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A Note On q-Integral Operators

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Abstract

In this article, a q-integral operator which is analogue to the well known Bernardi integral operator is investigated. Integral preserving property for a subclass of analytic functions defined by this q-operator is proved. Moreover, special new q-integral operators are obtained as consequences.

Keywords: q-integral operators, analytic functions.

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1 Introduction

Let A denote the usual class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (1)$$

which are analytic in the unit disk $\mathbb{U} = \{z \in C : |z| < 1\}$ and normalized with $f(0) = 0$ and $f'(0) - 1 = 0$. Also, we denote the subclass of A consisting of analytic and univalent functions $f(z)$ in the unit disk $\mathbb{U}$ by S.

In [1], [2], Jackson defined the q-derivative operator D_q of a function as follows:

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \quad (z \neq 0, q \neq 0) \qquad (2)$$

and $D_q f(z) = f'(0)$. In case $f(z) = z^k$ for k is a positive integer, the q-derivative of $f(z)$ is given by

$$D_q z^k = \frac{z^k - (zq)^k}{z(1-q)} = [k]_q z^{k-1}.$$

As $q \to 1^-$ and $k \in \mathbb{N}$, we have

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + ... + q^k \to k. \qquad (3)$$

The q-Jackson definite integral of the function f is defined by

$$\int_0^z f(t) d_q t = (1-q)z \sum_{n=1}^{\infty} f(zq^n) q^n, \quad z \in \mathbb{C},$$

provided that the series converges. For a function $h(z) = z^k$, we observe that

$$\int_0^z h(t) d_q t = \int_0^z t^k d_q t = \frac{z^{k+1}}{[k+1]_q} (k \neq -1)$$
$$\lim_{q \to 1} \int_0^z h(t) d_q t = \lim_{q \to 1} \frac{z^{k+1}}{[k+1]_q} = \frac{z^{k+1}}{k+1} = \int_0^z h(t) dt,$$

where $\int_0^z h(t)dt$ is the ordinary integral.

The following q-starlike class of functions has been studied by Aldweby and Darus in [3], [4] and Seoudy and Aouf in [5], [6].

$$\mathcal{S}_q^*(\alpha) = \left\{ f \in A : Re\left(\frac{zD_q(f(z))}{f(z)} \right) > \alpha, z \in \mathbb{U} \right\} \tag{4}$$

2 Main results

Theorem 2.1 *Let the function f be defined by (1) be in the class $\mathcal{S}_q^*(\alpha)$ and μ be a real number such that $\mu > -1$. Then the q-integral operator $F_q^\mu(z)$ defined by*

$$F_q^\mu(z) = \frac{[\mu+1]_q}{z^\mu} \int_0^z t^{\mu-1} f(t) d_q t \quad (\mu > -1), \tag{5}$$

also belongs to the class $\mathcal{S}_q^*(\alpha)$.

Proof. It follows from (5) that

$$\begin{aligned} F_q^\mu(z) &= \frac{[\mu+1]_q}{z^\mu} \int_0^z \left(t^\mu + a_2 t^{\mu+1} + a_3 t^{\mu+2} + \cdots \right) d_q t \\ &= \frac{[\mu+1]_q}{z^\mu} \left[\int_0^z t^\mu d_q t + a_2 \int_0^z t^{\mu+1} d_q t + a_3 \int_0^z t^{\mu+2} d_q t + \cdots \right] \\ &= \frac{[\mu+1]_q}{z^\mu} \left\{ \left[(1-q) z \sum_{m=0}^\infty (zq^m)^\mu q^m \right] + \left[(1-q) z \sum_{m=0}^\infty (zq^m)^{\mu+1} q^m \right] + \cdots \right\} \\ &= \frac{[\mu+1]_q}{z^\mu} \left\{ \left[\frac{1-q}{1-q^{\mu+1}} z^{\mu+1} \right] + \left[\frac{1-q}{1-q^{\mu+2}} a_2 z^{\mu+2} \right] + \cdots \right\} \\ &= z + \sum_{k=2}^\infty \frac{[\mu+1]_q}{[\mu+k]_q} a_k z^k, \end{aligned}$$

then, we have

$$d_k = \frac{[\mu+1]_q}{[\mu+k]_q} a_k \leq a_k \quad (k \geq 2).$$

Hence, $F_q^\mu \in \mathcal{S}_q^*(\alpha)$.

Remark 2.2 *When $q \to 1$ in (5), the q-integral operator $F_q^\mu(z)$ reduces to the well known Bernardi integral operator (see [7])*

For $\mu = 0$, we obtain the following corollary

Corollary 2.3 *Let the function f defined by (1) be in the class $\mathcal{S}_q^*(\alpha)$. Then* *the q-analogue of Alexander integral operator $F_q(z)$ defined by*

$$F_q(z) = \int_0^z \frac{f(t)}{t} d_q t = z + \sum_{k=2}^{\infty} \frac{1}{[k]_q} a_k z^k, \tag{6}$$

also belongs to the class $\mathcal{S}_q^*(\alpha)$.

For $\mu = 1$, we obtain the following corollary

Corollary 2.4 *Let the function f defined by (1) be in the class $\mathcal{S}_q^*(\alpha)$. Then the q-analogue of Libera integral operator $\mathcal{L}_q(z)$ defined by*

$$F_q(z) = \frac{[2]_q}{z} \int_0^z f(t) d_q t = z + \sum_{k=2}^{\infty} \frac{[2]_q}{[k+1]_q} a_k z^k \tag{7}$$

also belongs to the class $\mathcal{S}_q^*(\alpha)$.

Theorem 2.5 *Let F_q^μ be defined by (5). If f of the form (1), $\alpha > 0$ and*

$$Re[D_q(f(z))] \geq \alpha |zD_q^2(f(z))|, \quad \textit{for all} \quad z \in \mathbb{U},$$

then

$$\left| \sum_{i=0}^{\mu} q^i D_q(F(z)) + q^{\mu+1} z D_q^2(F(z)) \right| \geq \alpha \left| \sum_{i=0}^{\mu+1} q^i z D_q^2(F(z)) + q^{\mu+2} z^2 D_q^3(f(z)) \right|.$$

Proof. Since we have

$$\frac{z^\mu}{[\mu+1]_q} F_q(z) = \int_0^z t^{\mu-1} f(t) d_q t$$

By taking the q-derivative, we have

$$\frac{1}{[1+\mu]_q} \left[q^\mu z^\mu D_q(F_q(z)) + [\mu]_q z^{\mu-1} F_q(z) \right] = z^{\mu-1} f(z).$$

This relation equivalent to

$$\frac{[\mu]_q}{[1+\mu]_q} F_q(z) + \frac{q^\mu}{[1+\mu]_q} z D_q(F_q(z)) = f(z),$$

which implies that

$$\frac{[\mu]_q}{[1+\mu]_q} D_q(F_q(z)) + \frac{q^{\mu+1}}{[1+\mu]_q} z D_q^2(F_q(z)) + \frac{q^{\mu}}{[1+\mu]_q} D_q(F_q(z)) = D_q(f(z)),$$

and this is equivalent to

$$D_q(F_q(z)) + \frac{q^{\mu+1}}{[1+\mu]_q} z D_q^2(F_q(z)) = D_q(f(z)).$$

We obtain that

$$\frac{[\mu]_q + q^{\mu}[2]_q}{[1+\mu]_q} D_q^2(F_q(z)) + \frac{q^{\mu+2}}{[1+\mu]_q} z D_q^3(F_q(z)) = D_q^2(f(z)).$$

If $Re(D_q(f(z))) \geq \alpha |z D_q^2 f(z)|$, for all $z \in \mathbb{U}$ this implies that $|D_q(f(z))| \geq \alpha |z D_q^2 f(z)|$. In this relation, we put the expression of $D_q f$ and $D_q^2 f$ obtaining that:

$$\left| D_q(F_q(z)) + \frac{q^{\mu+1}}{[1+\mu]_q} z D_q^2(F_q(z)) \right| \geq \alpha \left| \frac{[\mu]_q + q^{\mu}[2]_q}{[1+\mu]_q} z D_q^2(F_q(z)) + \frac{q^{\mu+2}}{[1+\mu]_q} z^2 D_q^3(F_q(z)) \right|$$

$$\Leftrightarrow \frac{1}{|[1+\mu]_q|} \left| [1+\mu]_q D_q(F_q(z)) + q^{\mu+1} z D_q^2(F_q(z)) \right| \geq \frac{\alpha}{|[1+\mu]_q|} \left| ([\mu]_q + q^{\mu}[2]_q) z D_q^2(F_q(z)) + q^{\mu+2} z^2 D_q^3(F_q(z)) \right|$$

$$\Leftrightarrow \left| [1+\mu]_q D_q(F_q(z)) + q^{\mu+1} z D_q^2(F_q(z)) \right| \geq \alpha \left| ([\mu]_q + q^{\mu}[2]_q) z D_q^2(F_q(z)) + q^{\mu+2} z^2 D_q^3(F_q(z)) \right|$$

Hence, the proof is complete.

3 Summary and Remarks

These are just small part of the results. More problems can be solved such as the Hankel determinant, subordination problems and few other results related to integral operators.

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