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OPTIMALITY CONDITIONS FOR E-DIFFERENTIABLE FRACTIONAL OPTIMIZATION PROBLEMS UNDER E-INVEXITY

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Abstract. In this paper, a new class of (not necessarily) differentiable scalar fractional optimization problems with both inequality and equality constraints is considered in which the functions involved are E-differentiable. The so-called parametric E-Karush-Kuhn-Tucker necessary optimality conditions are established for the considered E-differentiable scalar programming problem. Also, the sufficient E-optimality conditions are derived for such nonsmooth scalar optimization problems under E-invexity hypotheses.

1. Introduction

Fractional optimization problems are frequently encountered in game theory, financial problem, decision theory, and all optimal decision problems with noncomparable criteria. Fractional programming problems have been studied extensively in the literature (see, for example, [8], [9], [10], [11], [12], [13], [16], [19], [20], [21], [22], [23], [24], [25], [26], and others).

During the past decades, various generalizations of convexity were introduced into optimization theory in order to weaken convexity assumptions in proving the fundamental results. One of such important generalizations of the convexity notion is the concept of invexity introduced by Hanson [14] in the case of differentiable scalar optimization problems. Namely, Hanson showed that, instead of the usual convexity assumption, if all functions are assumed to be invex (with respect to the same function η), then the sufficient optimality conditions and Wolfe weak duality can be proved. Classes of nonconvex sets and nonconvex functions, called E-convex sets and E-convex functions, respectively, were introduced and studied by Youness [27]. This kind of generalized convexity is based on the effect of an operator $E : R^n \to R^n$ on the sets and the domains of functions. However, some results and proofs presented by Youness [27] were incorrect as it was pointed out by Yang [28]. Megahed et al. [17] presented the concept of an E-differentiable convex function which transforms a (not necessarily) differentiable convex function to a differentiable function based on the effect of an operator $E : R^n \to R^n$. Recently,

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Abdulaleem [1] introduced a new concept of generalized convexity as a generalization of the notion of E-differentiable E-convexity and the notion of differentiable invexity. Namely, he defined the concept of E-differentiable E-invexity in the case of (not necessarily) differentiable vector optimization problems with E-differentiable functions.

In this paper, we consider a new class of (not necessarily) differentiable scalar fractional programming problems with both inequality and equality constraints in which the involved functions are E-differentiable. For the considered E-differentiable fractional programming problem, its equivalent scalar fractional E-programming problem is defined. Then, we use the Dinkelbach parametric approach for the scalar fractional E-programming problem. Since it is equivalent to the considered E-differentiable scalar fractional programming problem, then, in fact, we use the parametric approach for solving the original fractional optimization problem. Then, we prove the parametric necessary E-optimality conditions for the considered E-differentiable fractional programming problem. Further, under E-invexity assumptions, we also prove sufficient optimality conditions for the considered E-differentiable scalar fractional programming problem.

2. Preliminaries

Definition 2.1. [1] *Let $E : R^n \to R^n$. A set $M \subseteq R^n$ is said to be an E-invex set if and only if there exists a vector-valued function $\eta : M \times M \to R^n$ such that the relation*

$$E(u) + \lambda\eta(E(x), E(u)) \in M$$

holds for all $x, u \in M$ and any $\lambda \in [0, 1]$.

Remark. *If η is a vector-valued function defined by $\eta(z, y) = z - y$, then the definition of an E-invex set reduces to the definition of an E-convex set (see Youness* [27]*).*

Remark. *If $E(a) = a$, then the definition of an E-invex set with respect to the function η reduces to the definition of an invex set with respect to η (see Mohan and Neogy* [18]*).*

Definition 2.2. [17] *Let $E : R^n \to R^n$ and $f : M \to R$ be a (not necessarily) differentiable function at a given point $u \in M$. The function f is said to be an E-differentiable function at u if and only if $f \circ E$ is a differentiable function at u (in the usual sense), that is,*

$$(f \circ E)(x) = (f \circ E)(u) + \nabla(f \circ E)(u)(x - u) + \theta(u, x - u)\|x - u\|, \tag{2.1}$$

where $\theta(u, x - u) \to 0$ as $x \to u$.

Definition 2.3. *Let $E : R^n \to R^n$, $M \subseteq R^n$ be an open E-invex set with respect to the vector-valued function $\eta : R^n \times R^n \to R^n$ and $f : R^n \to R$ be an E-differentiable function on M. The function f is said to be an E-invex function with respect to η if, for all $x \in M$,*

$$f(E(x)) - f(E(u)) \geq \nabla f(E(u))\eta(E(x), E(u)). \tag{2.2}$$

If inequality (2.2) *holds for any $u \in M$, then f is E-invex with respect to η on M.*

Remark. *From Definition 2.3, there are special cases:*

a) *If f is a differentiable function and $E(x) \equiv x$ (E is an identity map), then the definition of an E-invex function reduces to the definition of an invex function introduced by Hanson* [14] *in the scalar case.*
b) *If $\eta : M \times M \to R^n$ is defined by $\eta(x, u) = x - u$, then we obtain the definition of an E-differentiable E-convex function introduced by Megahed et al.* [17].
c) *If f is differentiable, $E(x) = x$ and $\eta(x, u) = x - u$, then the definition of an E-invex function reduces to the definition of a differentiable convex function.*
d) *If f is E-differentiable and $\eta(x, u) = x - u$, then we obtain the definition of a differentiable E-convex function introduced by Youness* [27].

Definition 2.4. *Let $E : R^n \to R^n$, $M \subseteq R^n$ be an open E-invex set with respect to the vector-valued function $\eta : R^n \times R^n \to R^n$ and $f : R^n \to R$ be an E-differentiable function on M. The function f is said to be a strictly E-invex function with respect to η if, for all $x \in M$ with $E(x) \neq E(u)$, the inequalities*

$$f(E(x)) - f(E(u)) > \nabla f(E(u))\eta(E(x), E(u)) \tag{2.3}$$

hold. If inequality (2.3) *is fulfilled for any $u \in M$ ($E(x) \neq E(u)$), then f is strictly E-invex with respect to η on M.*

3. E-optimality conditions for E-differentiable programming

In the paper, we consider the following constrained fractional optimization problem with both inequality and equality constraints:

$$\begin{aligned} &\varphi(x) = \tfrac{f(x)}{q(x)} \quad \to \min \\ &\text{subject to } g_i(x) \leq 0, \ \ i \in I = \{1, ..., k\}, \\ &h_j(x) = 0, \ \ j \in J = \{1, ..., s\}, \end{aligned} \qquad \text{(FP)}$$

where $f : R^n \to R$, $q : R^n \to R$, $g_i : R^n \to R$, $i \in I$, $h_j : R^n \to R$, $j \in J$, are E-differentiable functions on R^n. We will write $g := (g_1, ..., g_k) : R^n \to R^k$ and $h := (h_1, ..., h_s) : R^n \to R^s$ for convenience. Further, we shall assume that $f(x) \geqq 0$, $q(x) > 0$, for all $x \in R^n$. Let $D := \{x \in R^n : g_i(x) \leq 0, \ i \in I, \ h_j(x) = 0, \ j \in J\}$ be the set of all feasible solutions for (FP).

Let $E : R^n \to R^n$ be a given one-to-one and onto operator. Now, for problem (FP), we define its associated scalar fractional programming problem (FP$_E$) as follows:

$$\begin{aligned} &\varphi(E(x)) = \tfrac{f(E(x))}{q(E(x))} \quad \to \min \\ &\text{subject to } g_i(E(x)) \leq 0, \ \ i \in I = \{1, ..., k\}, \\ &h_j(E(x)) = 0, \ \ j \in J = \{1, ..., s\}, \end{aligned} \qquad (\text{FP}_E)$$

where the functions $f \circ E : R^n \to R$, $q \circ E : R^n \to R$, $g_i \circ E : R^n \to R$, $i = 1, 2, ..., k$, $h_j \circ E : R^n \to R$, $j = 1, 2, ..., s$, are differentiable real-valued functions on R^n. We will write $g \circ E := (g_1 \circ E, ..., g_k \circ E) : R^n \to R^k$ and $h \circ E := (h_1 \circ E, ..., h_s \circ E) : R^n \to R^s$ for convenience. Let

$$D_E := \{x \in R^n : (g_i \circ E)(x) \leq 0, \ \ i \in I, \ (h_j \circ E)(x) = 0, \ j \in J\}$$

be the set of all feasible solutions of (FP$_E$). We call (FP$_E$) the scalar fractional E-programming problem.

We now give the result established by Antczak and Abdulaleem [4] which is useful in proving the main results of this paper.

Lemma 3.1. [4] *Let $E : R^n \to R^n$ be an one-to-one and onto and* $D_E = \{x \in R^n : g_i(E(x)) \leq 0,\ i \in I,\ h_j(E(x)) = 0,\ \ j \in J\}$*. Then $E(D_E) = D$.*

Now, we give the definitions of optimal solutions for scalar fractional optimization problems (FP) and (FP$_E$).

Definition 3.2. *Let $\overline{x} \in D_E$. If the inequality $(\varphi \circ E)(\overline{x}) \leq (\varphi \circ E)(x)$ holds for all $x \in D_E$, then $\overline{x}$ is called an optimal solution of the differentiable fractional E-optimization problem (FP$_E$).*

Definition 3.3. *Let $E(\overline{x}) \in D$. If the inequality $(\varphi \circ E)(\overline{x}) \leq (\varphi \circ E)(x)$ holds for all $E(x) \in D$, then $E(\overline{x})$ is called an E-optimal solution of the considered E-differentiable fractional optimization problem (FP).*

Lemma 3.4. *Let $E : R^n \to R^n$ be an one-to-one and onto operator. If $E(\overline{x}) \in D$ is an E-optimal solution of the considered optimization problem (FP), then $\varphi(E(\overline{x})) \leq \varphi(E(x))$ for all $x \in D_E$.*

Proof. Let $E(\overline{x}) \in D$ is an E-optimal solution of problem (FP). Hence, by Lemma 3.1, we have that $\overline{x} \in D_E$. We proceed by contradiction. Suppose, contrary to the result, that there exists $\widetilde{x} \in D_E$ such that $\varphi(E(\widetilde{x})) < \varphi(E(\overline{x}))$. By Lemma 3.1, it follows that $E(\widetilde{x}) \in D$. Thus, the inequality $\varphi(E(\widetilde{x})) < \varphi(E(\overline{x}))$ contradicts E-optimality of $E(\overline{x})$ of problem (FP). □

Lemma 3.5. *Let $\overline{x} \in D_E$ and the inequality $\varphi(E(\overline{x})) \leq \varphi(E(x))$ be satisfied for all $x \in D_E$. Then, $E(\overline{x})$ is an E-optimal of problem (FP).*

Proof. Since $\widetilde{x} \in D_E$, by Lemma 3.1, it follows that $E(\widetilde{x}) \in D$. We proceed by contradiction. Suppose, contrary to the result, that there exists $E(\widetilde{x}) \in D$ such that $\varphi(E(\widetilde{x})) < \varphi(E(\overline{x}))$. Since $\widetilde{x} \in D_E$, by Lemma 3.1, it follows that $E(\widetilde{x}) \in D$. Hence, the inequality $\varphi(E(\widetilde{x})) < \varphi(E(\overline{x}))$ holds for $\widetilde{x} \in D_E$, contradicting the assumption that the inequality $\varphi(E(\overline{x})) \leq \varphi(E(x))$ is satisfied for all $x \in D_E$. □

The following result follows directly from Lemmas 3.4 and 3.5.

Proposition 3.6. *$E(\overline{x}) \in D$ is an E-optimal solution of problem (FP) if and only if $\varphi(E(\overline{x})) \leq \varphi(E(x))$ for all $x \in D_E$.*

Proposition 3.7. *Let $E : R^n \to R^n$ be an one-to-one and onto operator. $\overline{x} \in D_E$ is an optimal solution of (FP$_E$) if and only if $E(\overline{x}) \in D$ is an E-optimal solution of (FP).*

Now, we use the parametric approach introduced by Dinkelbach [11] for solving scalar fractional optimization problems (FP) and (FP$_E$). For the foregoing scalar fractional programming problems (FP) and (FP$_E$), we define their associated non-fractional parametric scalar optimization problem (P$_E^\gamma$) for γ^E as follow

$$\begin{aligned} & f(E(x)) - \gamma^E q(E(x)) \to \min \\ & \text{subject to } g_i(E(x)) \leq 0,\ \ i \in I = \{1, ..., k\}, \\ & \qquad h_j(E(x)) = 0,\ \ j \in J = \{1, ..., s\}, \end{aligned} \qquad (\mathrm{P}_E^{\gamma^E})$$

Note that the set of feasible solutions of the nonfractional parametric scalar optimization problem $(\mathrm{P}_E^{\gamma^E})$ is the same as the set of all feasible solutions of (FP_E).

Using the following lemmas, we can obtain the Karush-Kuhn-Tucker type necessary E-optimality conditions for the considered nonsmooth scalar fractional programming problem (FP).

The following results follows directly from Proposition 3.7.

Lemma 3.8. *$\overline{x}$ is an optimal solution of the scalar fractional E-programming problem (FP_E) if and only if $\overline{x}$ is an optimal solution of the nonfractional parametric scalar optimization problem ($P_E^{\overline{\gamma}^E}$).*

Lemma 3.9. *$E(\overline{x})$ is an E-optimal solution of the considered scalar fractional programming problem (FP) if and only if $\overline{x}$ is an optimal solution of the nonfractional parametric scalar optimization problem ($P_E^{\overline{\gamma}^E}$).*

Subsequently, necessary optimality conditions similar to the well-known Karush-Kuhn-Tucker necessary optimality conditions for nonlinear programming were presented for fractional programming problems (for example, see [7], [16], [29]). One of such optimality criteria are the following parametric Karush-Kuhn-Tucker optimality conditions, which are necessary for optimality of a feasible solution $\overline{x}$ in the scalar optimization problem $(\mathrm{P}_E^{\gamma^E})$. In order to prove the so-called parametric necessary optimality conditions, it can be utilized the equivalence between the problems (FP) and $(\mathrm{P}_E^{\gamma^E})$ (see Proposition 3.7).

Theorem 3.10. *(Parametric necessary optimality conditions for (FP_E)). Let $\overline{x} \in D_E$ be an optimal solution of the scalar fractional E-programming problem (FP_E)). Further, assume that the so-called Guignard constraint qualification (GCQ)* [3] *is satisfied at $\overline{x}$. Then, there exist $\overline{\mu} \in R^k$, $\overline{\xi} \in R^s$ and $\overline{\gamma}^E \in R$ such that the following conditions*

$$\nabla f\left(E(\overline{x}) - \overline{\gamma}^E \nabla q(E(\overline{x}))\right) + \sum_{i=1}^{k} \overline{\mu}_i \nabla g_i(E(\overline{x})) + \sum_{j=1}^{s} \overline{\xi}_j \nabla h_j(E(\overline{x})) = 0, \quad (3.1)$$

$$\overline{\mu}_i g_i(E(\overline{x})) = 0, \quad i = 1, ..., k, \quad (3.2)$$

$$\overline{\mu}_i \geq 0, \ i \in I \quad (3.3)$$

hold.

Proof. Since $\overline{x} \in D_E$ is an optimal solution of the scalar fractional E-programming problem (FP_E)), by Lemma 3.8, $\overline{x} \in D_E$ is also an optimal solution of its associated nonfractional parametric scalar optimization problem (P_E^{γ}). Note that all hypotheses of Theorem 38 [3] are fulfilled. Then, there exist Lagrange multipliers $\overline{\mu} \in R^k$ and $\overline{\xi} \in R^s$ such that

$$\frac{1}{q(E(\overline{x}))}\left[\nabla f(E(\overline{x})) - \overline{\gamma}^E \nabla q(E(\overline{x}))\right] + \sum_{i=1}^{k} \mu_i^* \nabla g_i(E(\overline{x})) + \sum_{J=1}^{s} \xi_j^* \nabla h_j(E(\overline{x})) = 0, \quad (3.4)$$

$$\mu_i^*(g_i \circ E)(\overline{x}) = 0, \ \ i \in I, \quad (3.5)$$

$$\mu^* \geq 0, \quad (3.6)$$

where $\overline{\gamma}^E = \frac{f(E(\overline{x}))}{q(E(\overline{x}))}$. Now, let us denote $\overline{\mu}_i = \frac{\mu_i^*}{\frac{1}{q(E(\overline{x}))}}$, $i \in I$, $\overline{\xi}_j = \frac{\xi_j^*}{\frac{1}{q(E(\overline{x}))}}$, $j \in J$. Note that $\overline{\mu} \geq 0$ and, moreover, the necessary optimality conditions (3.1)-(3.3) are satisfied at $\overline{x}$ with Lagrange multipliers $\overline{\mu} \in R^k$ and $\overline{\xi} \in R^s$. This completes the proof of this theorem. □

Theorem 3.11. *(Parametric necessary E-optimality conditions for (FP)). Let $E(\overline{x})$ be an E-optimal solution of problem (FP). Further, assume that the so-called Guignard constraint qualification (GCQ)* [3] *is satisfied at $\overline{x}$. Then, there exist $\overline{\mu} \in R^k$, $\overline{\xi} \in R^s$ and $\overline{\gamma}^E$ such that the conditions (3.1)-(3.3) are satisfied.*

Remark. *The conditions (3.1)-(3.3) are the parametric Karush-Kuhn-Tucker necessary optimality conditions for the scalar fractional E-programming problem (FP_E). Thus, they are also the necessary E-optimality conditions for the original scalar fractional programming problem (FP). Therefore, we call them the parametric E-Karush-Kuhn-Tucker necessary optimality conditions of problem (FP). Note that, although the functions involved in the scalar fractional programming problem (FP) are not necessarily differentiable at an optimal solution $\overline{x}$ (since they are assumed to be E-differentiable only), we formulate the E-Karush-Kuhn-Tucker necessary optimality conditions for such a nonsmooth extremum problem by using tools for differentiable optimization problems.*

Now, under E-invexity hypotheses, we prove the sufficient E-optimality conditions of problem (FP). First, we prove the sufficient optimality conditions for the scalar fractional E-programming problem (FP_E) and we use them in proving the foregoing sufficient conditions for (FP).

Theorem 3.12. *(Sufficient optimality conditions for (FP_E) and also sufficient optimality conditions for (FP)). Let $\overline{x}$ be a feasible solution of the scalar fractional E-programming problem (FP_E), $\overline{\gamma}^E = \frac{f(E(\overline{x}))}{q(E(\overline{x}))}$, and the necessary optimality conditions (3.1)-(3.3) be satisfied at $\overline{x}$ with Lagrange multipliers $\overline{\mu} \in R^k$ and $\overline{\xi} \in R^s$. Further, assume that the following hypotheses are fulfilled:*

a) *function f, is E-invex with respect to η at $\overline{x}$ on D_E,*
b) *function $-q$, is E-invex with respect to η at $\overline{x}$ on D_E,*
c) *each function g_i, $i \in I_E(\overline{x})$, is E-invex with respect to η at $\overline{x}$ on D_E,*
d) *each function h_j, $j \in J_E^+(\overline{x}) := \{j \in J : \overline{\xi} > 0\}$, is E-invex with respect to η at $\overline{x}$ on D_E,*
e) *each function $-h_j$, $j \in J_E^-(\overline{x}) := \{j \in J : \overline{\xi} < 0\}$, is E-invex with respect to η at $\overline{x}$ on D_E.*

Then $\overline{x}$ is an optimal solution of the scalar fractional E-programming problem (FP_E) and, at the same time, $E(\overline{x})$ is an E-optimal solution of the original scalar fractional programming problem (FP).

Proof. Let the necessary optimality conditions (3.1)-(3.3) be satisfied at $\overline{x} \in D$ with Lagrange multipliers $\overline{\mu} \in R^k$ and $\overline{\xi} \in R^s$. Suppose, contrary to the result, that $\overline{x}$ is not an optimal solution of the problem (FP_E). Hence, by Definition 3.2, there exists $\widetilde{x} \in D_E$ such that

$$\frac{f(E(\widetilde{x}))}{q(E(\widetilde{x}))} \leq \frac{f(E(\overline{x}))}{q(E(\overline{x}))}.$$

The inequality above implies

$$f(E(\widetilde{x})) - \overline{\gamma}^E q(E(\widetilde{x})) \leq f(E(\overline{x})) - \overline{\gamma}^E q(E(\overline{x})). \tag{3.7}$$

Using the assumptions a)-e), we have, by Definition 2.3, that the inequalities

$$f(E(x)) - f(E(\overline{x})) \geq \nabla f(E(\overline{x}))\,\eta(E(x), E(\overline{x})), \tag{3.8}$$

$$-q(E(x)) + q(E(\overline{x})) \geq -\nabla q(E(\overline{x}))\,\eta(E(x), E(\overline{x})), \tag{3.9}$$

$$g_i(E(x)) - g_i(E(\overline{x})) \geq \nabla g_i(E(\overline{x}))\,\eta(E(x), E(\overline{x})),\ i \in I_E(\overline{x}), \tag{3.10}$$

$$h_j(E(x)) - h_j(E(\overline{x})) \geq \nabla h_j(E(\overline{x}))\,\eta(E(x), E(\overline{x})),\ j \in J_E^+(\overline{x}), \tag{3.11}$$

$$-h_j(E(x)) + h_j(E(\overline{x})) \geq -\nabla h_j(E(\overline{x}))\,\eta(E(x), E(\overline{x})),\ j \in J_E^-(\overline{x}) \tag{3.12}$$

hold for all $x \in D_E$. Therefore, they are also satisfied for $x = \widetilde{x} \in D_E$. Thus,

$$f(E(\widetilde{x})) - f(E(\overline{x})) \geq \nabla f(E(\overline{x}))\,\eta(E(\widetilde{x}), E(\overline{x})), \tag{3.13}$$

$$-q(E(\widetilde{x})) + q(E(\overline{x})) \geq -\nabla q(E(\overline{x}))\,\eta(E(\widetilde{x}), E(\overline{x})), \tag{3.14}$$

$$g_i(E(\widetilde{x})) - g_i(E(\overline{x})) \geq \nabla g_i(E(\overline{x}))\,\eta(E(\widetilde{x}), E(\overline{x})),\quad i \in I_E(\overline{x}), \tag{3.15}$$

$$h_j(E(\widetilde{x})) - h_j((E(\overline{x})) \geq \nabla h_j((E(\overline{x}))\,\eta(E(\widetilde{x}), E(\overline{x})),\ j \in J_E^+(\overline{x}), \tag{3.16}$$

$$-h_j(E(\widetilde{x})) + h_j(E(\overline{x})) \geq -\nabla h_j((E(\overline{x}))\,\eta(E(\widetilde{x}), E(\overline{x})),\ j \in J_E^-(\overline{x}). \tag{3.17}$$

We multiple the inequalities above by corresponding Lagrange multipliers and, (3.14) by $\overline{\gamma}^E = \frac{f(E(\overline{x}))}{q(E(\overline{x}))} \geq 0$. After summing the resulting inequalities and taking into account Lagrange multipliers equal to 0, (3.13)-(3.17) yield

$$\begin{gathered} f(E(\widetilde{x})) - \overline{\gamma}^E q(E(\widetilde{x})) + \sum_{i=1}^{k} \overline{\mu}_i g_i(E(\widetilde{x})) + \sum_{j=1}^{s} \xi_j h_j((E(\widetilde{x})) - \\ \left\{ f(E(\overline{x})) - \overline{\gamma}^E q(E(\overline{x})) + \sum_{i=1}^{k} \overline{\mu}_i g_i(E(\overline{x})) + \sum_{j=1}^{s} \xi_j h_j((E(\overline{x})) \right\} \geq \\ \left[\nabla f(E(\overline{x})) - \overline{\gamma}^E \nabla q(E(\overline{x})) + \sum_{i=1}^{k} \overline{\mu}_i \nabla g_i(E(\overline{x})) + \sum_{j=1}^{s} \xi_j \nabla h_j((E(\overline{x})) \right] \eta(E(\widetilde{x}), E(\overline{x})). \end{gathered} \tag{3.18}$$

Hence, by the parametric necessary optimality condition (3.1), (3.18) implies

$$\begin{gathered} f(E(\widetilde{x})) - \overline{\gamma}^E q(E(\widetilde{x})) + \sum_{i=1}^{k} \overline{\mu}_i g_i(E(\widetilde{x})) + \sum_{j=1}^{s} \xi_j h_j((E(\widetilde{x})) \geq \\ f(E(\overline{x})) - \overline{\gamma}^E q(E(\overline{x})) + \sum_{i=1}^{k} \overline{\mu}_i g_i(E(\overline{x})) + \sum_{j=1}^{s} \xi_j h_j((E(\overline{x})). \end{gathered}$$

Using the parametric necessary optimality condition (3.2) together with the feasibility of $\widetilde{x}$ and $\overline{x}$ in the problem (FP$_E$), we have that the inequality

$$f(E(\widetilde{x})) - \overline{\gamma}^E q(E(\widetilde{x})) \geq f(E(\overline{x})) - \overline{\gamma}^E q(E(\overline{x}))$$

holds, contradicting (3.7). The proof in the case of an E-optimal solution is similar and, therefore, it has been omitted in the paper. Thus, the proof of this theorem is completed. □

Now, we give an example of a scalar fractional programming problem involving E-differentiable E-invex functions.

Example 3.13. *Consider the following nondifferentiable fractional programming problem:*

$$\begin{aligned} &\textit{minimize } \tfrac{f(x_1,x_2)}{q(x_1,x_2)} = \tfrac{\sqrt[3]{x_1^2}+\sqrt[3]{x_2^2}+1}{-\sqrt[3]{x_1^2}+\sqrt[3]{x_2}+1} \\ &s.t. \quad g\left(x_1,x_2\right) = -x_1 - \sqrt[3]{x_1^2} + \sqrt[3]{x_2} \leq 0, \\ &\qquad\quad h\left(x_1,x_2\right) = \sqrt[3]{x_1^2} - \sqrt[3]{x_2} = 0. \end{aligned} \tag{FP1}$$

Note that $D = \left\{(x_1,x_2) \in R^2 : -x_1 - \sqrt[3]{x_1^2} + \sqrt[3]{x_2} \leq 0 \wedge \sqrt[3]{x_1^2} - \sqrt[3]{x_2} = 0\right\}$. *Let* $\eta(x_1,x_2) = (\sqrt[3]{x_1^2} + \sqrt[3]{x_2},\ x_1 + \sqrt[3]{x_2})$ *and* $E : R^2 \to R^2$ *be an one-to-one and onto mapping defined by* $E\left(x_1,x_2\right) = \left(x_1^3, x_2^3\right)$. *Now, for problem (FP1), we define its associated differentiable fractional programming problem (FP1$_E$) as follows:*

$$\begin{aligned} &\textit{minimize } \tfrac{f(E(x_1,x_2))}{q(E(x_1,x_2))} = \tfrac{x_1^2+x_2^2+1}{-x_1^2+x_2+1} \\ &s.t. \quad g(E\left(x_1,x_2\right)) = -x_1^3 - x_1^2 + x_2 \leq 0, \\ &\qquad\quad h(E\left(x_1,x_2\right)) = x_1^2 - x_2 = 0. \end{aligned} \tag{FP1$_E$}$$

Note that $D_E = \left\{(x_1,x_2) \in R^2 : x_1 \geq 0 \wedge x_2 = x_1^2\right\}$ *and* $\overline{x} = (0,0)$ *is a feasible solution of the fractional programming problem (FP1$_E$). Further, note that all functions constituting the problem (FP1) are* E*-differentiable at* $\overline{x} = (0,0)$. *Then, it can also be shown that the parametric necessary optimality conditions (3.1)-(3.3) are fulfilled at* $\overline{x} = (0,0)$ *with Lagrange multipliers* $\overline{\mu} = \frac{3}{2}$ *and* $\overline{\xi} = \frac{1}{2}$. *Further, it can be proved that* f, $-q$, g, *and* h *are* E*-invex functions at* $\overline{x}$ *on* D_E. *Since all hypotheses of Theorem 3.12 are fulfilled, it is possible to use the sufficient conditions formulated in this theorem to show that* $\overline{x} = (0,0)$ *is an optimal solution of the problem (FP1$_E$) and, thus,* $E\left(\overline{x}\right)$ *is also an* E*-optimal solution of the problem (FP1).*

4. Conclusions

In this paper, a new class of nonconvex nondifferentiable scalar fractional optimization problems has been investigated. Namely, the class of E-differentiable scalar fractional programming problem with both inequality and equality constraint has been considered. The so-called E-Karush-Kuhn-Tucker necessary optimality conditions have been established for the considered E-differentiable scalar fractional optimization problem. Further, under appropriate E-invexity hypotheses, sufficient E-optimality conditions have been derived for such nonconvex nonsmooth scalar fractional optimization problems.

However, some interesting topics for further research remain. It would be of interest to investigate whether it is possible to prove similar results for other classes of E-differentiable fractional optimization problems. We shall investigate these questions in subsequent papers.

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