &\leqq 0,\, t\in T=\{1,\dots,m\}, \end{aligned} \qquad \text{(VP)}$$

where $f_i : R^n \to R,\ i \in I = \{1,\dots,p\},\ g_t : R^n \to R,\ t \in T$, are (not necessarily) differentiable functions on R^n. Let D denote the set of all feasible solutions in problem (VP), that is,

$$D := \{x \in R^n : g_t(x) \leqq 0,\, t \in T\}.$$

Further, we denote by $T(\overline{x}) = \{t \in T : g_t(\overline{x}) = 0\}$ the set of indices of inequality constraints that are active at the arbitrary feasible point $\overline{x}$.

The following convention for equalities and inequalities will be used in the paper. If $x = (x_1, x_2, \dots, x_n)^T$ and $y = (y_1, y_2, \dots, y_n)^T$ in R^n, then, $x = y \Leftrightarrow x_i = y_i$, $i = 1,2,\cdots,n$; $x > y \Leftrightarrow x_i > y_i$, $i = 1,2,\cdots,n$; $x \geqq y \Leftrightarrow x_i \geqq y_i$, $i = 1,2,\cdots,n$; $x \geq y \Leftrightarrow x \geqq y$ and $x \neq y$.

For the multiobjective programming problem (VP), the following concepts of a weakly efficient solution (a weak Pareto solution) and an efficient solution (a Pareto solution) are defined as follows:

Definition 1.1 : [1] A point $\overline{x}$ is said to be a weakly efficient solution (weak Pareto solution) of (VP) if there exists no x such that

$$f(x) < f(\overline{x}).$$

Definition 1.2 : [1] A point $\overline{x}$ is said to be an efficient solution (a Pareto solution) of (VP) if there exists no x such that

$$f(x) \leq f(\overline{x}).$$

During the last few years, there has been an increasing interest in generalizations of convexity in connection with sufficiency and duality in optimization problems. Optimality conditions and duality results for differentiable and nondifferentiable multiobjective programming problems have been studied extensively in the literature (see, for example, [1], [2], [4], [5], [6], [8], [9], [11], [12], [13], [14], [15] and others).

Hanson and Mond [10] introduced type I and type II invexities which have been generalized to pseudo type I and quasi type I functions by Reuda and Hanson [13]. Later, Aghezzaf and Hachimi [8] introduced vector-valued generalized type I functions. Megahed et al. [12] introduced a new definition of an E-differentiable convex function which transforms a (not necessarily) differentiable function to a differentiable function under an operator $E : R^n \to R^n$. Abdulaleem [3] introduced the E-differentiable

E -invexity notion as a generalization of the concept of E-differentiable E-convexity. Recently, Abdulaleem [7] introduced a new concept of the so-called E-type I functions for E-differentiable vector optimization problems. Namely, Abdulaleem defined several classes of generalized vector-valued E-type I functions for E-differentiable functions as a generalization of the concept of differentiable vector-valued type I functions introduced by Hanson and Mond [10].

In this paper, we consider a new class of (not necessarily differentiable) multiobjective programming problems in which the involved functions are E-differentiable. For the considered E-differentiable programming problem, its vector E-dual problems in the sense of Wolfe and mixed are defined. Then various Wolfe and mixed E-duality theorems are established between the considered E-differentiable multicriteria optimization problem and its vector Wolfe and mixed E-dual problems under E-type I functions.

2. Preliminaries

We now give the definition of an E-invex set introduced by Abdulaleem [3].

Definition 2.1 : [3] Let $E : R^n \to R^n$. A set $M \subseteq R^n$ is said to be an E-invex set if and only if there exists a vector-valued function $\eta : M \times M \to R^n$ such that the relation

$$E(u) + \lambda\eta\left(E(x), E(u)\right) \in M$$

holds for all $x, u \in M$ and any $\lambda \in [0,1]$.

We now recall for a common reader the definition of an E-differentiable function introduced by Megahed et al. [12].

Definition 2.2 : [12] Let $E : R^n \to R^n$ and $f : M \to R$ be a (not necessarily) differentiable function at a given point $u \in M$. It is said that f is an E-differentiable function at u if and only if $f \circ E$ is a differentiable function at u (in the usual sense), that is,

$$(f \circ E)(x) = (f \circ E)(u) + \nabla(f \circ E)(u)(x-u) + \theta(u, x-u)\left\|x-u\right\|, \tag{1}$$

where $\theta(u, x-u) \to 0$ as $x \to u$.

Let $E : R^n \to R^n$ be a given one-to-one and onto operator. Throughout the paper, we shall assume that the functions constituting the problem (VP) are E-differentiable at any feasible solution. Now, for the problem

(VP), we define its associated differentiable vector optimization problem (VP_E) as follows:

$$\begin{aligned} &\text{minimize } f(E(x)) = (f_1(E(x)),\dots,f_p(E(x))) \\ &\text{subject to } g_t(E(x)) \leqq 0,\ t\in T=\{1,\dots,m\}. \end{aligned} \qquad (VP_E)$$

Let D_E denote the set of all feasible solutions in problem (VP_E), that is,

$$D_E := \{x \in R^n : g_t(E(x)) \leqq 0,\ t \in T\}.$$

Further, we denote by $T_E(\overline{x}) = \{t \in T : g_t(E(\overline{x})) = 0\}$ the set of indices of inequality constraints that are active at the arbitrary feasible point $\overline{x} \in D_E$.

Definition 2.3 : [1] A point $E(\overline{x}) \in D$ is said to be a weakly E-efficient solution (weak E-Pareto solution) of (VP) if there exists no $E(x) \in D$ such that

$$f(E(x)) < f(E(\overline{x})).$$

Definition 2.4 : [1] A point $E(\overline{x}) \in D$ is said to be an E-efficient solution (an E-Pareto solution) of (VP) if there exists no $E(x) \in D$ such that

$$f(E(x)) \leq f(E(\overline{x})).$$

Lemma 2.5 : [1] *Let $E : R^n \to R^n$ be a one-to-one and onto. Then $E(D_E) = D$.*

Lemma 2.6 : [1] *Let $\overline{x} \in D$ be a weakly efficient solution (an efficient solution) of the problem (VP). Then, there exists $\overline{z} \in D_E$ such that $\overline{x} = E(\overline{z})$ and $\overline{z}$ is a weakly efficient solution (an efficient solution) of the problem (VP_E).*

Lemma 2.7 : [1] *Let $\overline{z} \in D_E$ be a weakly efficient solution (an efficient solution) of the problem (VP_E). Then $E(\overline{z})$ is a weakly efficient solution (an efficient solution) of the problem (VP).*

Definition 2.8 : [7] Let $E : R^n \to R^n$, $M \subseteq R^n$ be an open E-invex set. (f, g) is said to be (strictly) E-type I with respect to η at $u \in M$ if there exists a vector-valued function $\eta : M \times M \to R^n$ such that, for all $x \in M$,

$$f_i(E(x)) - f_i(E(u)) \ \geqq\ \nabla f_i(E(u))\eta(E(x), E(u)),\ (>) \qquad (2)$$

$$-g_t(E(u)) \ \geqq\ \nabla g_t(E(u))\eta(E(x), E(u)). \qquad (3)$$

Theorem 2.9 : [1] *(E-Karush-Kuhn-Tucker necessary optimality conditions). Let* $\overline{x} \in D_E$ *be a weakly efficient solution of the problem* (VP_E) *(and, thus,* $E(\overline{x})$ *be a weakly E-efficient solution of the considered multiobjective programming problem (VP)). Further, f and g are E-differentiable at* $\overline{x}$ *and the Guignard constraint qualification [3] is satisfied at* $\overline{x}$*. Then there exist Lagrange multipliers* $\overline{\tau} \in R^p$, $\xi \in R^m$ *such that*

$$\sum_{i=1}^{p} \overline{\tau}_i \nabla f_i \left(E(\overline{x})\right) + \sum_{t=1}^{m} \overline{\xi}_t \nabla g_t \left(E(\overline{x})\right) = 0, \tag{4}$$

$$\overline{\xi}_t g_t \left(E(\overline{x})\right) = 0, t \in T(E(\overline{x})), \tag{5}$$

$$\overline{\tau} \geq 0, \overline{\xi} \geqq 0. \tag{6}$$

3. Wolfe *E*-duality

In this section, for the considered *E*-differentiable multiobjective programming problem (VP), we define the following dual problem in the sense of Wolfe. Further, under *E*-type I functions, we prove *E*-duality results between the multiobjective programming problem (VP) and its vector *E*-dual problem in the sense of Wolfe. Let $E : R^n \to R^n$ be a one to one and onto operator. We define for problem (VP_E) its *E* -differentiable vector *E*-dual problem in the sense of Wolfe as follows:

$$\text{maximize } f(E(y)) + \left[\sum_{t=1}^{m} \xi_t g_t (E(y)) \right] e$$

$$\text{subject to} \sum_{i=1}^{p} \tau_i \nabla (f_i \circ E)(y) + \sum_{t=1}^{m} \xi_t \nabla (g_t \circ E)(y) = 0, \qquad (\text{WD}_E)$$

$$y \in R^n,\ \tau \geq 0, \tau \in R^p, \tau e = 1,\ e = (1,1,...,1)^T \in R^p,$$

$$\xi \in R^m, \xi \geqq 0,$$

where $e = (1,...,1) \in R^p$. Denote by

$$\Gamma_E = \left\{ (y, \tau, \xi) \in R^n \times R^p \times R^m : \sum_{i=1}^{p} \tau_i \nabla (f_i \circ E)(y) + \sum_{t=1}^{m} \xi_t \nabla (g_t \circ E)(y) = 0, \tau e = 1, \tau \geq 0, \xi \geqq 0 \right\}.$$

the feasible solutions set of (WD$_E$). Further, $Y_E = \{y \in R^n : (y, \tau, \xi) \in \Gamma_E\}$.

Theorem 3.1 : *(Wolfe weak duality between (VP$_E$) and (WD$_E$)). Let* $x \in D_E$ *and* $(y, \tau, \xi) \in \Gamma_E$ *such that* (f, g) *is E-type I at y. Then*

$$f(E(x)) \not< f(E(y)) + \left[\sum_{t=1}^{m} \xi_t g_t(E(y))\right] e. \tag{7}$$

***Proof* :** Let $x \in D_E$ and $(y, \tau, \xi) \in \Gamma_E$. We proceed by contradiction. Suppose, contrary to the result, that the inequality

$$f(E(x)) < f(E(y)) + \left[\sum_{t=1}^{m} \xi_t g_t(E(y))\right] e \tag{8}$$

holds. Thus,

$$f(E(x)) < f(E(y)) + \left[\sum_{t=1}^{m} \xi_t g_t(E(y))\right] \sum_{i=1}^{p} \tau_i, i \in I \tag{9}$$

holds. Multiplying the above inequality by τ_i with taking that $\sum_{i=1}^{p} \tau_i = 1$, the following inequality

$$\sum_{i=1}^{p} \tau_i f_i(E(x)) < \sum_{i=1}^{p} \tau_i f_i(E(y)) + \sum_{t=1}^{m} \xi_t g_t(E(y)) \tag{10}$$

holds. From $x \in D_E$, it follows that

$$\sum_{i=1}^{p} \tau_i f_i(E(x)) + \sum_{t=1}^{m} \xi_t g_t(E(x)) < \sum_{i=1}^{p} \tau_i f_i(E(y)) + \sum_{t=1}^{m} \xi_t g_t(E(y)), \tag{11}$$

$$\sum_{i=1}^{p} \tau_i f_i(E(x)) < \sum_{i=1}^{p} \tau_i f_i(E(y)), \tag{12}$$

$$\sum_{t=1}^{m} \xi_t g_t(E(x)) \leqq \sum_{t=1}^{m} \xi_t g_t(E(y)). \tag{13}$$

By assumption, $x \in D_E$ and $(y, \tau, \xi) \in \Gamma_E$. Since (f, g) is E-type I with respect to η at y and by Definition 2.8, the following inequalities

$$f_i(E(x)) - f_i(E(y)) \geqq \nabla f_i(E(y)) \eta(E(x), E(y)), \quad i \in I \tag{14}$$

$$0 = -g_t(E(y)) \geqq \nabla g_t(E(y)) \eta(E(x), E(y)). \quad t \in T(E(y)) \tag{15}$$

hold, respectively. By $\tau_i > 0,\ \ i = 1,2,..,p,\ \ \xi_t \geqq 0,\ \ t \in T(E(y)),$ using above inequalities, we obtain that the inequality

$$\sum_{i=1}^{p}\tau_i f_i(E(x)) - \sum_{i=1}^{p}\tau_i f_i(E(y)) \geqq \left[\sum_{i=1}^{p}\tau_i \nabla f_i(E(y)) + \sum_{t\in T(E(y))}\xi_t \nabla g_t(E(y))\right]\eta(E(x),E(y)) \quad (16)$$

holds. Thus, by the first constraint of (WD_E), it follows that the following inequality

$$\sum_{i=1}^{p}\tau_i f_i(E(x)) \geqq \sum_{i=1}^{p}\tau_i f_i(E(y)) \quad (17)$$

holds, contradicting (12). Thus, the proof of this theorem is completed. □

Theorem 3.2 : *(Wolfe weak E-duality between (VP) and (WD_E)). Let* $E(x) \in D$ *and* $(y,\tau,\xi) \in \Gamma_E$. *Further, suppose that all hypotheses of Theorem 3.1 are fulfilled. Then, Wolfe weak E-duality between (VP) and (WD_E) holds, that is,*

$$f(E(x)) \nless f(E(y)) + \left[\sum_{t=1}^{m}\xi_t g_t(E(y))\right]e.$$

Proof : Let $E(x) \in D$ and $(y,\tau,\xi) \in \Gamma_E$. Then, Lemma 2.5 implies that $x \in D_E$. Since all hypotheses of Theorem 3.1 are fulfilled, from Theorem 3.1 the Wolfe weak E-duality between (VP) and (WD_E) follows directly. □

Theorem 3.3 : *(Wolfe weak duality between (VP_E) and (WD_E)). Let* $x \in D_E$ *and* $(y,\tau,\xi) \in \Gamma_E$ *such that* (f,g) *is strictly E-type I at y. Then*

$$f(E(x)) \nless f(E(y)) + \left[\sum_{t=1}^{m}\xi_t g_t(E(y))\right]e. \quad (18)$$

Theorem 3.4 : *(Wolfe weak E-duality between (VP) and (WD_E)). Let* $E(x) \in D$ *and* $(y,\tau,\xi) \in \Gamma_E$. *Further, suppose that all hypotheses of Theorem 3.3 are fulfilled. Then, weak E-duality between (VP) and (WD_E) holds, that is,*

$$f(E(x)) \nless f(E(y)) + \left[\sum_{t=1}^{m}\xi_t g_t(E(y))\right]e.$$

Theorem 3.5 : *(Wolfe strong duality between (VP_E) and (WD_E) and also strong E-duality between (VP) and (WD_E)). Let* $\bar{x} \in D_E$ *be a weakly efficient solution (an efficient solution) of the problem (VP_E) (and, thus,* $E(\bar{x}) \in D$ *be a weakly E-efficient solution (an E-efficient solution) of the problem (VP)). Further, assume that the Guignard constraint qualification [3] be satisfied at* $\bar{x}$. *Then there exist* $\tau \in R^p$, $\xi \in R^m$, $\xi \geqq 0$ *such that* $(\bar{x}, \tau, \xi) \in \Gamma_E$. *If all hypotheses of (Theorem 3.1) Theorem 3.3 are satisfied, then* $(\bar{x}, \tau, \xi)$ *is a weakly efficient solution (an efficient solution) of a maximum type in the problem (WD_E).*

In other words, if $E(\bar{x}) \in D$ is a weakly *E*-efficient solution (an *E*-efficient solution) of the problem (VP), then $(\bar{x}, \tau, \xi)$ is a weakly efficient solution (an efficient solution) of a maximum type in the dual problem (WD_E).

Proof : Since $\bar{x} \in D_E$ is a weakly efficient solution of the problem (VP_E) and the Guignard constraint qualification [3] is satisfied at $\bar{x}$, there exist $\tau \in R^p$, $\xi \in R^m$, $\xi \geqq 0$ such that

$$\sum_{i=1}^{p} \tau_i \nabla (f_i \circ E)(\bar{x}) + \sum_{t=1}^{m} \xi_t \nabla (g_t \circ E)(\bar{x}) = 0,$$

$$\xi_t (g_t \circ E)(\bar{x}) = 0, t \in T,$$

$$\tau \geq 0, \xi \geqq 0.$$

Thus, $(\bar{x}, \tau, \xi)$ is a feasible solution for (WD_E). If $(\bar{x}, \tau, \xi)$ is not a weakly efficient solution (an efficient solution) for (WD_E), then there exists a feasible solution $(\tilde{x}, \tilde{\tau}, \tilde{\xi})$ of (WD_E) such that $f(E(\tilde{x})) < f(E(\bar{x}))$, which contradicts the Theorem 3.1. Hence $(\bar{x}, \tau, \xi)$ is a weakly efficient solution (an efficient solution) for (WD_E).

In addition, according to Lemma 2.5, we have $E(\bar{x}) \in D$. Since $\bar{x} \in D_E$ is an efficient solution of the problem (VP_E), Lemma 2.7 implies that $E(\bar{x})$ is a weakly *E*-efficient solution in the problem (VP). Then, through the strong duality between (VP_E) and (WD_E), we conclude that the strong *E*-duality also holds between the problems (VP) and (WD_E). This implies that if $E(\bar{x}) \in D$ is a weakly *E*-efficient solution of the problem (VP), there exist $\tau \in R^p$, $\xi \in R^m$, $\xi \geqq 0$ such that $(\bar{x}, \tau, \xi)$ is a weakly efficient solution of a maximum type in the problem (WD_E). □

4. Mixed *E*-duality

In this section, for the considered *E*-differentiable multiobjective programming problem (VP), we define the following dual problem in the sense of mixed. Further, under *E*-type I functions, we prove *E*-duality results between the multiobjective programming problem (VP) and its vector *E*-dual problem in the sense of mixed.

Let the index set T be divided into two different subsets, T_1 and T_2, so that $T = T_1 \cup T_2$. Let T_1 be a subset of T and $T_2 = T \setminus T_1$, $\xi_{T_K} g_{T_K} = \sum_{t\in T_k} \xi_t g_t$, $k = 1,2$ and, in addition, the cardinality of index sets T_1 and T_2, respectively, is denoted by $|T_1|$ and $|T_2|$. Let

$$D^2 \ = \ \{x \in R^n : g_t(x) \leqq 0, t \in T_2\}.$$

Now, for the problem (VP), we introduce the definition of the scalar Lagrange function $L : D \times R^p_+ \times R^{|T_1|}_+ \to R$ as follows

$$L(x,\tau,\xi_{T_1}) \ := \ \sum_{i=1}^{p} \tau_i f_i(x) + \sum_{t\in T_1} \xi_t g_t(x). \tag{19}$$

Further, let $E : R^n \to R^n$ be a given one-to-one and onto operator. Furthermore, let

$$D_E^2 \ := \ \left\{x \in R^n : (g_t \circ E)(x) \leqq 0, t \in T_2\right\}.$$

Now, we define for problem (VP) its *E*-differentiable vector *E*-dual problem in the sense of mixed as follows:

$$\begin{aligned} &\text{maximize } f(E(y)) + \left[\xi_{T_1}(g_{T_1} \circ E)(y)\right] e \\ &\text{subject to } \tau \nabla f(E(y)) + \xi \nabla g(E(y)) = 0, \\ &\qquad \xi_{T_2}(g_{T_2} \circ E)(y) \geqq 0, \\ &\qquad \tau \in R^p, \tau \geq 0, \tau e = 1, \xi \in R^m, \xi \geqq 0. \end{aligned} \qquad (\text{MVD}_E)$$

Let Ψ_E denote the set of all feasible solutions of (MVD_E), that is,

$$\Psi_E = \left\{ (y,\tau,\xi) \in R^n \times R^p \times R^q : \tau \nabla f(E(y)) + \xi \nabla g(E(y)) = 0, \right.$$

$$\left. \sum_{t\in T_2} \xi_t (g_t \circ E)(y) \geqq 0, \tau \geq 0, \tau e = 1, \xi \geqq 0 \right\}.$$

Further, let $Y_E = \{y \in R^n : (y,\tau,\xi) \in \Psi_E\}$. Note that we get a vector Wolfe E-dual problem [4] for $T_1 = \varnothing$ and a vector Mond-Weir E-dual problem [1], [5] for $T_2 = \varnothing$ in the vector mixed E-dual problem (MVD$_E$), respectively.

Theorem 4.1 : *(Mixed weak duality between (VP$_E$) and (MVD$_E$)). Let* $x \in D_E$ *and* $(y,\tau,\xi) \in \Psi_E$ *such that* (f,g) *is E-type I at y. Then*

$$f(E(x)) \not< f(E(y)) + [\xi_{T_1}(g_{T_1} \circ E)(y)]e. \tag{20}$$

Proof : Let $x \in D_E$ and $(y,\tau,\xi) \in \Psi_E$. By means of contradiction, suppose that

$$(f \circ E)(x) < (f \circ E)(y) + \left[\xi_{T_1}(g_{T_1} \circ E)(y)\right]e.$$

Thus,

$$(f_i \circ E)(x) < (f_i \circ E)(y) + \left[\sum_{t \in T_1} \xi_t (g_t \circ E)(y)\right], i \in I. \tag{21}$$

Multiplying (21) by τ_i and then adding both sides of the resulting inequalities, we get

$$\sum_{i=1}^{p} \tau_i (f_i \circ E)(x) < \sum_{i=1}^{p} \tau_i (f_i \circ E)(y) + \left[\sum_{t \in T_1} \xi_t (g_t \circ E)(y)\right] \sum_{i=1}^{p} \tau_i .$$

Since $\sum_{i=1}^{p} \tau_i = 1$, , the following inequality

$$\sum_{i=1}^{p} \tau_i (f_i \circ E)(x) < \sum_{i=1}^{p} \tau_i (f_i \circ E)(y) + \sum_{t \in T_1} \xi_t (g_t \circ E)(y)$$

holds. By $x \in D_E$ and $(y,\tau,\xi) \in \Psi_E$, we have

$$\sum_{i=1}^{p} \tau_i (f_i \circ E)(x) + \sum_{t \in T_1} \xi_t (g_t \circ E)(x) < \sum_{i=1}^{p} \tau_i (f_i \circ E)(y) + \sum_{t \in T_1} \xi_t (g_t \circ E)(y), \tag{22}$$

$$\sum_{i=1}^{p} \tau_i (f_i \circ E)(x) < \sum_{i=1}^{p} \tau_i (f_i \circ E)(y), \tag{23}$$

$$\sum_{t \in T_2} \xi_t (g_t \circ E)(x) \leqq \sum_{t \in T_2} \xi_t (g_t \circ E)(y), \tag{24}$$

Since (f,g) is E-type I with respect to η at y and by Definition 2.8, the following inequalities

$$f_i(E(x)) - f_i(E(y)) \geqq \nabla f_i(E(y))\eta(E(x),E(y)), \quad i \in I \tag{25}$$

$$0 = -g_t(E(y)) \geqq \nabla g_t(E(y))\eta(E(x),E(y)). \quad t \in T(E(y)) \tag{26}$$

hold, respectively. By $\tau_i > 0,\ i = 1,2,..,p,\ \xi_t \geqq 0,\ t \in T(E(y)),$ using above inequalities, we obtain that the inequality

$$\sum_{i=1}^{p}\tau_i f_i(E(x)) - \sum_{i=1}^{p}\tau_i f_i(E(y)) \geqq \left[\sum_{i=1}^{p}\tau_i \nabla f_i(E(y)) + \sum_{t\in T(E(y))}\xi_t \nabla g_t(E(y))\right]\eta(E(x),E(y)) \tag{27}$$

holds. Thus, by the first constraint of (MVD$_E$), it follows that the following inequality

$$\sum_{i=1}^{p}\tau_i f_i(E(x)) \geqq \sum_{i=1}^{p}\tau_i f_i(E(y)) \tag{28}$$

holds, contradicting (23). Thus, the proof of this theorem is completed.

Theorem 4.2 : *(Mixed weak E-duality between (VP) and (MVD$_E$)). Let $E(x) \in D$ and $(y,\tau,\xi) \in \Psi_E$. In addition, assume that all hypotheses of Theorem 4.1 are satisfied. Then, mixed weak E-duality between (VP) and (MVD$_E$) holds, that is,*

$$f(E(x)) \not< f(E(y)) + \left[\xi_{T_1}(g_{T_1} \circ E)(y)\right]e.$$

Theorem 4.3 : *(Mixed strong duality between (VP$_E$) and (MVD$_E$) and also strong E-duality between (VP) and (MVD$_E$)). Let $\overline{x} \in D_E$ be a weakly efficient solution (an efficient solution) of the problem (VP$_E$) (and, thus, $E(\overline{x}) \in D$ be a weakly E-efficient solution (an E-efficient solution) of the problem (VP)). Further, assume that the Guignard constraint qualification [3] be satisfied at $\overline{x}$. Then there exist $\tau \in R^p$, $\xi \in R^m$, $\xi \geqq 0$ such that $(\overline{x},\tau,\xi) \in \Psi_E$. If all hypotheses of (Theorem 4.1) are satisfied, then $(\overline{x},\tau,\xi)$ is a weakly efficient solution (an efficient solution) of a maximum type in the problem (MVD$_E$).*

In other words, if $E(\overline{x}) \in D$ is a weakly E-efficient solution (an E-efficient solution) of the problem (VP), then $(\overline{x},\tau,\xi)$ is a weakly efficient solution (an efficient solution) of a maximum type in the dual problem (MVD$_E$).

5. Concluding remarks

In this paper, new classes of E-differentiable multiobjective programming problems have been considered. For such E-differentiable multiobjective programming problems, the so-called vector Wolfe and mixed E-dual problems have been formulated. Then, under E-type I functions, various E-duality theorems have been proven between the considered E-differentiable vector optimization problem and its vector Wolfe and mixed E-duals. Vector Wolfe type and vector Mond-Weir type E-duals are special cases of a vector mixed type E-dual.

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