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ORIGINAL RESEARCH

Mixed *E*-Duality for *E*-differentiable Vector Optimization Problems Under (Generalized) *V*-*E*-Invexity

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Abstract
In this paper, a class of E-differentiable vector optimization problems with both inequality and equality constraints is considered. The so-called vector mixed E-dual problem is defined for the considered E-differentiable vector optimization problem with both inequality and equality constraints. Then, several mixed E-duality theorems are established under (generalized) V-E-invexity hypotheses.

Keywords E-differentiable function · V-E-invex function · Generalized convexity · Mixed E-duality

Mathematics Subject Classification (2000) 90C26 · 90C30 · 90C46

1 Introduction

Multiobjective programming duality has been of much interest in the recent past. Many authors have established duality results for various classes of multiobjective programming problems (see, for example, [1–3, 5, 7–11, 13–15, 17, 19–23], and others). Youness [24] introduced the concepts of an E-convex set and an E-convex function. Megahed et al. [18] presented a new concept of an E-differentiable convex function and they established optimality conditions for mathematical programming problems in which the functions involved are E-differentiable. Abdulaleem [4] introduced a new concept of generalized convexity as a

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generalization of the notion of E-differentiable E-convexity and the notion of differentiable invexity introduced by Hanson [12]. Namely, Abdulaleem defined the concept of E-differentiable E-invexity in the case of (not necessarily) differentiable vector optimization problems with E-differentiable functions. Recently, Abdulaleem [5] introduced a new concept of generalized convexity as a generalization of the E-differentiable E-invexity notion and the concept of V-invexity given by Jeyakumar and Mond [16]. Namely, Abdulaleem defined the concept of E-differentiable V-E-invexity in the case of (not necessarily) differentiable vector optimization problems with E-differentiable functions and used this concept to prove sufficient optimality conditions for a new class of nonconvex E-differentiable vector optimization problems.

In this paper, a class of E-differentiable V-E-invex vector optimization problems with both inequality and equality constraints is considered. A mixed E-dual problem is defined for the considered E-differentiable V-E-invex vector optimization problem with both inequality and equality constraints. Then, various mixed E-duality theorems are established between the considered E-differentiable multicriteria optimization problem and its vector mixed E-dual problem under appropriate (generalized) V-E-invexity hypotheses.

2 Definitions and Preliminaries

Let R^n be the n-dimensional Euclidean space and R^n_+ be its nonnegative orthant. The following convention for equalities and inequalities will be used in the paper. For any vectors $x = \left(x_1, x_2, ..., x_n\right)^T$ and $y = \left(y_1, y_2, ..., y_n\right)^T$ in R^n, $x > y$ means that the vector x is componentwise greater than the vector y. Similarly, the same convention has been used for $x = y$, $x \geqq y$ and $x \geq y$.

First, we recall for a common reader the definition of E-differentiable function introduced by Megahed et al. [18].

Definition 1 [18] Let $E : R^n \to R^n$ and $f : R^n \to R$ be a (not necessarily) differentiable function at a given point $u \in R^n$. It is said that f is an E-differentiable function at u if and only if $f \circ E$ is a differentiable function at u (in the usual sense), that is,

$$(f \circ E)(x) = (f \circ E)(u) + \nabla(f \circ E)(u)(x - u) + \theta(u, x - u) \|x - u\|, \tag{1}$$

where $\theta(u, x - u) \to 0$ as $x \to u$.

Now, we give the definition of a V-E-invex function introduced by Abdulaleem [5].

Definition 2 [5] Let $E : R^n \to R^n$ and $f : R^n \to R^k$ be an E-differentiable function on R^n. It is said that f is a V-E-invex function (a strictly V-E-invex function) with respect to η at $u \in R^n$ on R^n if, there exist functions $\eta : R^n \times R^n \to R^n$ and $\alpha_i : R^n \times R^n \to R_+ \setminus \{0\}$, $i = 1, 2, ..., k$, such that, for all $x \in R^n$ ($E(x) \neq E(u)$), the inequalities

$$f_i(E(x)) - f_i(E(u)) \geqq \alpha_i(E(x), E(u))\nabla f_i(E(u))\eta(E(x), E(u)) \quad (>) \tag{2}$$

hold. If inequalities (2) are fulfilled for any $u \in R^n$ $(E(x) \neq E(u))$, then f is a V-E-invex (strictly V-E-invex) function at u on R^n. Each function f_i, $i = 1, ..., k$, satisfying (2), is said to be α_i-E-invex (strictly α_i-E-invex) with respect to η at u on R^n.

Remark 1 Note that the Definition 2 generalizes and extends several generalized convexity notions, previously introduced in the literature. Indeed, there are the following special cases:

(a) In the case when $\alpha_i(x, u) = 1, i = 1, ..., k$, then the definition of a V-E-invex function reduces to the definition of an E-invex function introduced by Abdulaleem [4].
(b) If f is differentiable and $E(x) \equiv x$ (E is an identity map), then the definition of a V-E-invex function reduces to the definition of a V-invex function introduced by Jeyakumar and Mond [16].
(c) If f is differentiable, $E(x) \equiv x$ (E is an identity map) and $\alpha_i(x, u) = 1$, $k = 1$, then the definition of a V-E-invex function reduces to the definition of an invex function introduced by Hanson [12].
(d) If η is defined by $\eta(x, u) = x - u$ and $\alpha_i(x, u) = 1, i = 1, ..., k$, then we obtain the definition of an E-differentiable E-convex vector-valued function introduced by Megahed et al. [18].
(e) If f is differentiable, $E(x) = x$ and $\eta(x, u) = x - u$ and $\alpha_i(x, u) = 1, i = 1, ..., k$, then the definition of a V-E-invex function reduces to the definition of a differentiable convex vector-valued function.
(f) If f is a differentiable scalar function, $\eta(x, u) = x - u$ and $\alpha_i(x, u) = 1$, then we obtain the definition of a differentiable E-convex function introduced by Youness [24].

Now, we give various classes of generalized E-differentiable V-E-invex functions.

Definition 3 Let $E : R^n \to R^n$ and $f : R^n \to R^k$ be an E-differentiable function on R^n. It is said that f is a V-E-pseudo-invex function with respect to η at $u \in R^n$ on R^n if, there exist functions $\eta : R^n \times R^n \to R^n$ and $\alpha_i : R^n \times R^n \to R_+ \setminus \{0\}$, $i = 1, 2, ..., k$, such that, for all $x \in R^n$, the relations

$$\begin{aligned} &\sum_{i=1}^{k} \alpha_i(E(x), E(u))f_i(E(x)) < \sum_{i=1}^{k} \alpha_i(E(x), E(u))f_i(E(u)) \\ &\qquad\qquad \Rightarrow \sum_{i=1}^{k} \nabla f_i(E(u))\eta(E(x), E(u)) < 0 \end{aligned} \tag{3}$$

hold. If (3) are fulfilled for any $u \in R^n$, then f is V-E-pseudo-invex with respect to η on R^n. Each function f_i, $i = 1, ..., k$, satisfying (3) is said to be α_i-E-pseudo-invex with respect to η at u on R^n.

Definition 4 Let $E : R^n \to R^n$ and $f : R^n \to R^k$ be an E-differentiable function on R^n. It is said that f is a V-E-quasi-invex function with respect to η at $u \in R^n$ on R^n if there exist functions $\eta : R^n \times R^n \to R^n$ and $\alpha_i : R^n \times R^n \to R_+ \setminus \{0\}$, $i = 1, 2, ..., k$, such that, for all $x \in R^n$, the relations

$$\sum_{i=1}^{k} \alpha_i(E(x), E(u))f_i(E(x)) \leqq \sum_{i=1}^{k} \alpha_i(E(x), E(u))f_i(E(u)) \Rightarrow \sum_{i=1}^{k} \nabla f_i(E(u))\eta(E(x), E(u)) \leqq 0 \tag{4}$$

hold. If (4) are fulfilled for any $u \in R^n$, then f is V-E-quasi-invex on R^n. Each function f_i, $i = 1, ..., k$, satisfying (4) is said to be α_i-E-quasi-invex with respect to η at u on R^n.

In this paper, we consider the following (not necessarily differentiable) multiobjective programming problem (MOP) with both inequality and equality constraints defined as follows:

$$\begin{aligned} &\text{minimize } f(x) = \left(f_1(x), ..., f_p(x)\right) \\ &\text{subject to } g_j(x) \leqq 0, \ j \in J = \{1, ..., m\}, \\ &\qquad h_t(x) = 0, \ t \in T = \{1, ..., q\}, \end{aligned} \tag{MOP}$$

where $f_i : R^n \to R$, $i \in I = \{1, ..., p\}$, $g_j : R^n \to R$, $j \in J$, $h_t : R^n \to R$, $t \in T$, are real-valued functions defined on R^n. We shall write $g := \left(g_1, ..., g_m\right) : R^n \to R^m$ and $h := \left(h_1, ..., h_q\right) : R^n \to R^q$ for convenience. Let

$$\Omega := \left\{x \in X : g_j(x) \leqq 0, \ j \in J, h_t(x) = 0, \ t \in T\right\}$$

be the set of all feasible solutions of (MOP). Further, we denote by $J(x)$ the set of inequality constraint indices that are active at a feasible solution x, that is, $J(x) = \left\{j \in J : g_j(x) = 0\right\}$.

Definition 5 A feasible point $\overline{x}$ is said to be a weak Pareto (weakly efficient) solution of (MOP) if and only if there exists no feasible point x such that

$$f(x) < f(\overline{x}).$$

Definition 6 A feasible point $\overline{x}$ is said to be a Pareto (efficient) solution of (MOP) if and only if there exists no feasible point x such that

$$f(x) \leq f(\overline{x}).$$

Let $E : R^n \to R^n$ be a given one-to-one and onto operator. Throughout the paper, we shall assume that the functions constituting the considered problem (MOP) are E-differentiable at any feasible solution.

Now, for the considered problem (MOP), we define its associated differentiable vector optimization problem (VP_E) as follows:

$$\begin{aligned} &\text{minimize}\, f(E(x)) = \left(f_1(E(x)), ..., f_p(E(x))\right) \\ &\text{subject to}\; g_j(E(x)) \leqq 0,\; j \in J = \{1, ..., m\}, (\mathrm{VP}_E) \\ &\qquad h_t(E(x)) = 0,\; t \in T = \{1, ..., q\}. \end{aligned}$$

We call the problem (VP_E) an E-vector optimization problem associated to (MOP). Let

$$\Omega_E := \left\{x \in R^n \,:\, g_j(E(x)) \leqq 0,\ j \in J,\ h_t(E(x)) = 0,\ t \in T\right\}$$

be the set of all feasible solutions of (VP_E). Further, we denote by $J_E(x)$ the set of inequality constraint indices that are active at a feasible solution $x \in \Omega_E$, that is, $J_E(x) = \left\{j \in J \,:\, \left(g_j \circ E\right)(x) = 0\right\}$.

Definition 7 A feasible point $E(\overline{x})$ is said to be a weak E-Pareto (weakly E-efficient) solution of (MOP) if and only if there exists no feasible point $E(x)$ such that

$$f(E(x)) < f(E(\overline{x})).$$

Definition 8 A feasible point $E(\overline{x})$ is said to be an E-Pareto (E-efficient) solution of (MOP) if and only if there exists no feasible point $E(x)$ such that

$$f(E(x)) \leq f(E(\overline{x})).$$

Lemma 1 [1] *Let $E \,:\, R^n \to R^n$ be a one-to-one and onto. Then $E\left(\Omega_E\right) = \Omega$.*

Lemma 2 [1] *Let $\overline{z} \in \Omega_E$ be a weak Pareto solution (a Pareto solution) of the E-vector optimization problem (VP_E). Then $E\left(\overline{z}\right)$ is a weak Pareto solution (a Pareto solution) of the considered multiobjective programming problem (MOP).*

Now, we give the Karush-Kuhn-Tucker necessary optimality conditions for a feasible solution $\overline{x}$ to be a weak Pareto solution in (VP_E). These conditions are, at the same time, the so-called E-Karush-Kuhn-Tucker necessary optimality conditions for $E\left(\overline{x}\right)$ to be a weak E-Pareto solution in (MOP).

Theorem 1 [4] (*E-Karush–Kuhn–Tucker necessary optimality conditions*). *Let $\overline{x} \in \Omega_E$ be a weak Pareto solution of the E-vector optimization problem (VP_E) (and, thus, $E\left(\overline{x}\right)$ be a weak E-Pareto solution of the considered problem (MOP)). Further, f, g, h be E-differentiable at $\overline{x}$ and the E-Guignard constraint qualification [4] be satisfied at $\overline{x}$. Then there exist Lagrange multipliers $\overline{\lambda} \in R^p$, $\overline{\mu} \in R^m$, $\overline{\xi} \in R^q$ such that*

$$\sum_{i=1}^{p} \overline{\lambda}_i \nabla f_i\left(E(\overline{x})\right) + \sum_{j=1}^{m} \overline{\mu}_j \nabla g_j\left(E(\overline{x})\right) + \sum_{t=1}^{q} \overline{\xi}_t \nabla h_t\left(E(\overline{x})\right) = 0, \tag{5}$$

$$\overline{\mu}_j g_j\left(E(\overline{x})\right) = 0,\ j \in J\left(E\left(\overline{x}\right)\right), \tag{6}$$

$$\overline{\lambda} \geq 0,\ \overline{\mu} \geqq 0. \tag{7}$$

3 Mixed *E*-Duality

In this section, a vector mixed E-dual problem is defined for the considered E-differentiable problem (MOP) with inequality and equality constraints.

Let the index set J be partitioned into two disjoint subset J_1 and J_2 such that $J = J_1 \cup J_2$ and the index set T be partitioned into two disjoint subset T_1 and T_2 such that $T = T_1 \cup T_2$. Let J_1 be an index set such that $J_1 = J \backslash J_2$ and $J_2 = J \backslash J_1$, let $\mu_{J_K} g_{J_K} = \sum_{j \in J_k} \mu_j g_j,\ k = 1, 2$ and, moreover, $\left|J_1\right|$ and $\left|J_2\right|$ denote the cardinality of the index sets J_1 and J_2, respectively. Further, let T_1 be an index set such that $T_1 = T \backslash T_2$ and $T_2 = T \backslash T_1$, let $\xi_{T_K} h_{T_K} = \sum_{t \in T_k} \xi_t h_t,\ k = 1, 2$ and, moreover, $\left|T_1\right|$ and $\left|T_2\right|$ denote the cardinality of the index sets T_1 and T_2, respectively. Let us denote the set

$$\Omega^2 = \left\{x \in R^n : g_j(x) \leqq 0,\ j \in J_2,\ h_t(x) = 0,\ t \in T_2\right\}.$$

Now, for the define E-differentiable problem (MOP), we introduce the definition of the scalar Lagrange function $L : \Omega \times R_+^p \times R_+^{\left|J_1\right|} \times R^{\left|T_1\right|} \to R$ as follows

$$L(x, \lambda, \mu, \xi) := \sum_{i=1}^{p} \lambda_i f_i(x) + \sum_{j \in J_1} \mu_j g_j(x) + \sum_{t \in T_1} \xi_t h_t(x). \tag{8}$$

Further, let $E : R^n \to R^n$ be a given one-to-one and onto operator. Further, let us define the following set

$$\Omega_E^2 := \left\{x \in R^n : \left(g_j \circ E\right)(x) \leqq 0,\ j \in J_2,\ \left(h_t \circ E\right)(x) = 0,\ t \in T_2\right\}.$$

Now, we define the following vector mixed E-dual problem (VMD_E) for the considered E-differentiable problem (MOP):

$$\begin{aligned}
&\text{maximize}\, f(E(y)) + \left[\mu_{J_1}\left(g_{J_1} \circ E\right)(y) + \xi_{T_1}\left(h_{T_1} \circ E\right)(y)\right] e \\
&\text{subject to} \quad \lambda \nabla f(E(y)) + \mu \nabla g(E(y)) + \xi \nabla h(E(y)) = 0, \\
&\qquad\qquad \mu_{J_2}\left(g_{J_2} \circ E\right)(y) \geqq 0, \qquad\qquad (\text{VMD}_E) \\
&\qquad\qquad \xi_{T_2}\left(h_{T_2} \circ E\right)(y) = 0, \\
&\lambda \in R^p, \lambda \geq 0, \lambda e = 1, \mu \in R^m, \mu \geqq 0, \xi \in R^q,
\end{aligned}$$

where all functions are defined in the similar way as for the considered vector optimization problem (MOP). Further, let Γ_E denote the set of all feasible solutions of (VMD_E), that is,

$$\Gamma_E = \Big\{ (y, \lambda, \mu, \xi) \in R^n \times R^p \times R^m \times R^q \; : \\ \lambda \nabla f(E(y)) + \mu \nabla g(E(y)) + \xi \nabla h(E(y)) = 0, \\ \sum_{j \in J_2} \mu_j \big(g_j \circ E\big)(y) \geqq 0, \sum_{t \in T_2} \xi_t \big(h_t \circ E\big)(y) = 0, \; \lambda \geq 0, \; \lambda e = 1, \; \mu \geqq 0 \Big\}.$$

Further, $Y_E = \{y \in R^n : (y, \lambda, \mu, \xi) \in \Gamma_E\}$. We call ($\mathrm{VMD}_E$) the vector mixed E-dual problem for the E-differentiable multiobjective optimization problem (MOP).

Note that if set $J_1 = \emptyset$ and $T_1 = \emptyset$ in (VMD_E), then we get a vector Mond-Weir E-dual problem for (MOP) [7] and, moreover, if we set $J_2 = \emptyset$ and $T_2 = \emptyset$ in (VMD_E), then we obtain a vector Wolfe E-dual problem for (MOP) (see, for example, [1, 6]).

Now, we shall prove several mixed duality results between E-vector optimization problems (VP_E) and (VMD_E) under (generalized) V-E-invexity assumptions. Then, we use these duality results in proving several mixed E-duality results between vector optimization problems (MOP) and (VMD_E).

Theorem 2 (Mixed weak duality between (VP_E) *and* (VMD_E)). *Let x and (y, λ, μ, ξ) be any feasible solutions of the problems* (VP_E) *and* (VMD_E), *respectively. Further, assume that at least one of the following hypotheses is fulfilled*:

(a) *each objective function f_i, $i \in I$, is α_i-E-invex with respect to η at y on $\Omega_E \cup Y_E$, each constraint function g_j, $j \in J$, is β_j-E-invex with respect to η at y on $\Omega_E \cup Y_E$, the functions h_t, $t \in T^+(E(y)) = \{t \in T : \xi_t > 0\}$ and the functions $-h_t$, $t \in T^-(E(y)) = \{t \in T : \xi_t < 0\}$, are γ_t-E-invex with respect to η at y on $\Omega_E \cup Y_E$.*

(b) $(f \circ E)(\cdot) + \Big[\mu_{J_1}\big(g_{J_1} \circ E\big)(\cdot) + \xi_{T_1}\big(h_{T_1} \circ E\big)(\cdot)\Big]e$ *is V-E-pseudo-invex at y on $\Omega_E \cup Y_E$, $\mu_{J_2}\big(g_{J_2} \circ E\big)(\cdot)$ is β_j-E-quasi-invex at y on $\Omega_E \cup Y_E$, $\xi_{T_2}\big(h_{T_2} \circ E\big)(\cdot)$ is γ_t -E-quasi-invex at y on $\Omega_E \cup Y_E$.*

Then

$$(f \circ E)(x) \nless (f \circ E)(y) + \Big[\mu_{J_1}\big(g_{J_1} \circ E\big)(y) + \xi_{T_1}\big(h_{T_1} \circ E\big)(y)\Big]e. \tag{9}$$

Proof Let x and (y, λ, μ, ξ) be any feasible solutions of the problems (VP_E) and (VMD_E), respectively.

The proof of this theorem under hypothesis (a). By means of contradiction, suppose that

$$(f \circ E)(x) < (f \circ E)(y) + \Big[\mu_{J_1}\big(g_{J_1} \circ E\big)(y) + \xi_{T_1}\big(h_{T_1} \circ E\big)(y)\Big]e.$$

Thus,

$$\left(f_i \circ E\right)(x) < \left(f_i \circ E\right)(y) + \left[\sum_{j\in J_1} \mu_j\left(g_j \circ E\right)(y) + \sum_{t\in T_1} \xi_t\left(h_t \circ E\right)(y)\right], i \in I. \tag{10}$$

Multiplying each inequality (10) by λ_i and then adding both sides of the resulting inequalities, we get

$$\sum_{i=1}^{p} \lambda_i\left(f_i \circ E\right)(x) < \sum_{i=1}^{p} \lambda_i\left(f_i \circ E\right)(y)$$
$$+\left[\sum_{j\in J_1} \mu_j\left(g_j \circ E\right)(y) + \sum_{t\in T_1} \xi_t\left(h_t \circ E\right)(y)\right] \sum_{i=1}^{p} \lambda_i.$$

Since $\sum_{i=1}^{p} \lambda_i = 1$, the following inequality

$$\sum_{i=1}^{p} \lambda_i\left(f_i \circ E\right)(x) < \sum_{i=1}^{p} \lambda_i\left(f_i \circ E\right)(y)$$
$$+\sum_{j\in J_1} \mu_j\left(g_j \circ E\right)(y) + \sum_{t\in T_1} \xi_t\left(h_t \circ E\right)(y)$$

holds. By $x \in \Omega_E$ and $(y, \lambda, \mu, \xi) \in \Gamma_E$, we have

$$\begin{aligned}\sum_{i=1}^{p} \lambda_i\left(f_i \circ E\right)(x) + \sum_{j\in J_1} \mu_j\left(g_j \circ E\right)(x) + \sum_{t\in T_1} \xi_t\left(h_t \circ E\right)(x) < \\ \sum_{i=1}^{p} \lambda_i\left(f_i \circ E\right)(y) + \sum_{j\in J_1} \mu_j\left(g_j \circ E\right)(y) + \sum_{t\in T_1} \xi_t\left(h_t \circ E\right)(y),\end{aligned} \tag{11}$$

$$\sum_{i=1}^{p} \lambda_i\left(f_i \circ E\right)(x) < \sum_{i=1}^{p} \lambda_i\left(f_i \circ E\right)(y), \tag{12}$$

$$\sum_{j\in J_2} \mu_j\left(g_j \circ E\right)(x) \leqq \sum_{j\in J_2} \mu_j\left(g_j \circ E\right)(y), \tag{13}$$

$$\sum_{t\in T_2} \xi_t\left(h_t \circ E\right)(x) = \sum_{t\in T_2} \xi_t\left(h_t \circ E\right)(y). \tag{14}$$

Combining (11)–(14), we get

$$\sum_{i=1}^{p} \lambda_i \left(f_i \circ E\right)(x) + \sum_{j=1}^{m} \mu_j \left(g_j \circ E\right)(x) + \sum_{t=1}^{q} \xi_t \left(h_t \circ E\right)(x) < \sum_{i=1}^{p} \lambda_i \left(f_i \circ E\right)(y) + \sum_{j=1}^{m} \mu_j \left(g_j \circ E\right)(y) + \sum_{t=1}^{q} \xi_t \left(h_t \circ E\right)(y). \tag{15}$$

By assumption, (y, λ, μ, ξ) is feasible in (VMD$_E$). Hence,

$$\sum_{j=1}^{m} \mu_j g_j(E(x)) \leqq \sum_{j=1}^{m} \mu_j g_j(E(y)). \tag{16}$$

Since $x \in \Omega_E$ and $y \in \Omega_E$, one has

$$\sum_{t=1}^{q} \xi_t h_t(E(x)) = \sum_{t=1}^{q} \xi_t h_t(E(y)). \tag{17}$$

Since the functions f_i, $i \in I$, g_j, $j \in J$, h_t, $t \in T^+$, and $-h_t$, $t \in T^-$, are V-E-invex at y on $\Omega_E \cup Y_E$, by Definition 2, the following inequalities

$$f_i(E(x)) - f_i(E(y)) \geqq \alpha_i(E(x), E(y)) \nabla f_i(E(y)) \eta(E(x), E(y)),\ i \in I, \tag{18}$$

$$g_j(E(x)) - g_j(E(y)) \geqq \beta_j(E(x), E(y)) \nabla g_j(E(y)) \eta(E(x), E(y)), j \in J(E(y)), \tag{19}$$

$$h_t(E(x)) - h_t(E(y)) \geqq \gamma_t(E(x), E(y)) \nabla h_t(E(y)) \eta(E(x), E(y)), t \in T^+(E(y)), \tag{20}$$

$$-h_t(E(x)) + h_t(E(y)) \geqq -\gamma_t(E(x), E(y)) \nabla h_t(E(y)) \eta(E(x), E(y)),\ \ t \in T^-(E(y)) \tag{21}$$

hold, respectively. Combining (12) and (18), we have

$$\alpha_i(E(x), E(y)) \lambda_i \nabla \left(f_i \circ E\right)(y) \eta(E(x), E(y)) < 0. \tag{22}$$

Since $\alpha_i(E(x), E(y)) > 0, i = 1, 2, ..., p$, the above inequalities yield

$$\lambda_i \nabla \left(f_i \circ E\right)(y) \eta(E(x), E(y)) < 0. \tag{23}$$

Multiplying (23) by the corresponding Lagrange multipliers, we get that the following inequality

$$\left[\sum_{i=1}^{p} \lambda_i \nabla \left(f_i \circ E\right)(y)\right] \eta(E(x), E(y)) < 0 \tag{24}$$

holds. Multiplying inequalities (19)–(21) by the corresponding Lagrange multipliers, respectively, we obtain

$$\mu_j g_j(E(x)) - \mu_j g_j(E(y)) \geqq \beta_j(E(x), E(y)) \mu_j \nabla g_j(E(y)) \eta(E(x), E(y)), j \in J(E(y)), \tag{25}$$

$$\xi_t h_t(E(x)) - \xi_t h_t(E(y)) \geqq \gamma_t(E(x), E(y))\xi_t \nabla h_t(E(y))\eta(E(x), E(y)),\ t \in T^+(E(y)), \tag{26}$$

$$\xi_t h_t(E(x)) - \xi_t h_t(E(y)) \geqq \gamma_t(E(x), E(y))\xi_t \nabla h_t(E(y))\eta(E(x), E(y)),\ \ t \in T^-(E(y)). \tag{27}$$

Combining the above inequalities with (16)–(17), we obtain that the inequalities

$$\beta_j(E(x), E(y))\mu_j \nabla g_j(E(y))\eta(E(x), E(y)) \leqq 0, j \in J(E(y)), \tag{28}$$

$$\gamma_t(E(x), E(y))\xi_t \nabla h_t(E(y))\eta(E(x), E(y)) \leqq 0, t \in T^+(E(y)), \tag{29}$$

$$\gamma_t(E(x), E(y))\xi_t \nabla h_t(E(y))\eta(E(x), E(y)) \leqq 0, t \in T^-(E(y)) \tag{30}$$

hold. Since $\beta_j(E(x), E(y)) > 0,\ \ j = 1, 2, ..., m,\ \gamma_t(E(x), E(y)) > 0,\ \ t = 1, 2, ..., q,$ the above inequalities yield, respectively

$$\mu_j \nabla g_j(E(y))\eta(E(x), E(y)) \leqq 0, j \in J(E(y)), \tag{31}$$

$$\xi_t \nabla h_t(E(y))\eta(E(x), E(y)) \leqq 0, t \in T^+(E(y)), \tag{32}$$

$$\xi_t \nabla h_t(E(y))\eta(E(x), E(y)) \leqq 0,\ \ t \in T^-(E(y)). \tag{33}$$

Adding both sides of the inequalities (31)–(33) and (24), we obtain that the inequality

$$\left[\sum_{i=1}^{p} \lambda_i \nabla \left(f_i \circ E\right)(y) + \sum_{j=1}^{m} \mu_j \nabla g_j(E(y)) + \sum_{t=1}^{q} \xi_t \nabla h_t(E(y))\right] \eta(E(x), E(y)) < 0 \tag{34}$$

holds, contradicts the first constraint of the vector mixed E-dual problem (VMD$_E$). This means that the proof of the mixed weak duality theorem between the E-vector optimization problems (VP$_E$) and (VMD$_E$) is completed under hypothesis (a).

The proof of this theorem under hypothesis (b). We proceed by contradiction. Suppose, contrary to the result, that (10) holds. Since the function $(f \circ E)(\cdot) + \left[\mu_{J_1}\left(g_{J_1} \circ E\right)(\cdot) + \xi_{T_1}\left(h_{T_1} \circ E\right)(\cdot)\right] e$ is V-E-pseudo-invex at y on $\Omega_E \cup Y_E$, by Definition 3, the inequality

$$\left[\sum_{i=1}^{p} \lambda_i \nabla \left(f_i \circ E\right)(y) + \sum_{j \in J_1} \mu_j \nabla \left(g_j \circ E\right)(y) + \sum_{t \in T_1} \xi_t \nabla \left(h_t \circ E\right)(y)\right] \eta(E(x), E(y)) < 0 \tag{35}$$

holds. By $x \in \Omega_E$ and $(y, \lambda, \mu, \xi) \in \Gamma_E$, it follows that the relations (12)–(14) are fulfilled. Since $\mu_{J_2}\left(g_{J_2} \circ E\right)(\cdot)$ and $\xi_{T_2}\left(h_{T_2} \circ E\right)(\cdot)$ are V-E-quasi-invex at y on $\Omega_E \cup Y_E$, by the foregoing above relations, Definition 4 implies that the inequalities

$$\sum_{j\in J_2} \mu_j \nabla\left(g_j \circ E\right)(y)\eta(E(x), E(y)) \leqq 0, \tag{36}$$

$$\sum_{t\in T_2} \xi_t \nabla\left(h_t \circ E\right)(y)\eta(E(x), E(y)) \leqq 0. \tag{37}$$

hold, respectively. Combining (35), (36) and (37), it follows that the inequality (34) is fulfilled, contradicting the first constraint of the vector mixed E-dual problem (VMD$_E$). This means that the proof of the mixed weak duality theorem between the E-vector optimization problems (VP$_E$) and (VMD$_E$) is completed under hypothesis (b).

Theorem 3 (*Mixed weak E-duality between* (*MOP*) *and* (*VMD*$_E$)). *Let* $E(x)$ *and* (y, λ, μ, ξ) *be any feasible solutions of the problems* (*MOP*) *and* (*VMD*$_E$), *respectively. Further, assume that all hypotheses of Theorem 2 are fulfilled. Then, mixed weak E-duality between* (*MOP*) *and* (*VMD*$_E$) *holds, that is,*

$$(f \circ E)(x) \nless (f \circ E)(y) + \left[\mu_{J_1}\left(g_{J_1} \circ E\right)(y) + \xi_{T_1}\left(h_{T_1} \circ E\right)(y)\right]e.$$

Proof Let $E(x)$ and (y, λ, μ, ξ) be any feasible solutions of the problems (MOP) and (VMD$_E$), respectively. Then, by Lemma 1, it follows that x is any feasible solution of (VP$_E$). Since all hypotheses of Theorem 2 are fulfilled, the mixed weak E-duality theorem between the problems (MOP) and (VMD$_E$) follows directly from Theorem 2.

If some stronger V-E-invexity hypotheses are imposed on the functions constituting the considered E-differentiable problem, then the following result is true.

Theorem 4 (Mixed weak duality between (VP$_E$) *and* (*VMD*$_E$)). *Let* x *and* (y, λ, μ, ξ) *be any feasible solutions of the problems* (*VP*$_E$) *and* (*VMD*$_E$), *respectively. Further, assume that at least one of the following hypotheses is fulfilled*:

(A) *each objective function* f_i, $i \in I$, *is strictly* α_i-*E-invex with respect to* η *at* y *on* $\Omega_E \cup Y_E$, *each constraint function* g_j, $j \in J$, *is* β_j-*E-invex with respect to* η *at* y *on* $\Omega_E \cup Y_E$, *the functions* h_t, $t \in T^+(E(y))$ *and the functions* $-h_t$, $t \in T^-(E(y))$, *are* γ_t-*E-invex with respect to* η *at* y *on* $\Omega_E \cup Y_E$.

(B) $(f \circ E)(\cdot) + \left[\mu_{J_1}\left(g_{J_1} \circ E\right)(\cdot) + \xi_{T_1}\left(h_{T_1} \circ E\right)(\cdot)\right]e$ *is strictly V-E-pseudo-invex at* y *on* $\Omega_E \cup Y_E$, $\mu_{J_2}\left(g_{J_2} \circ E\right)(\cdot)$ *is* β_j-*E-quasi-invex at* y *on* $\Omega_E \cup Y_E$, $\xi_{T_2}\left(h_{T_2} \circ E\right)(\cdot)$ *is* γ_t-*E-quasi-invex at* y *on* $\Omega_E \cup Y_E$.

Then

$$(f \circ E)(x) \nleq (f \circ E)(y) + \left[\mu_{J_1}\left(g_{J_1} \circ E\right)(y) + \xi_{T_1}\left(h_{T_1} \circ E\right)(y)\right]e. \tag{38}$$

Theorem 5 (*Mixed weak E-duality between* (*MOP*) *and* (VMD_E)). *Let* $E(x)$ *and* (y, λ, μ, ξ) *be any feasible solutions of the problems* (*MOP*) *and* (VMD_E), *respectively. Further, assume that all hypotheses of Theorem 4 are fulfilled. Then, mixed weak E-duality between* (*MOP*) *and* (VMD_E) *holds, that is,*

$$(f \circ E)(x) \nleq (f \circ E)(y) + \left[\mu_{J_1}\left(g_{J_1} \circ E\right)(y) + \xi_{T_1}\left(h_{T_1} \circ E\right)(y)\right]e.$$

Theorem 6 (*Mixed strong duality between* (VP_E) *and* (VMD_E) *and also strong E-duality between* (*MOP*) *and* (VMD_E)). *Let* $\overline{x} \in \Omega_E$ *be a weak Pareto solution* (*a Pareto solution*) *of the E-vector optimization problem* (VP_E) *and the E-Guignard constraint qualification* [4] *be satisfied at* $\overline{x}$. *Then there exist* $\overline{\lambda} \in R^p$, $\overline{\lambda} \neq 0$, $\overline{\mu} \in R^m$, $\overline{\mu} \geqq 0$, $\overline{\xi} \in R^q$, $\overline{\xi} \geqq 0$ *such that* $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ *is feasible for the problem* (VMD_E) *and the objective functions of* (VP_E) *and* (VMD_E) *are equal at these points. If also weak duality* (*Theorem* 2) *holds between* (VP_E) *and* (VMD_E), *then* $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ *is a* (*weak*) *maximum point for* (VMD_E).*In other words, if* $E(\overline{x}) \in \Omega$ *is a* (*weak*) *E-Pareto solution of the problem* (*MOP*), *then* $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ *is a* (*weak*) *efficient solution of a maximum type in the vector mixed dual problem* (VMD_E). *This means that the mixed strong E-duality holds between the problems* (*MOP*) *and* (VMD_E).

Proof Since $\overline{x} \in \Omega_E$ is a (weak) Pareto solution of the problem (VP_E) and the E-Guignard constraint qualification [4] is satisfied at $\overline{x}$, by Theorem 1, there exist $\overline{\lambda} \in R^p$, $\overline{\lambda} \neq 0$, $\overline{\mu} \in R^m$, $\overline{\mu} \geqq 0$, $\overline{\xi} \in R^q$, $\overline{\xi} \geqq 0$ such that $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a feasible solution of the problem (VMD_E). This means that the objective functions of (VP_E) and (VMD_E) are equal. If we assume that weak duality (Theorem 2) holds between (VP_E) and (VMD_E), $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a (weak) maximum point for (VMD_E) in the sense of mixed.

Moreover, we have, by Lemma 1, that $\overline{x} \in \Omega_E$. Since $\overline{x} \in \Omega_E$ is a weak Pareto solution of the problem (VP_E), by Lemma 2, it follows that $E\left(\overline{x}\right)$ is a weak E-Pareto solution in the problem (MOP). Then, by the strong duality between (VP_E) and (VMD_E), we conclude that also the mixed strong E-duality holds between the problems (MOP) and (VMD_E). This means that if $E\left(\overline{x}\right) \in \Omega$ is a weak E-Pareto solution of the problem (MOP), there exist $\overline{\lambda} \in R^p$, $\overline{\lambda} \neq 0$, $\overline{\mu} \in R^m$, $\overline{\mu} \geqq 0$, $\overline{\xi} \in R^q$, $\overline{\xi} \geqq 0$ such that $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a weakly efficient solution of a maximum type in the mixed dual problem (VMD_E).

Theorem 7 (*Mixed converse duality between* (VP_E) *and* (VMD_E)). *Let* $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ *be a* (*weakly*) *efficient solution of a maximum type in mixed E-dual problem* (VMD_E) *such that* $\overline{x} \in \Omega_E$. *Moreover, that the objective functions* f_i, $i \in I$, *are* α_i-*E-invex with respect to* η *at* $\overline{x}$ *on* $\Omega_E \cup Y_E$, *the constraint functions* g_j, $j \in J$, *are* β_j-*E-invex with respect to* η *at* $\overline{x}$ *on* $\Omega_E \cup Y_E$, *the functions* h_t, $t \in T^+\left(E\left(\overline{x}\right)\right)$ *and the functions* $-h_t$, $t \in T^-\left(E\left(\overline{x}\right)\right)$, *are* γ_t-*E-invex with respect to* η *at* $\overline{x}$ *on* $\Omega_E \cup Y_E$. *Then* $\overline{x}$ *is a* (*weak*) *Pareto solution of the problem* (VP_E).

Proof Let $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ be a (weakly) efficient solution of a maximum type in mixed E-dual problem (VMD$_E$) such that $\overline{x} \in \Omega_E$. By contradiction, suppose that

$$(f \circ E)(x) < (f \circ E)(\overline{x}) + \left[\overline{\mu}_{J_1}\left(g_{J_1} \circ E\right)(\overline{x}) + \overline{\xi}_{T_1}\left(h_{T_1} \circ E\right)(\overline{x})\right]e.$$

Thus,

$$\left(f_i \circ E\right)(x) < \left(f_i \circ E\right)(\overline{x}) + \left[\sum_{j\in J_1} \overline{\mu}_j\left(g_j \circ E\right)(\overline{x}) + \sum_{t\in T_1} \overline{\xi}_t\left(h_t \circ E\right)(\overline{x})\right], i \in I. \quad (39)$$

Multiplying each inequality (39) by $\overline{\lambda}_i$ and then adding both sides of the resulting inequalities, we get

$$\sum_{i=1}^{p} \overline{\lambda}_i\left(f_i \circ E\right)(x) < \sum_{i=1}^{p} \overline{\lambda}_i\left(f_i \circ E\right)(\overline{x}) + \left[\sum_{j\in J_1} \overline{\mu}_j\left(g_j \circ E\right)(\overline{x}) + \sum_{t\in T_1} \overline{\xi}_t\left(h_t \circ E\right)(\overline{x})\right] \sum_{i=1}^{p} \overline{\lambda}_i.$$

Since $\sum_{i=1}^{p} \overline{\lambda}_i = 1$, the following inequality

$$\sum_{i=1}^{p} \overline{\lambda}_i\left(f_i \circ E\right)(x) < \sum_{i=1}^{p} \overline{\lambda}_i\left(f_i \circ E\right)(\overline{x}) + \sum_{j\in J_1} \overline{\mu}_j\left(g_j \circ E\right)(\overline{x}) + \sum_{t\in T_1} \overline{\xi}_t\left(h_t \circ E\right)(\overline{x})$$

holds. By $x \in \Omega_E$ and $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Gamma_E$, we have

$$\begin{aligned}\sum_{i=1}^{p} \overline{\lambda}_i\left(f_i \circ E\right)(x) + \sum_{j\in J_1} \overline{\mu}_j\left(g_j \circ E\right)(x) + \sum_{t\in T_1} \overline{\xi}_t\left(h_t \circ E\right)(x) < \\ \sum_{i=1}^{p} \overline{\lambda}_i\left(f_i \circ E\right)(\overline{x}) + \sum_{j\in J_1} \overline{\mu}_j\left(g_j \circ E\right)(\overline{x}) + \sum_{t\in T_1} \overline{\xi}_t\left(h_t \circ E\right)(\overline{x}),\end{aligned} \quad (40)$$

$$\sum_{i=1}^{p} \overline{\lambda}_i\left(f_i \circ E\right)(x) < \sum_{i=1}^{p} \overline{\lambda}_i\left(f_i \circ E\right)(\overline{x}), \quad (41)$$

$$\sum_{j\in J_2} \overline{\mu}_j\left(g_j \circ E\right)(x) \leqq \sum_{j\in J_2} \overline{\mu}_j\left(g_j \circ E\right)(\overline{x}), \quad (42)$$

$$\sum_{t\in T_2} \overline{\xi}_t\left(h_t \circ E\right)(x) = \sum_{t\in T_2} \overline{\xi}_t\left(h_t \circ E\right)(\overline{x}). \quad (43)$$

Combining (40)–(43), we get

$$\sum_{i=1}^{p} \overline{\lambda}_i \big(f_i \circ E\big)(x) + \sum_{j=1}^{m} \overline{\mu}_j \big(g_j \circ E\big)(x) + \sum_{t=1}^{q} \overline{\xi}_t \big(h_t \circ E\big)(x) < \sum_{i=1}^{p} \overline{\lambda}_i \big(f_i \circ E\big)(\overline{x}) + \sum_{j=1}^{m} \overline{\mu}_j \big(g_j \circ E\big)(\overline{x}) + \sum_{t=1}^{q} \overline{\xi}_t \big(h_t \circ E\big)(\overline{x}). \tag{44}$$

By assumption, $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is feasible in (VMD$_E$). Hence,

$$\sum_{j=1}^{m} \overline{\mu}_j g_j(E(x)) \leqq \sum_{j=1}^{m} \overline{\mu}_j g_j(E(\overline{x})). \tag{45}$$

Since $x \in \Omega_E$ and $\overline{x} \in \Omega_E$, one has

$$\sum_{t=1}^{q} \overline{\xi}_t h_t(E(x)) = \sum_{t=1}^{q} \overline{\xi}_t h_t(E(\overline{x})). \tag{46}$$

Since the functions f_i, $i \in I$, g_j, $j \in J$, h_t, $t \in T^+$, and $-h_t$, $t \in T^-$, are V-E-invex at $\overline{x}$ on $\Omega_E \cup Y_E$, by Definition 2, the following inequalities

$$f_i(E(x)) - f_i\big(E\big(\overline{x}\big)\big) \geqq \alpha_i(E(x), E(\overline{x}))\nabla f_i\big(E\big(\overline{x}\big)\big)\eta\big(E(x), E\big(\overline{x}\big)\big),\ i \in I, \tag{47}$$

$$g_j(E(x)) - g_j(E\big(\overline{x}\big)) \geqq \beta_j(E(x), E(\overline{x}))\nabla g_j\big(E\big(\overline{x}\big)\big)\eta\big(E(x), E\big(\overline{x}\big)\big), j \in J\big(E\big(\overline{x}\big)\big), \tag{48}$$

$$h_t(E(x)) - h_t(E\big(\overline{x}\big)) \geqq \gamma_t(E(x), E(\overline{x}))\nabla h_t\big(E\big(\overline{x}\big)\big)\eta\big(E(x), E\big(\overline{x}\big)\big),\ t \in T^+\big(E\big(\overline{x}\big)\big), \tag{49}$$

$$-h_t(E(x)) + h_t(E\big(\overline{x}\big)) \geqq -\gamma_t(E(x), E(\overline{x}))\nabla h_t\big(E\big(\overline{x}\big)\big)\eta\big(E(x), E\big(\overline{x}\big)\big),\ t \in T^-\big(E\big(\overline{x}\big)\big) \tag{50}$$

hold, respectively. Combining (41) and (47), we have

$$\alpha_i(E(x), E(\overline{x}))\overline{\lambda}_i \nabla\big(f_i \circ E\big)\big(\overline{x}\big)\eta\big(E(x), E\big(\overline{x}\big)\big) < 0. \tag{51}$$

Since $\alpha_i(E(x), E(\overline{x})) > 0$, $i = 1, 2, ..., p$, the above inequalities yield

$$\overline{\lambda}_i \nabla\big(f_i \circ E\big)\big(\overline{x}\big)\eta\big(E(x), E\big(\overline{x}\big)\big) < 0. \tag{52}$$

Multiplying (52) by the corresponding Lagrange multipliers, we get that the following inequality

$$\left[\sum_{i=1}^{p} \overline{\lambda}_i \nabla\big(f_i \circ E\big)\big(\overline{x}\big)\right]\eta\big(E(x), E\big(\overline{x}\big)\big) < 0 \tag{53}$$

holds. Multiplying inequalities (48)–(50) by the corresponding Lagrange multipliers, respectively, we obtain

$$\overline{\mu}_j g_j(E(x)) - \overline{\mu}_j g_j(E(y)) \geqq \beta_j(E(x), E(\overline{x}))\overline{\mu}_j \nabla g_j\left(E\left(\overline{x}\right)\right)\eta\left(E(x), E\left(\overline{x}\right)\right), j \in J\left(E\left(\overline{x}\right)\right), \tag{54}$$

$$\overline{\xi}_t h_t(E(x)) - \overline{\xi}_t h_t\left(E\left(\overline{x}\right)\right) \geqq \gamma_t(E(x), E(\overline{x}))\overline{\xi}_t \nabla h_t\left(E\left(\overline{x}\right)\right)\eta\left(E(x), E\left(\overline{x}\right)\right), t \in T^+\left(E\left(\overline{x}\right)\right), \tag{55}$$

$$\overline{\xi}_t h_t(E(x)) - \overline{\xi}_t h_t\left(E\left(\overline{x}\right)\right) \geqq \gamma_t(E(x), E(\overline{x}))\overline{\xi}_t \nabla h_t\left(E\left(\overline{x}\right)\right)\eta\left(E(x), E\left(\overline{x}\right)\right), \ t \in T^-\left(E\left(\overline{x}\right)\right). \tag{56}$$

Combining the above inequalities with (45)–(46), we obtain that the inequalities

$$\beta_j(E(x), E(\overline{x}))\overline{\mu}_j \nabla g_j\left(E\left(\overline{x}\right)\right)\eta\left(E(x), E\left(\overline{x}\right)\right) \leqq 0, j \in J\left(E\left(\overline{x}\right)\right), \tag{57}$$

$$\gamma_t(E(x), E(\overline{x}))\overline{\xi}_t \nabla h_t\left(E\left(\overline{x}\right)\right)\eta\left(E(x), E\left(\overline{x}\right)\right) \leqq 0, t \in T^+\left(E\left(\overline{x}\right)\right), \tag{58}$$

$$\gamma_t(E(x), E(\overline{x}))\overline{\xi}_t \nabla h_t\left(E\left(\overline{x}\right)\right)\eta\left(E(x), E\left(\overline{x}\right)\right) \leqq 0, \ t \in T^-\left(E\left(\overline{x}\right)\right) \tag{59}$$

hold. Since $\beta_j(E(x), E(\overline{x})) > 0$, $j = 1, 2, ..., m$, $\gamma_t(E(x), E(\overline{x})) > 0$, $t = 1, 2, ..., q$, the above inequalities yield, respectively

$$\overline{\mu}_j \nabla g_j\left(E\left(\overline{x}\right)\right)\eta\left(E(x), E\left(\overline{x}\right)\right) \leqq 0, j \in J\left(E\left(\overline{x}\right)\right), \tag{60}$$

$$\overline{\xi}_t \nabla h_t\left(E\left(\overline{x}\right)\right)\eta\left(E(x), E\left(\overline{x}\right)\right) \leqq 0, t \in T^+\left(E\left(\overline{x}\right)\right), \tag{61}$$

$$\overline{\xi}_t \nabla h_t\left(E\left(\overline{x}\right)\right)\eta\left(E(x), E\left(\overline{x}\right)\right) \leqq 0, t \in T^-\left(E\left(\overline{x}\right)\right). \tag{62}$$

Adding both sides of the inequalities (60)–(62) and (53), we obtain that the inequality

$$\left[\sum_{i=1}^{p} \overline{\lambda}_i \nabla\left(f_i \circ E\right)\left(\overline{x}\right) + \sum_{j=1}^{m} \overline{\mu}_j \nabla g_j\left(E\left(\overline{x}\right)\right) + \sum_{t=1}^{q} \overline{\xi}_t \nabla h_t\left(E\left(\overline{x}\right)\right)\right]\eta\left(E(x), E\left(\overline{x}\right)\right) < 0 \tag{63}$$

holds, contradicts the first constraint of the vector mixed E-dual problem (VMD$_E$). This means that the proof of this theorem is completed.

Theorem 8 (*Mixed converse E-duality between* (*MOP*) *and* (*VMD$_E$*)). *Let* $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ *be a* (*weakly*) *efficient solution of a maximum type in mixed dual problem* (*VMD$_E$*). *Further, assume that all hypotheses of Theorem 7 are fulfilled. Then* $E(\overline{x}) \in \Omega$ *is a* (*weak*) *E-Pareto solution of the problem* (*MOP*).

Proof The proof of this theorem follows directly from Lemma 2 and Theorem 7.

Theorem 9 *(Mixed restricted converse duality between* (VP_E) *and* (VMD_E)*). Let* $\overline{x}$ *and* $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ *be feasible solutions for the problems* (VP_E) *and* (VMD_E)*, respectively, such that*

$$(f \circ E)\,(\overline{x}) < (f \circ E)\,(\overline{y}) + \left[\sum_{j\in J_1} \overline{\mu}_j\left(g_j \circ E\right)(\overline{y}) + \sum_{t\in T_1} \overline{\xi}_t\left(h_t \circ E\right)(\overline{y})\right] e.\ (\leq) \tag{64}$$

Moreover, assume that $\overline{\lambda}_i f_i$*,* $i \in I$*, are* α_i*-E-pseudo-invex at* $\overline{y}$ *on* $\Omega_E \cup Y_E$*, the functions* $\overline{\mu}_{J_2}\left(g_{J_2} \circ E\right)(\cdot)$ *is* β_j*-E-quasi-invex at* $\overline{y}$ *on* $\Omega_E \cup Y_E$*,* $\overline{\xi}_{T_2}\left(h_{T_2} \circ E\right)(\cdot)$ *is* γ_t*-E-quasi-invex at* $\overline{y}$ *on* $\Omega_E \cup Y_E$*. Then* $\overline{x} = \overline{y}$*, that is,* $\overline{x}$ *is a (weak) Pareto solution of the problem* (VP_E) *and* $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ *is a (weakly) efficient point of maximum type for the problem* (VMD_E)*.*

Proof The proof of this theorem follows directly from Theorem 2 and Theorem 4.

Theorem 10 *(Mixed restricted converse E-duality between (MOP) and* (VMD_E)*). Let* $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ *be a feasible solution of the problem* (VMD_E)*. Further, assume that there exist* $E\left(\overline{x}\right) \in \Omega$ *such that* $\overline{x} = \overline{y}$*. If all hypotheses of Theorem 9 are fulfilled, then* $E\left(\overline{x}\right)$ *is an E-Pareto solution of the problem (MOP) and* $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ *is a weakly efficient solution of maximum type for the problem* (VMD_E)*.*

Proof The proof of this theorem follows directly from Lemma 2 and Theorem 9.

4 Conclusion

This paper analyzes mixed E-duality results for E-differentiable V-E-invex multiobjective programming problems with both inequality and equality constraints. The so-called vector mixed E-dual problem has been formulated for such nonconvex (not necessarily)differentiable multiobjective programming problems. Then, various mixed E-duality theorems between the considered E-differentiable vector optimization problem and its mixed dual problem have been proved under (generalized) V-E-invexity hypotheses. The results established in this paper for E-differentiable vector optimization problems extend and generalize similar duality results in the sense of mixed established under other concepts of E-differentiable (generalized) convexity and also duality results in the sense of Mond-Weir and in the sense of Wolfe established for such multicriteria optimization problems.

However, some interesting topics for further research remain. It would be of interest to investigate whether it is possible to prove similar results for other classes of E-differentiable vector optimization problems. We shall investigate these questions in subsequent papers.

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Declarations

Competing Interests The author declares to have no competing interests.

References

1. Antczak T, Abdulaleem N (2019) *E*-optimality conditions and Wolfe *E*-duality for *E*-differentiable vector optimization problems with inequality and equality constraints. J Nonlinear Sci Appl 12:745–764
2. Ahmad I, Gulati TR (2005) Mixed type duality for multiobjective variational problems with generalized (F, ρ)-convexity. J Math Anal Appl 306(2):669–683
3. Ahmad I (2005) Multiobjective mixed symmetric duality with invexity. N Z J Math 34(1):1–9
4. Abdulaleem N (2019) *E*-invexity and generalized *E*-invexity in *E*-differentiable multiobjective programming, In: ITM Web of Conferences, (Vol. 24, p. 01002), EDP Sciences
5. Abdulaleem N (2021) *V*-*E*-invexity in *E*-differentiable multiobjective programming. Numer Algebra, Control Optim. https://doi.org/10.3934/naco.2021014
6. Abdulaleem N (2018) Wolfe *E*-duality for *E*-differentiable *E*-invex vector optimization problems with inequality and equality constraints. 2018 Int Conf Appl Math Comput Sci (ICAMCS.NET). Budapest, Hungary, 156–163
7. Abdulaleem N (2019) *E*-duality results for *E*-differentiable vector optimization problems under (generalized) *E*-convexity. 6th International Conference on Recent Adv Pure Appl Math. Istanbul, 9–26
8. Abdulaleem N (2019) *E*-duality results for *E*-differentiable *E*-invex multiobjective programming problems. In J Phys Conf Ser 1294(3):032027. IOP Publishing
9. Bector CR, Chandra S (2001) On mixed duality in mathematical programming. J Math Anal Appl 259:346–356
10. Bot RI, Grad SM, Wanka G (2009) Duality in vector optimization. Springer Science & Business Media
11. Cambini R, Carosi L (2010) Mixed Type Duality for Multiobjective Optimization Problems with Set Constraints, in Optimality Conditions in Vector Optimization, M. A. Jimenéz, G. Ruiz-Garzón and A. Rufián-Lizana eds., Bentham Science Publishers, 119–142
12. Hanson MA (1981) On sufficiency of the Kuhn-Tucker conditions. J Math Anal Appl 80(2):545–550
13. Habibi S, Kanzi N, Ebadian A (2020) Weak slater qualification for nonconvex multiobjective semi-infinite programming. Iranian Journal of Science and Technology, Transactions A: Science 44(2):417–424
14. Jahn J (1983) Duality in vector optimization. Math Program 25(3):343–353
15. Jayswal A, Jha S, Prasad AK, Ahmad I (2018) Second-order symmetric duality in variational control problems over cone constraints. Asia Pac J Oper Res 35(4):1850028
16. Jeyakumar V, Mond B (1992) On generalised convex mathematical programming. Anziam J 34(1):43–53
17. Kanzi N, Shaker AJ, Caristi G (2018) Optimality, scalarization and duality in linear vector semi-infinite programming. Optimization 67(5):523–536

18. Megahed AA, Gomma HG, Youness EA, El-Banna AZ (2013) Optimality conditions of *E*-convex programming for an *E*-differentiable function. J Inequal Appl 2013:246
19. Mukherjee RN, Rao CP (2000) Mixed type duality for multiobjective variational problems. J Math Anal Appl 252(2):571–586
20. Piao GR, Jiao L (2015) Optimality and mixed duality in multiobjective *E*-convex programming. J Inequal Appl 2015:1–13
21. Treanţă S, Mititelu Ş (2019) Duality with (ρ, *b*)-quasiinvexity for multidimensional vector fractional control problems. J Inf Optim Sci 40(7):1429–1445
22. Treanţă S, Mititelu Ş (2020) Efficiency for variational control problems on Riemann manifolds with geodesic quasiinvex curvilinear integral functionals. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 114(3):1-15
23. Xu Z (1996) Mixed type duality in multiobjective programming problems. J Math Anal Appl 198(3):621–635
24. Youness EA (1999) *E*-convex sets, *E*-convex functions, and *E*-convex programming. J Optim Theory Appl 102:439–450