Journal of Mathematical Analysis
ISSN: 2217-3412, URL: http://ilirias.com/jma
Volume 9 Issue 4 (2018), Pages 44-52.

TOPOLOGICAL SPACES ASSOCIATED WITH SIMPLE GRAPHS

ADEM KILICMAN, KHALID ABDULKALEK

Abstract. The aim of this article is to associate a topology with a set of vertices for any simple graph (finite or infinite), called incidence topology. A subbasis family to generate the topology is introduced. Some properties of this topology were investigated and were shown that this topology satisfies the property of Alexandroff, i.e. the intersection of each collection of open sets is open. Giving a fundamental step toward studying some properties of simple graphs by their corresponding topology is our motivation. Furthermore, a comparison between two different subbases to generate a topology is presented.

1. Introduction

Graph theory is a prominent mathematical tool in many subjects [9] and it is considered as a substantial structure in discrete mathematics for two reasons. First, graphs are mathematically elegant from theoretical viewpoint. Even though graphs are simple relational combinations, they can be used to represent topological spaces, combinatorial objects and many other mathematical combinations. Many concepts will be very useful from practical perspective when they are abstractly represented by graphs and this is the second reason [11]. Topology is an interesting and important field of mathematics because it is a powerful tool that leading to such beneficial concepts as connectivity, continuity, and homotopy. Its influence in most other branches of mathematics is evident [6].

Topologizing discrete structures is a problem that many publications concerned with. One of these discrete structures is graph theory. The investigation of topology on graphs is inspired by the representation of the digital image using a graph model; the points of the image and the connectivity between them are represented by the vertices and the edges of the graph respectively. Therefore, topological properties of the digital images can be studied through topologies on the vertices of graphs [5].In 2013, S. M. Amiri et al. [3] associated a topology with the vertex set V of any simple locally finite graph $G = (V, E)$, called graphic topology. They introduced a subbasis family S_G=$\{A_x|x \in V\}$ to generate the topology on V such that A_x represents all adjacent vertices of x and they showed that this topology is an Alexandroff space. In [1], a new approach of applying the topology on digraphs associated topologies with set of edges of directed graph, which are known as compatible and incompatible edge topologies, and some related properties in bitopological spaces, see [2].

The previous work of topology on graphs was associated with the vertex set of a locally finite graph, a graph in which every vertex is adjacent with finite number of edges. Therefore, this topology is not appropriate to be associated with simple graphs that have vertices of infinite degree. So, our target is to generalize the topology on the vertex set of any simple graph (not only locally finite graph) by introducing a new subbasis family to generate the topology and present a fundamental step toward studying some properties of simple graphs by their corresponding topology. Furthermore, a comparison in some results between graphic topology and our new model (incidence topology) are presented. In this paper, we defined a topology on the

2010 *Mathematics Subject Classification.* 05C99, 54A05.
Key words and phrases. simple graphs; isomorphic graphs; Alexandroff topology.

Submitted June 6, 2018. Published August 6, 2018.
Communicated by Siraj Uddin.

set of vertices for any simple graph. Properties of incidence topology and the relation between this topology and corresponding graphs is presented. In section 2 some definitions of graph theory and topology is shown. Subbasis to generate the topology on the set of vertices of simple graphs is introduced. Section 3 is dedicated to some introductory results of incidence topology. More properties of incidence topology are presented in section 4. Section 5 is to show the investigation of some stipulations on topological spaces to be incidence topology. The subject of section 6 is the connectivity of incidence topology. The last section of the article is devoted to dense subsets of incidence topology.

2. Preliminaries

In this part, some basic notions of graph theory [9, 11] and topology [8] are presented. Furthermore, introduce a subbasis to generate a topology on the set of vertices for any simple graph.

A graph G consist of a non-empty set $V(G)$ of nodes (or vertices), and a set $E(G)$ of arcs (or edges). Usually the graph is denoted by $G = (V, E)$. If v and u are vertices and e is an edge such that $e = vu$, then e is said to join v and u; each vertex (v and u) is incident with e. If there is no edges incident with a vertex v, then v is called isolated vertex. The number of the edges $e \in E$ such that v incident with e is called the degree of the vertex v and denoted by $d(v)$. If the number of the vertices also the number of the edges in a graph G is finite, then G is finite graph; otherwise it is an infinite graph. If any vertex can be reached from any other vertex in a graph G by travelling along the edges, then G is called connected graph and disconnected otherwise.

A topology T on a set A is a combination of subsets of A, called open, such that the union of the members of any subset of T is a member of T, the intersection of the members of any finite subset of T is a member of T, and both empty set and A are in T. The ordered pair (A, T) is called a topological space. The topology $T = P(A)$ on A is called discrete topology while the topology $T = \{A, \phi\}$ on A is called trivial (or indiscrete) topology. A topology in which arbitrary intersection of open set is open called Alexandroff space.

Now, we introduce our new subbasis family to generate a topology on the set of vertices V of a simple graph $G = (V, E)$ without isolated vertex. Let I_e be the incident vertices with the edge e. Define S_{IG} as follows: $S_{IG} = \{I_e | e \in E\}$.
Since there is no isolated vertex in G, we have $V = \bigcup_{e \in E} I_e$. Hence S_{IG} forms a subbasis for a topology T_{IG} on V, called incidence topology of G.

Example 2.1. *Let $G = (V, E)$ be a simple graph as in Figure (1) such that $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, e_3\}$.*
We have

$$I_{e_1} = \{v_1, v_2\}, \quad I_{e_2} = \{v_2, v_3\}, \quad I_{e_3} = \{v_3, v_4\}.$$

By taking finitely intersection the basis obtained is:

$$\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_2\}, \{v_3\}, \phi\}$$

Then by taking all unions the topology can be written as:

$$T_{IG} = \{\phi, V, \{v_2\}, \{v_3\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}\}.$$

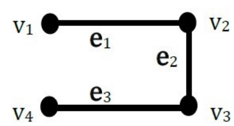

Figure 1.

It is obvious that, the incidence topologies of the cycle $C_n; n \geq 3$, the complete graph $K_n; n \geq 3$ and the complete bipartite graph $K_{n,m}; n, m > 1$ are discrete since each vertex incident with at least two edges, then the finitely intersection of the subbasis give all singleton subsets of V and this basis generates the discrete topology, but the incidence topology of P_n is not discrete because P_n contains two vertices incident with one edge is not open. In addition, the incidence topology of every simple graph differs from graphic topology, but in the graphs C_n and $K_n; n \geq 3$, both are discrete topologies.

3. Introductory Results

In this part, we present some preliminary results and show that the incidence topology is an Alexandroff space as the graphic topology.

Proposition 3.1. *Suppose that T_{IG} is the incidence topology of the graph $G = (V, E)$. If $d(v) \geq 2$, then $\{v\} \in T_{IG}$ for every $v \in V$.*

Proof. Since G is a simple graph and for any degree of v, we have $\bigcap_{i=2}^{\infty} I_{e_i} = \{v\}$ such that $v \in I_{e_i}$ for all $i = 2, 3, ...$Now by the definition of T_{IG}, $\{v\}$ is an element in the basis of T_{IG}. Hence $\{v\} \in T_{IG}$. □

The following corollary is a trivial result for the previous proposition.

Corollary 3.2. *Let $G = (V, E)$ be a graph. If $d(v) \geq 2$ for all $v \in V$, then T_{IG} is a discrete topology.*

Proposition 3.3. *The topological space (V, T_{IG}) of a graph $G = (V, E)$ satisfies the property of Alexandroff.*

Proof. It is adequate to show that arbitrary intersection of elements of S_{IG} is open. Let $A \subseteq E$. There are two cases:

(i). If G has only one edge, then $A = E = \{e\}$ and $\bigcap_{e \in A} I_e = I_e \in T_{IG}$ from definition of T_{IG}. Hence $\bigcap_{e \in A} I_e$ is open.

(ii). If G has at least two edges, then either $\bigcap_{e \in A} I_e = \phi$ is open or $\bigcap_{e \in A} I_e = \{v\}$ such that $v \in I_e$ for all $e \in A$. This means v is incident with at least two edges. Then by proposition (3.1), $\{v\} \in T_{IG}$. Hence $\bigcap_{e \in A} I_e$ is open.

□

In any graph $G = (V, E)$ since (V, T_{IG}) is Alexandroff space, for each $v \in V$, the intersection of all open sets containing v is the smallest open set containing v and denoted by U_v. Also the family $M_G = \{U_v | v \in V\}$ is the minimal basis for the topological space (V, T_{IG}) (see[10]).

Remark. *Let $G = (V, E)$ be a graph, then I_v is the set of all edges incident with the vertex v.*

Proposition 3.4. *In any graph $G = (V, E)$, $U_v = \bigcap_{e \in I_v} I_e$ for every $v \in V$.*

Proof. Since S_{IG} is the subbasis of T_{IG} and U_v is the intersection of all open set containing v, we have $U_v = \bigcap_{e \in A} I_e$ for some subset A of E. This leads to $v \in I_e$ for each $e \in A$. Therefore, $e \in I_v$ for all $e \in A$. Hence $A \subseteq I_v$ and so $v \in \bigcap_{e \in I_v} I_e \subseteq U_v$. From the definition of U_v the proof is complete. □

Remark. *Let $G = (V, E)$ be a graph. For any $v \in V$,*

(i). *If $d(v) \geq 2$, then by proposition (3.4), $U_v = \bigcap_{i=2}^{\infty} I_{e_i}$ such that $e_i \in I_v$ for all $i = 2, 3, ...$ Since G is simple graph, $\bigcap_{i=2}^{\infty} I_{e_i} = \{v\}$ for all $i = 2, 3, ...$ Hence $U_v = \{v\}$.*

(ii). *If* $d(v)=1$*, then by proposition (3.4),* $U_v = \bigcap_{e\in I_v} I_e = I_e$ *since* I_e *is the only open set in the subbasis that containing* v*.*

Corollary 3.5. *For any* $u, v \in V$ *in a graph* $G=(V,E)$*, we have* $u \in U_v$ *if and only if* $I_v \subseteq I_u$*. Equivalently* $U_v = \{u \in V | I_v \subseteq I_u\}$*.*

Proof. By proposition (3.4), $U_v = \bigcap_{e\in I_v} I_e$. Therefore,

$$u \in U_v \Longleftrightarrow u \in \bigcap_{e\in I_v} I_e \Longleftrightarrow u$$

incident with e for all $e \in I_v \Longleftrightarrow e \in I_u$ for all $e \in I_v \Longleftrightarrow I_v \subseteq I_u$. ☐

Remark. *By corollary (3.5), the topological space* (V, T_{IG}) *in any graph* $G=(V,E)$ *is discrete if and only if* $I_u \nsubseteq I_v$ *and* $I_v \nsubseteq I_u$ *for every distinct pair of vertices* $u, v \in V$*.*

Remark. *From ([7]) the Alexandroff topological space* (X,T) *is* T_1 *if and only if* $U_x = \{x\}$*. It follows that* (X,T) *is discrete. Therefore, the incidence topology* (V, T_{IG}) *which is an Alexandroff space is* T_1 *if and only if it is discrete. Now from ([4]) we have, if* (V, T_{IG}) *is an Alexandroff space, then* (V, T_{IG}) *is* T_0 *if and only if* $U_u = U_v$ *implies* $u=v$*. This means* $U_u \neq U_v$ *for all distinct pair of vertices* $u, v \in V$*. Then from corollary (3.5), the incidence topology is* T_0 *if and only if* $I_u \neq I_v$ *for every distinct pair of vertices* $u, v \in V$*.*

Corollary 3.6. *Let* $G=(V,E)$ *be a graph. For every* $v \in V$ *we have,* $U_v \subseteq I_e$ *for all* $e \in I_v$ *and so* $\overline{U_v} \subseteq \overline{I_e}$ *for all* $e \in I_v$*.*

Proof. From proposition (3.4), $U_v = \bigcap_{e\in I_v} I_e$ for every $v \in V$. Therefore, $U_v \subseteq I_e$ for all $e \in I_v$. Now to prove $\overline{U_v} \subseteq \overline{I_e}$ for all $e \in I_v$, let $u \in \overline{U_v}$ this implies $U \bigcap U_v \neq \phi$ for all open set U containing u. Since $U_v \subseteq I_e$, this implies $U \bigcap I_e \neq \phi$ for all open set U containing u. Hence $u \in \overline{I_e}$ and so $\overline{U_v} \subseteq \overline{I_e}$ for all $e \in I_v$. ☐

Corollary 3.7. *Given a graph* $G=(V,E)$*. For every* $v \in V, \overline{\{v\}} \subseteq \overline{U_v} \subseteq \overline{I_e}$ *for all* $e \in I_v$*.*

Proof. Let $u \in \overline{\{v\}}$, this implies $U \bigcap \{v\} \neq \phi$ for all open set U containing u. Since $\{v\} \subseteq U_v$ this implies $U \bigcap U_v \neq \phi$ for all open set U containing u. Hence $u \in \overline{U_v}$ and so $\overline{\{v\}} \subseteq \overline{U_v}$. Also by corollary (3.6), $\overline{\{v\}} \subseteq \overline{U_v} \subseteq \overline{I_e}$ for all $e \in I_v$. ☐

Corollary 3.8. *For any* $u, v \in V$ *in a graph* $G=(V,E)$ *we have,* $u \in \overline{\{v\}}$ *if and only if* $I_u \subseteq I_v$*.*

Proof. $u \in \overline{\{v\}} \Longleftrightarrow U \bigcap \{v\} \neq \phi$ for all open set U containing $u \Longleftrightarrow v \in U_u \Longleftrightarrow I_u \subseteq I_v$ by corollary (3.5). ☐

4. Properties of Incidence topology

Proposition 4.1. *Let* T_{IG} *be the incidence topology of the graph* $G=(V,E)$*,then we have the following:*

(i). *If* $M = \{v \in V | d(v) \geq 2\}$*, then* $M \in T_{IG}$*.*

(ii). *If* $L = \{v \in V | d(v) = 1\}$*, then* L *is closed in* T_{IG}*.*

Proof. (i). Let $v \in M$. Since $d(v) \geq 2$, then by proposition (3.4), $U_v = \{v\}$. As a result $v \in U_v \subseteq M$ and so v is an interior point of M. Hence $M \in T_{IG}$.

(ii). By assumption $L = \bigcup_{v\in L} \{v\}$ and so

$$\overline{L} = \overline{\bigcup_{v\in L} \{v\}} = \bigcup_{v\in L} \overline{\{v\}}$$

by properties of closure (see[8]). Let $u \in \overline{L}$, then $u \in \overline{\{v\}}$ for some $v \in L$. By corollary (3.8), $I_u \subseteq I_v$. Since $d(v) = 1$, then $I_v = \{e\}$ such that $e \in E$. Therefore, $d(u) = 1$ because $I_u \subseteq I_v$ and so $u \in L$. Hence $\overline{L} \subseteq L$ and the proof is complete.

□

Proposition 4.2. *Let $G = (V, E)$ be a graph. Then (V, T_{IG}) is a compact topological space if and only if V is finite.*

Proof. Let (V, T_{IG}) be a compact topological space. By contradiction, suppose that V is infinite. Then $M_G = \{U_v | v \in V\}$ is an open covering of (V, T_{IG}) which has no finite subcover. Therefore, (V, T_{IG}) is not compact which is a contradiction. For the converse, it follows directly that (V, T_{IG}) is compact since there are only finitely many open subsets on finite space. □

Definition 4.1. *Given a graph $G = (V, E)$, if the number of components of G increases by the removal of a vertex v and all edges incident with it, then v is called a cut-point (or cut-vertex). If $G - C$ has more than one component such that $C \subseteq V(G)$ and G is connected, then C is called a vertex cut. If every proper subset of the vertex cut C of G is not a vertex cut, then C is called a minimal vertex cut (see*[9]*).*

In any graph $G = (V, E)$ (not necessary connected), if v is a cut-vertex, then $d(v) \geq 2$ because the deletion of a vertex of degree one and the edge incident with it does not increases the number of components of G. Consequently by proposition (3.1), $\{v\} \in T_{IG}$.

Proposition 4.3. *Let C be a minimal vertex cut in a connected graph $G = (V, E)$. Then $C \in T_{IG}$.*

Proof. Since C is a minimal vertex cut in G, every vertex $v \in C$ must be adjacent to vertices of at least two different components. Therefore, $d(v) \geq 2$ for all $v \in C$. By proposition (3.4) , $U_v = \{v\}$. As a result $v \in U_v \subseteq C$ and so v is an interior point of C. Hence $C \in T_{IG}$. □

Definition 4.2. *Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic to each other, and written $G_1 \cong G_2$, if there is a bijection $\Psi : V_1 \longrightarrow V_2$ with $\{a, b\} \in E_1$ if and only if $\{\Psi(a), \Psi(b)\} \in E_2$ for all $a, b \in V_1$. The function Ψ is called an isomorphism (see*[11]*).*

Remark. *It is clear that the topological spaces (V_1, T_{IG_1}) and (V_2, T_{IG_2}) are homeomorphic, if the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic but in general the opposite is not true. For example, the incidence topologies of K_n and C_n for $n \geq 4$ are homeomorphic since both are discrete, but they are not isomorphic graphs.*

Remark. *Suppose that (V_1, T_1) and (V_2, T_2) are two topological spaces. Then a map $\Psi : (V_1, T_1) \longrightarrow (V_2, T_2)$ is continuous if and only if $\Psi(\overline{B}) \subseteq \overline{\Psi(B)}$ for every subset B of V_1 and closed if and only if $\overline{\Psi(B)} \subseteq \Psi(\overline{B})$ for every subset B of V_1 (see*[6]*).*

Proposition 4.4. *Let Ψ be a function defined from V_1 to V_2, such that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two undirected graphs. Consider Ψ as a function between (V_1, T_{IG_1}) and (V_2, T_{IG_2}). Then we get the following:*

(i). *For every $u, v \in V_1$, $I_u \subseteq I_v$ implies $I_{\Psi(u)} \subseteq I_{\Psi(v)}$ if and only if the function Ψ is continuous.*

(ii). *For every $u, v \in V_1$, $I_{\Psi(u)} \subseteq I_{\Psi(v)}$ implies $I_u \subseteq I_v$ and Ψ is onto, then Ψ is closed. For the opposite, for every $u, v \in V_1$, $I_{\Psi(u)} \subseteq I_{\Psi(v)}$ implies $I_u \subseteq I_v$ if the function Ψ is closed and one-to-one.*

Proof. (i). From corollary (3.8) we have $I_u \subseteq I_v$ if and only if $u \in \overline{\{v\}}$. Then it is enough to prove that Ψ is continuous if and only if $u \in \overline{\{v\}}$ implies $\Psi(u) \in \overline{\Psi(v)}$ for every $u, v \in V_1$. Suppose that Ψ is continuous and $u \in \overline{\{v\}}$. Then $\Psi(u) \in \Psi(\overline{\{v\}})$. Through continuity of Ψ,$\Psi(\overline{\{v\}}) \subseteq \overline{\{\Psi(v)\}}$ and thus $\Psi(u) \in \overline{\{\Psi(v)\}}$. For the opposite, let B be a subset of V_1 and $u \in \overline{B}$. It is obvious that $B = \bigcup_{v \in B} \{v\}$ and thus

$$\overline{B} = \overline{\bigcup_{v \in B} \{v\}} = \bigcup_{v \in B} \overline{\{v\}}$$

by properties of closure (see[8]). For this reason there exists an element $v \in B$ such that $u \in \overline{\{v\}}$. By the hypothesis $\Psi(u) \in \overline{\{\Psi(v)\}} \subseteq \overline{\Psi(B)}$. Thus $\Psi(\overline{B}) \subseteq \overline{\Psi(B)}$ and so Ψ is continuous.

(ii). Suppose that Ψ is onto and Φ be the right inverse of Ψ. For each $u, v \in V_2$, suppose that $I_u \subseteq I_v$. Hence $I_{\Psi(\Phi(u))} \subseteq I_{\Psi(\Phi(v))}$, because $\Psi \circ \Phi = \mathrm{id}_{V_2}$. By the hypothesis $I_{\Phi(u)} \subseteq I_{\Phi(v)}$. Then by (i) Φ is continuous. Therefore, if $B \subseteq V_1$ and B is closed, we have $\Psi(B) = \Phi^{-1}(B)$ which is closed, because Φ is continuous. Hence Ψ is closed. For the opposite, if Ψ is closed and one-to-one, then Ψ^{-1} is continuous on $\Psi(V_1)$. Therefore, by (i) for all $u, v \in V_1$, $I_{\Psi(u)} \subseteq I_{\Psi(v)}$ implies $I_{\Psi^{-1}(\Psi(u))} \subseteq I_{\Psi^{-1}(\Psi(v))}$ and so $I_u \subseteq I_v$.

□

Corollary 4.5. *By notation of proposition (4.4), Ψ is a homeomorphism if and only if it is bijective and $I_u \subseteq I_v$ if and only if $I_{\Psi(u)} \subseteq I_{\Psi(v)}$ for every $u, v \in V_1$.*

5. Stipulations on Topological Spaces to be Incidence Topology

In this part, we display the requisite conditions for topological spaces to be incidence topology whereas conditions for being graphic topology are considered as an open problem. In addition, a topological property for topological spaces to be incidence topology is introduced as in graphic topology.

Definition 5.1. *Any topological space (A, T) is called incidence topology, if $T = T_{IG}$ for some simple graph G with vertex set A and without isolated vertex.*

Remark. *It is easy to see that the topology $T = \{\phi, A\}$ is the only incidence topology on a set A with two elements. Now suppose that A has at least three elements. Then there are two cases:*

(i). *If T is a discrete topology on A, then by corollary (3.2), $T = T_{IG}$ for some simple graph G with a vertex set A such that $d(v) \geq 2$ for all $v \in V$ and without isolated vertex. Hence T is an incidence topology.*

(ii). *If T is not a discrete topology on A, then from the definition of T_{IG}, all open sets $U_i \in T_{IG}$ for some i that contain two elements are the edges of the simple graph G and form a subbasis for the topology T_{IG}. Therefore, if there exist open sets $U_i \in T_{IG}$ for some i contain two elements such that these open sets are the edges of a simple graph without isolated vertex and form a subbasis for T, then T is an incidence topology on A.*

Example 5.1. *Let $A = \{1, 2, 3, 4\}$, such that*

$$T_1 = \{\phi, A, \{2\}, \{1, 2\}, \{2, 4\}, \{2, 3\}, \{1, 2, 4\}, \{1, 2, 3\}, \{2, 3, 4\}\}$$

and

$$T_2 = \{\phi, A, \{1\}, \{2\}, \{1, 2\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}.$$

According to this example T_1 is an incidence topology since $\{1, 2\}, \{2, 4\}$ and $\{2, 3\}$ are edges of a simple graph without isolated vertex as in Figure 2 and these edges form a subbasis for T_1, but T_2 is not an incidence topology because $\{1, 2\}$ and $\{3, 4\}$ are not a subbasis for T_2.

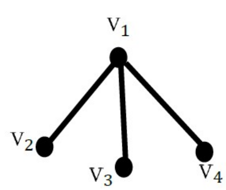

Figure 2.

The next proposition gives the topological property for topological spaces to be an incidence topology.

Proposition 5.1. *Let T_{IG} be the incidence topology of the graph $G = (V, E)$. The topological space (V^*, T) is an incidence topology if it is homeomorphic to (V, T_{IG}).*

Proof. Suppose that $\Psi : (V, T_{IG}) \longrightarrow (V^*, T)$ is a homeomorphism. Since (V, T_{IG}) is an Alexandroff space and $(V, T_{IG}) \cong (V^*, T)$, (V^*, T) is an Alexandroff space. To construct on V^* a graph $G^* = (V^*, E^*)$ we put $\{\Psi(u), \Psi(v)\}$ is an edge in E^* if and only if $\{u, v\}$ is an edge in E for every $u, v \in V$. Then we have $\Psi(\{u, v\}) = \{\Psi(u), \Psi(v)\}$ and so $T = T_{IG^*}$. As a result, $U^*_u = W_u$ such that U^*_u and W_u are the smallest open set containing u in (V^*, T_{IG^*}) and (V^*, T) respectively. Since Ψ is a homeomorphism, $\Psi(U_v) = W_{\Psi(v)}$ such that U_v is the smallest open set containing v in (V, T_{IG}). Also Ψ is an isomorphism between G and G^*, then $\Psi(U_v) = U^*_{\Psi(v)}$. □

6. Connectivity

The sufficient conditions for connectivity of incidence topology are presented in this section while the connectivity of graphic topology remains an open problem. The incidence topology of every graph $G = (V, E)$ such that $d(v) \geq 2$ for all $v \in V$ is disconnected since it is a discrete topology by corollary (3.2).

Definition 6.1. *A topology T_{IG} on a set V is connected if there is no two non-empty open disjoint subsets B_1 and B_2 such that $V = B_1 \bigcup B_2$ (see[8]).*

Proposition 6.1. *The topological space (V, T_{IG}) of every disconnected graph $G = (V, E)$ is disconnected topological space.*

Proof. Suppose that $\{G_i, i \in N\}$ is the set of all components (connected subgraphs) of G such that $G_i = (V_i, E_i)$. For every component $G_i, i \in N$ we have, $\bigcup_{e \in E_i} I_e = V(G_i)$ and I_e is an open set for all $e \in E_i$. As a result, $V(G_i) \in T_{IG}$. Since $[V(G_i)]^c$ in $V(G)$ is the union of vertices of other components, thus $[V(G_i)]^c \in T_{IG}$. Then we have $V(G) = V(G_i) \bigcup [V(G_i)]^c$ for every $i \in N$ and so $V(G)$ is the union of two non-empty open disjoint subsets. Hence (V, T_{IG}) is disconnected. □

Now, suppose that $G = (V, E)$ is a connected graph and T_{IG} is not a discrete topology, i.e. G contains at least one vertex $v \in V$ such that $d(v) = 1$. It is easy to see that the topological space (V, T_{IG}) of G such that $n(V) < 4$ is a connected topological space.

The next proposition gives the sufficient condition for connectedness of the incidence topology of a connected graph such that T_{IG} is not a discrete topology and $n(V) \geq 4$.

Proposition 6.2. *Let $G = (V, E)$ be a connected graph such that T_{IG} is not a discrete topology and $n(V) \geq 4$. Then (V, T_{IG}) is a connected topological space if and only if $I_e \bigcap I_f \neq \phi$ for every pair of edges $e, f \in E$.*

Proof. (V, T_{IG}) is a connected topological space if and only if V is not the union $V = B_1 \bigcup B_2$ of two non-empty open disjoint subsets B_1 and B_2 if and only if by the definition of T_{IG}, there are no open subsets in the subbasis of T_{IG} such that

$$\left(\bigcup_{g \in E} I_g\right) \bigcup \left(\bigcup_{h \in E} I_h\right) = V \text{ and } \left(\bigcup_{g \in E} I_g\right) \bigcap \left(\bigcup_{h \in E} I_h\right) = \phi$$

for some $g, h \in E$ if and only if $I_e \bigcap I_f \neq \phi$ for every distinct pair of edges $e, f \in E$. □

Remark. *By notation of proposition (6.2), the edges of the connected graph $G = (V, E)$ that has a connected incidence topology T_{IG} are the edges that join one vertex of the graph to other vertices in the graph. For example, let $V = \{v_1, v_2, v_3, v_4\}$. Then the graphs G_1, G_2, G_3 and G_4 in Figure 3 are connected graphs with connected incidence topologies. The incidence topology of G_1 is*

$$T_{IG_1} = \{\phi, V, \{v_1\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}\}$$

and similar for other graphs. Also all graphs in Figure 3 are isomorphic graphs.

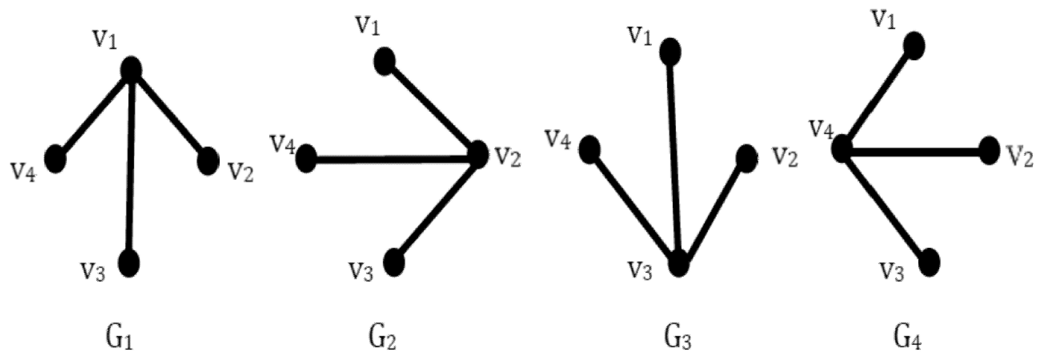

Figure 3.

7. Density in Incidence Topology

Some necessary conditions for dense subsets of incidence topology associated to simple graphs are investigated in this part. The only dense subset in (V, T_{IG}) of every graph $G = (V, E)$ such that $d(v) \geq 2$ for all $v \in V$ is V since T_{IG} is a discrete topology. Also any non-empty subset in (V, T_{IG}) of a graph G with two vertices is dense since T_{IG} is a trivial topology and I_e in S_{IG} is dense because $I_e = V$.

Remark. *It is Known that in (V, T_{IG}) the subset $M \subseteq V$ is dense in V if and only if the complement of M has empty interior (see[6]).*

Proposition 7.1. *Let $G = (V, E)$ be a connected graph with at least one vertex $v \in V$ such that $d(v) = 1$ and $n(V) \geq 3$. The set $M = \{v \in V | d(v) \geq 2\}$ is dense in (V, T_{IG}).*

Proof. By previous remark, it is enough to prove that the complement of M has empty interior. For every $v \in M^c$, v is a vertex such that $d(v) = 1$. Therefore, $I_e \bigcap I_f \neq \{v\}$ for every $e, f \in E$, and any two distinct vertices in M^c are not adjacent. As a result, for every $B \subseteq M^c$, B cannot be written as a union of finitely intersection of elements of S_{IG}, i,e. $B \notin T_{IG}$. Hence $\mathrm{Int}(M^c = \phi)$ and this means M is dense subset in (V, T_{IG}). □

Corollary 7.2. *Let $G = (V, E)$ be a connected graph such that T_{IG} is not a discrete topology and $n(V) \geq 3$. Then a subset B of V is dense in (V, T_{IG}) if and only if $M \subseteq B$ such that $M = \{v \in V | d(v) \geq 2\}$.*

Proof. $(\Longrightarrow)$ If B is dense in (V, T_{IG}), then by previous remark, B^c has empty interior. By proposition (3.1), $\{v\} \in T_{IG}$ for every $v \in M$ and so $M \in T_{IG}$. Hence $M \subseteq B$ because B^c has empty interior.

$(\Longleftarrow)$ By proposition (7.1), $\overline{M} = V$. From assumption, $M \subseteq B$. Hence $\overline{B} = V$ and so B is dense in (V, T_{IG}). □

Remark. *Let $G = (V, E)$ be a connected graph such that T_{IG} is not a discrete topology and $n(V) \geq 3$. By corollary (7.2), I_e in S_{IG} is dense subset of V for every $e \in E$ if and only if $M \subseteq I_e$ such that $M = \{v \in V | d(v) \geq 2\}$.*

Proposition 7.3. *Let $M \subseteq V$ in a finite connected graph $G = (V, E)$. If $\overline{M} = V$, then the induced subgraph on M is connected.*

Proof. Let K be the induced subgraph on M. If K is not connected, then there is a minimal cut set $S \subseteq V \setminus M$. By proposition (4.3), S is open. However, $S \bigcap M = \phi$ which is a contradiction with density of M. □

8. Conclusion

A synthesis between graph theory and topology has been made. A topology with the set of vertices for any simple graph (finite or infinite) has been associated, called incidence topology. The study of some properties of this topology has been presented in details. It has been shown that this topology is an Alexandroff topology. Useful comparisons of some results between graphic topology and incidence topology have been introduced. Therefore, this article can be considered as a starting point of studying another topological concept on graphs, which could lead to significant applications in the future.

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

References

[1] K. A. Abdu, A. Kılıçman, *Topologies on the edges set of directed graphs*, International Journal of Mathematical Analysis, **12**(2), (2018), 71–84.
[2] K. A. Abdu, A. Kılıçman, *Bitopological spaces on undirected graphs*, J. Math. Computer Sci., 18 (2018), 232-241.
[3] S. M. Amiri, A. Jafarzadeh, H. Khatibzadeh, *An Alexandroff topology on graphs*, Bulletin of the Iranian Mathematical Society. **39**(4) (2013), 647-662.
[4] F. J. Arenas, *Alexandroff spaces*, Acta Math. Univ. Comenian. **68**(1) (1999), 17-25.
[5] A. Bretto, *Digital topologies on graphs.* In Applied Graph Theory in Computer Vision and Pattern Recognition. (2007), 65-82.
[6] J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, (1966).
[7] A. El-Fattah El-Atik, M. E. Abd El-Monsef, E. I. Lashin, *On finite T_0 topological spaces*, Proceedings of the Ninth Prague Topological Symposium, Topol. Atlas, North Bay, Prague. (2001), 75-90.
[8] J. M. Moller, *General topology*, Authors' Notes, (2007).
[9] S. S. Ray, *Graph Theory with Algorithms and its Applications: In Applied Sciences and Technology*, Springer, New Delhi, (2013).
[10] T. Speer, *A short study of Alexandroff space*, arXiv preprint arXiv:0708.2136, (2007).
[11] C. Vasudev, *Graph theory with applications*, New Age International (P) Ltd, New Delhi, (2006).

Adem Kilicman
Department of Mathematics, Universiti Putra Malaysia (UPM), 4300 Serdang, Selangor, Malaysia
E-mail address: akilic@upm.edu.my

Khalid Abdulkalek
Department of Mathematics, Universiti Putra Malaysia (UPM), 4300 Serdang, Selangor, Malaysia.
Department of Accounting, Al-Iraqia University, Adhmia, Baghdad, Iraq
E-mail address: khalid74aso@yahoo.co.uk