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Asymptotic Regional Gradient Reduced-Order Observer

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Asymptotic Regional Gradient Reduced-Order Observer

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Abstract. The aim of this paper is to extend the notion of general asymptotic regional gradient observer ($GARG$-observer) to the reduced order case. More precisely, we study and discuss the existing of this approach in a sub-region of the considered domain. Thus, we show that the approach is enables to estimate the unknown part of the state gradient when the output function gives part of information about the region state in ω. The characterization of this notion depend on regional gradient strategic sensors (RGS-sensor) concept in order that asymptotic regional gradient reduced-order observability ($ARGRO$-observability) to be achieved and analyzed. An application presented to various situation cases of strategic sensors.

Key words: RGS-sensors, ARG-detectability, $ARGRO$-observers, Exchange system.

1. Introduction

The basic concept of an observer theory was introduced by Luenberger in [1-5]. In 1975, the observer approach is extended to a linear system of infinite dimensional case described by SCS-group operators [6]. The observation concepts given a main purpose for introducing the observer notions [7-8]. Therefore, the study of observer theory depending to the notions of sensors characterizations as in [9-11]. Thus the regional asymptotic reconstruction has been presented and explored in [12-14]. So the motivation behind this study is related to many real world problem when one cannot estimate the state of the system in whole the domain **Ω**, but only in a region ω of this domain as in "Figure 1" [15-16].

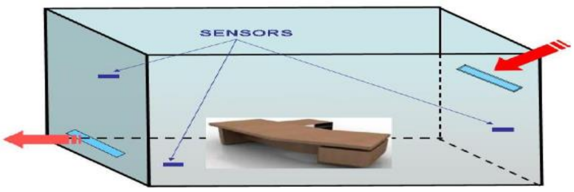

Figure 1. Real model (locate in-out-vents, sensors, workspace)

The objective of present research is to extend the work in [17-18] to regional gradient case for observer type reduced. The rest of paper is ordered as follows: In section2 some preliminaries and formulation of the considered problem are stated. Section 3, recalls the definitions of ARG-stability and ARG-detectability. Section 4, introduces an approach how to build an $\boldsymbol{ARGRO}$-observer and gives a sufficient condition of such estimator by using ARG-detectability and RGS-sensors. Numerous applications about different sensors locations has been given in the last section.

2. Some Preliminaries and Formulation of the Considered Problem Footnotes

Let $\mho$ be a regular bounded open subset of R^n, with smooth boundary $\partial\mho$ and ω be subregion of $\mho$. We denoted $\mathcal{Q} = \mho \times]0,\infty[$, $\Sigma = \partial\mho \times]0,\infty[$. Considering systems which are designated by the following

$$\begin{cases} \frac{\partial x}{\partial t}(\zeta,t) = \mathcal{A}\,x(\zeta,t) + \mathcal{B}\,u(t) & \mathcal{Q} \\ x(\zeta,0) = x(\zeta) & \mho \\ x(\eta,t) = 0 & \Sigma \end{cases} \quad (1)$$

Augmented by the output function

$$\mathcal{Y}(.,t) = \mathcal{C}\,x(\,,t) \quad (2)$$

where $\mathcal{A}$ is a second order linear differential operator, which is a generator of a SCS-group on the Hilbert space $\mathbb{X}$ [4-5]. So $\mathcal{B} \in \mathcal{L}\,(R^p, X)$ and $\mathcal{C} \in \mathcal{L}\,(R^q, X)$, may be bounded or unbounded operators [8]. Thus $\mathbb{X}, \mathbb{U}$ and $\mathbb{Y}$ be the Sobolev spaces of type separable where $\mathbb{X}$ is a state space, $\mathbb{U} = \mathcal{L}^2(0,T,R^p)$ is the input space and $\mathbb{Y} = \mathcal{L}^2(0,T,R^q)$ is the output space. The mathematical model (1)-(2) in Fig.2 is more general than the real model in "Figure 1"

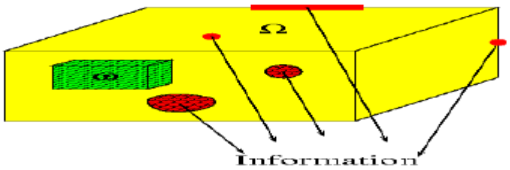

Figure 2. General mathematical model**.**

The solution of the above system is specified by [6]

$$x(\zeta,t) = S_{\mathcal{A}}(t)\,x_0(\zeta) + \int_0^t S_{\mathcal{A}}(t-\tau)B\,u(\tau)d\tau \quad (3)$$

Then the related output is can be achieved by the following form in several cases [10-11]

$$\mathcal{Y}(.,t) = \mathcal{C}\,x(\zeta,t) \quad (4)$$

Define the operator

$$K: x \in \mathbb{X} \to Kx = \mathcal{C}\,S_{\mathcal{A}}(.)\,x \in \mathbb{Y}$$

with adjoint $K^*: \mathbb{Y} \to X$ obtained by

$$K^*\,\mathcal{Y}^* = \int_0^t S_{\mathcal{A}}^*(\ell)\,\mathcal{C}^*\,\mathcal{Y}^*(\ell)\,d\ell, \quad \ell \in [0,\ t]$$

Contemplate the following application

$$\nabla: H^1(\mho) \to (H^1(\mho))^n$$

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$$x \to \nabla x = (\frac{\partial x}{\partial \xi_1}, \dots, \frac{\partial x}{\partial \xi_n})$$

and it's adjoint denotes by ∇^* given by

$$\nabla^*: (H^1(\mho))^n \to H^1(\mho)$$
$$x \to \nabla^* x = v$$

For $\omega \subset \mho$ deliberate the following application

$$\chi_\omega: (H^1(\mho))^n \to (H^1(\omega))^n$$
$$x \quad \to \quad \chi_\omega x = x|_\omega$$

with χ_ω^* the adjoint [4-5]. Let $H = \chi_\omega \nabla K^*$ from $\mathbb{O}$ into $(H^1(\omega))^n$.

3. *ARG*-Detectability and Sensor

Now we discuss and analysis the link between the concept of ARG-Detectability and a RGS-sensor structure in order to build an ARG-observer for the gradient of the system state in reduced order case [13,19-20].

3.1. Definitions and characterizations

This subsection presents some definitions and characterization are needed in our research [11].

- System (1)-(2) is so-called exactly regionally gradient observable (ERG-observable) if

$$Im\, H = Im\, \chi_\omega \nabla K^* = (H^1(\omega))^n$$

- System (1)-(2) is so-called weakly regionally gradient observable (WRG-observable) if

$$\overline{ImH} = \overline{Im\, \chi_\omega \nabla K^*} = (H^1(\omega))^n$$

- So system (1)-(2) may be WRG-observable if

$$ker\ H^* = ker\, K\nabla^* \chi_\omega{}^* = \{0\} \text{ [11]}$$

- A sensor (D, f) is RGS-sensor if the related system is WRG-observable .
- System (1) is ARG-stable if $\mathcal{A}$ produces a semigroup which is ARG-stable. Further, the system (1) is ARG-stable, if and only if there exists M_{ARG}, $\alpha_{ARG} \geq 0$ such that,

$$\|\chi_\omega \nabla S_{\mathcal{A}}(.)\|_{(H^1(\omega))^n} \leq M_{ARG} e^{-\alpha_{ARG} t}, \forall t \geq 0. \quad (5)$$

- If the semigroup $(S_A(t))_{t\geq 0}$ is ARG-stable , then $\forall\, x_0 \in H^1(\mho)$, we can get

$$\lim_{t\to\infty} \|\nabla x(.,t)\|_{(H^1(\omega))^n} = \lim_{t\to\infty} \|\chi_\omega\, \nabla S_A(.)\, x_0\|_{(H^1(\omega))^n} = 0 \quad (6)$$

- Systems (1)-(2) is ARG-detectable if $H_{ARG}: R^q \to (H^1(\omega))^n$ represents an operator such that $(A - H_{ARG}\,\mathcal{C})$ products a SCS-group $(S_{H_{ARG}}(t))_{t\geq 0}$ it is ARG-stable.
- Then equation (6) implies $\lim_{t\to\infty}\|\nabla x(.,t)\|_{(H^1(\omega))^n} = 0$.

Theorem 3.1. Assume $(D_i, f_i)_{1\leq i\leq q}$ the measurement sensors. Then system (1)-(2) is ARG-detectable if and only if :

(I) $q \geq r$

(II) rank $G_m = r_m$, $\forall m, m = 1, \dots, J$ with

$$G_m = (G_m)_{ij} = \begin{cases} \psi_{mj}(b_i), f_i(.) >_{L^2(D_i)} & \text{for zone sensors} \\ \psi_{mj}(b_i) & \text{for pointwise sensors} \\ < \dfrac{\partial \psi_{mj}}{\partial v}, f_i(.) >_{L^2(\Gamma_i)} & \text{for boundary zone sensors} \end{cases}$$

where sup $r_m = r < \infty$ and $\mathrm{j} = 1, \dots, r_m$, ψ_{mj} is eigenfunctions in $(H^1(\mho))^n$ orthonormal in $(H^1(\omega))^n$, and λ_m are the eigenvalues with multiplicity r_m.

Proof: The proof is the same of [16] with some modifications.

Remark 3.2. The important purpose of asymptotic ARG-detectability that is related to the possibility for defining an ARG-estimator of the system state from the knowledge of the output and input function.

4. Reconstruction of $ARGRO$-Observer

In this section we extend the approach of general ARG-observer studied in [4, 18] to the reduced type. Now deliberate the following systems

$$\begin{cases} \frac{\partial x}{\partial t}(\zeta, t) = \mathcal{A}x(\zeta, t) + \mathcal{B}\,u(t) & \mathcal{Q} \\ x(\zeta, 0) = x_0(\zeta) & \mho \\ x(\eta, t) = 0 & \Sigma \end{cases} \tag{7}$$

where the output function

$$\mathcal{Y}(t) = \mathcal{C}x(.,t) \tag{8}$$

In this case, by using regional reduced order forms as in [5], we have

$$\begin{cases} \frac{\partial x_1}{\partial t}(\zeta,t) = \mathcal{A}_{11}x_1(\zeta,t) + \mathcal{A}_{12}\,x_2(\zeta,t) + \mathcal{B}_1 u(t) & Q \\ x_1(\zeta,0) = x_{01}(\zeta) & \mho \\ x_1(\eta,t) = 0 & \Sigma \end{cases} \quad (9)$$

and

$$\begin{cases} \frac{\partial x_2}{\partial t}(\zeta,t) = \mathcal{A}_{21}x_1(\zeta,t) + \mathcal{A}_{22}\,x_2(\zeta,t) + \mathcal{B}_2\, u(t) & Q \\ x_2(\zeta,0) = x_{02}(\zeta) & \mho \\ x_2(\eta,t) = 0 & \Sigma \end{cases} \quad (10)$$

Provided with the information function

$$\mathcal{Y}(.,t) = Cx_1(\zeta,t). \quad (11)$$

The aim of this section is concentrated to construct a dynamic estimator for subsystem (10) [5] which formulated via the form

$$\begin{cases} \frac{\partial a}{\partial t}(\zeta,t) = \mathcal{A}_{22}\,a(\zeta,t) + [\mathcal{B}_2\, u(t) + \mathcal{A}_{21}\, y(\zeta,t)] & Q \\ a(\zeta,0) = a(\zeta) & \mho \\ a(\eta,t) = 0 & \Sigma \end{cases} \quad (12)$$

together with the output function

$$\tilde{\mathcal{Y}}(.,t) = \mathcal{A}_{12}\, a(\zeta,t). \quad (13)$$

So that the state a in (12) represents x_2 in (10).

In this case, the system (12)-(13) is ARG-detectable, there exists an operator $H_{ARG} \in \mathcal{L}(\mathbb{X}|_{\Omega|\omega}, \mathbb{X}|_{\omega}) \ni (\mathcal{A}_{22} - H_{ARG}\mathcal{A}_{12})$ generates an ARG-stable semi-group $(S_{\mathcal{A}_{22}-H_{ARG}\mathcal{A}_{12}}(t))_{t\geq 0}$ on the space $\mathbb{X}_2 = \mathbb{X}|_{\omega}$. So that we can deduce that

$$\exists M_{ARG}, \alpha_{ARG} > 0 \ni \left\|\chi_{ARG}(S_{\mathcal{A}_{22}-H_{ARG}\mathcal{A}_{12}}(.))\right\|_{(H^1(\omega))^n} \leq M_{ARG}e^{-\alpha_{ARG}t}.$$

Therefor we can define an $ARGRO$-observer for (12)-(13) depending on [5, 9]

$$\begin{cases} \frac{\partial \hat{z}}{\partial t}(\zeta,t) = \mathcal{A}_{22}\,\hat{z}(\zeta,t) + [\mathcal{B}_2\, u(t) + \mathcal{A}_{21}\,\mathcal{Y}(\xi,t)] + H_{ARG}(\widetilde{\mathcal{Y}}(.,t) - \mathcal{A}_{12}\,\hat{z}(\zeta,t)) & Q \\ \hat{z}(\zeta,0) = \hat{z}_0(\zeta) & \mho \\ \hat{z}(\eta,t) = 0 & \Sigma \end{cases} \quad (14)$$

The main result which presents the characterization of $ARGRO$-observer notion may be given in the next impotent result.

Theorem 4.1. Let (12)-(13) be an ARG-detectable system, therefore,

$$\lim_{t\to\infty} \|w(\zeta,t) + H_{ARG}\mathcal{Y}(\zeta,t) - x_2(\zeta,t)\|_{(H^1(\omega))^n} = 0, \quad \forall\, \zeta \in \omega$$

and the system of dynamic type (14) is $ARGRO$-observer for the equations (12)-(13) where $\mathcal{Y}(\zeta,t)$ is the output given by (11) and $w(\zeta,t)$ is the resolution of the following

$$\begin{cases}\frac{\partial w}{\partial t}(\zeta,t) = (\mathcal{A}_{22} - H_{ARG}\mathcal{A}_{12})\, w(\zeta,t) + [\mathcal{A}_{22}\, H_{\omega_{GRO}} - H_{\omega_{GRO}}\, \mathcal{A}_{12}\, H_{ARG} \\ \quad -H_{ARG}\mathcal{A}_{11} + \mathcal{A}_{21}]\, \mathcal{Y}(\zeta,t) + [\mathcal{B}_2 - H_{ARG}\, \mathcal{B}_1]u(t) & \mathcal{Q} \\ w(\zeta,0) = w_0(\zeta) & \mho \\ w(\eta,t) = 0 & \Sigma\end{cases}$$

Proof. For the moment the resolution of the (14) is stated by

$$\hat{z}(\zeta,t) = (S_{(\mathcal{A}_{22}-H_{ARG}\mathcal{A}_{12})}(t)\,\hat{z}_0(\xi) + \{\int_0^t S_{(\mathcal{A}_{22}-H_{ARG}\mathcal{A}_{12})}(t-\mathcal{T}).$$

$$[\mathcal{B}_2\, u(\mathcal{T}) + \mathcal{A}_{21}\,\mathcal{Y}(\zeta,t) + H_{ARG}\,\tilde{\mathcal{Y}}(.,t)]\, d\mathcal{T}\} \tag{15}$$

From the use of equations (10) and (9) , we have

$$\tilde{\mathcal{Y}}(.,t) = \mathcal{A}_{12}\, a(\zeta,t) = \frac{\partial x_1}{\partial t}(\zeta,t) - \mathcal{A}_{11}\, x_1(\zeta,t) + \mathcal{B}_1\, u(t) \tag{16}$$

Inserting (16) into (15), we obtain

$$\hat{z}(\zeta,t) = (S_{(\mathcal{A}_{22}-H_{ARG}\mathcal{A}_{12})}(t)\,\hat{z}_0(\zeta) + \int_0^t (S_{(\mathcal{A}_{22}-H_{ARG}\mathcal{A}_{12})}(t-\mathcal{T})\, H_{ARG}\,\frac{\partial x_1}{\partial t}(\zeta,\mathcal{T})\, d\mathcal{T} +$$
$$+ \int_0^t (S_{(\mathcal{A}_{22}-H_{ARG}\mathcal{A}_{12})}(t-\mathcal{T})\,[\mathcal{B}_2\, u(\mathcal{T}) + \mathcal{A}_{21}\,\mathcal{Y}(\zeta,t) - H_{ARG}\,\mathcal{A}_{11}\, x_1(.,\mathcal{T}) -$$
$$- H_{ARG}\,\mathcal{B}_1 u(\mathcal{T})]d\mathcal{T} \tag{17}$$

and we can get

$$\int_0^t (S_{(\mathcal{A}_{22}-H_{ARG}\mathcal{A}_{12})}(t-\mathcal{T})\, H_{ARG}\,\frac{\partial x_1}{\partial t}(\zeta,\mathcal{T})d\mathcal{T} = H_{ARG}\, x_1(.,\mathcal{T}) -$$
$$S_{(\mathcal{A}_{22}-H_{ARG}\mathcal{A}_{12})}(t)\, H_{ARG} x_{01}(.) + (\mathcal{A}_{22} - H_{ARG}\,\mathcal{A}_{21})\int_0^t S_{(\mathcal{A}_{22}-H_{ARG}\mathcal{A}_{12})}(t-\mathcal{T})\, H_{ARG}\, x_1(.,\mathcal{T})\, d\mathcal{T}$$

(18) Consequently the equation (18) becomes

$$\int_0^t S_{(\mathcal{A}_{22}-H_{ARG}\mathcal{A}_{12})}(t-\mathcal{T})\, H_{ARG}\,\frac{\partial x_1}{\partial t}(\zeta,\mathcal{T})\, d\mathcal{T} = H_{ARG}\, x_1(.,\mathcal{T}) -$$
$$S_{(\mathcal{A}_{22}-H_{ARG}\mathcal{A}_{12})}(t)\, H_{ARG}\, x_{01}(.) + \int_0^t S_{(\mathcal{A}_{22}-H_{ARG}\mathcal{A}_{12})}(t-\mathcal{T})\,(\mathcal{A}_{22} - H_{ARG}\mathcal{A}_{21})\, H_{ARG} x_1(.,\mathcal{T})\, d\mathcal{T}$$

(19) Substituting (19) into (17), we have

$$\hat{z}(\zeta,t) = (S_{(\mathcal{A}_{22}-H_{ARG}\mathcal{A}_{12})}(t)\,\hat{z}_0(\zeta) - S_{\mathcal{A}-H_{ARG}\mathcal{A}_{12})}(t) H_{ARG} x_{01}(.) + H_{ARG} x_1(.,t) +$$
$$\int_0^t S_{\mathcal{A}-H_{ARG}\mathcal{A}_{12})}(t-\mathcal{T})\,[\mathcal{A}_{22}\, H_{ARG} - H_{ARG} A_{21}\, H_{ARG} - H_{ARG}\,\mathcal{A}_{11} + \mathcal{A}_{21}]\, x_1(.,\mathcal{T})\, d\mathcal{T} +$$
$$\int_0^t S_{(\mathcal{A}_{22}-H_{ARG}A_{12})}(t-\mathcal{T})\,[\mathcal{B}_2 - H_{ARG}\mathcal{B}_1]u(\mathcal{T})d\mathcal{T}$$

(20)

Setting

$$\psi(\xi,t) = \hat{z}(\zeta,t) + H_{ARG}\,\mathcal{Y}\,(\zeta,t), \text{ with } \psi(\xi,0) = \hat{z}(\zeta,0) + H_{ARG}y(\zeta,0)$$

In this case deliberate $(\mathcal{A}_{22}H_{ARG} - H_{ARG}\mathcal{A}_{21}H_{ARG} - H_{ARG}\mathcal{A}_{11} + \mathcal{A}_{21})$ together with $(\mathcal{B}_2 - H_{ARG}\mathcal{B}_1)$ satisfied the boundedness property in (20) , the we can get

$$\begin{cases} \frac{\partial\psi}{\partial t}(\zeta,t) = (\mathcal{A}_{22} - H_{ARG}A_{12})\,\psi\,(\zeta,t) + (\mathcal{A}_{22}H_{ARG} & \\ -H_{ARG}\mathcal{A}_{12}\,H_{ARG} - H_{ARG}\,\mathcal{A}_{11} + \mathcal{A}_{21})\,\mathcal{Y}(\zeta,t) + (H_{ARG}\mathcal{B}_1)u(t) & \mathcal{Q} \\ \psi\,(\zeta,0) = \psi_{\,0}(\zeta) & \mho \\ \psi\,(\eta,t) = 0 & \Sigma \end{cases} \quad (21)$$

and therefore

$$\frac{\partial z}{\partial t}\,(\zeta,t) - \frac{\partial x_2}{\partial t}(\,\zeta,t) = \psi\,(\zeta,t) + H_{ARG} - x_2(\zeta,t) = \mathcal{A}_{22}\hat{z}\,(\zeta,t) + \mathcal{B}_2u\,(t) + \mathcal{A}_{21}y\,(.,t) + H_{ARG}\hat{\mathcal{Y}}(\zeta,t) - \mathcal{A}_{12}\,\hat{z}\,(\zeta,t)) - \mathcal{A}_{21}x_1(\zeta,t) - \mathcal{A}_{22}x_2(\zeta,t) - \mathcal{B}_2u(t) = (\mathcal{A}_{22} - H_{ARG}\mathcal{A}_{12})(\,\hat{z}(\zeta,t) - x_2(\zeta,t))$$

Now, if the sensors is a RG-strategic then from theorem 3.1 we have the following conditions holds:

(I) $q \geq r_2$.

(II) $rank\ \mathrm{G}_{2m} = \mathrm{r}_{2m}, \forall m,\ m = 1,\dots,J$,

So we can deduced that the system (12)-(13) is ARG-detectable, *i.e.*, there exists an operator $H_{ARG} \in \mathcal{L}(\mathbb{O},(H^1(\omega))^n$ such that $(\mathcal{A}_{22} - H_{ARG}\mathcal{A}_{12})$ generates an ARG-stable semi-group $(S_{\mathcal{A}-H_{ARG}\,\mathcal{A}_{12}}(t))_{t\geq 0}$ on a Hilbert space $\mathbb{X}_2$.

$$\exists M_{ARG}, \alpha_{ARG} > 0 \text{ such that } \left\|\chi_\omega\nabla(S_{\mathcal{A}_{22}-H_{ARG}\mathcal{A}_{12}}(t))\right\|_{(H^1(\omega))^n} \leq M_{ARG}\boldsymbol{e}^{-\alpha_{ARG}(t)}$$

Thus, we obtain

$$\|\hat{z}(\zeta,t) - x_2(,t)\|_{((H^1(\omega))^n} \leq \left\|\chi_\omega\nabla(S_{A_{22}-H_{ARG}A_{12}}(t))\right\|_{(H^1(\omega))^n}\|\hat{z}(\zeta,0) - x_2(\zeta,0)\|_{(H^1(\omega))^n}$$

$$\leq M_{ARG}\boldsymbol{e}^{-\alpha_{ARG}(t)}\|\hat{z}(\zeta,0) - x_2(\zeta,o)\|_{(H^1(\omega))^n} \to 0 \text{ as } \quad t \to \infty$$

Then we have the system (14) is an asymptotic $ARGRO$-observer for the systems (12)-(13).□

5. Some Exchange Systems Application to $ARGRO$-Observer

In the subsequent we reflect two-phase systems of type exchange

$$\begin{cases} \frac{\partial x_1}{\partial t}(\zeta_1,\zeta_2,t) = \frac{x_1}{\partial \zeta^2}\big((\zeta_1,\zeta_2,t)\big) + \big(x_1(\zeta_1,\zeta_2,t) - x_2(\zeta_1,\zeta_2,t)\big) & Q \\ \frac{\partial x_2}{\partial t}(\zeta_1,\zeta_2,t) = \frac{\partial^2 x_2}{\partial \zeta^2}(\zeta_1,\zeta_2,t) + \big(x_2(\zeta_1,\zeta_2,t) - x_1(\zeta_1,\zeta_2,t)\big) & Q \\ x_1(\zeta_1,\zeta_2,0) = x_{01}(\zeta_1,\zeta_2),\ x_2(\zeta_1,\zeta_2,0) = x_{02}(\zeta_1,\zeta_2) & \mho \\ x_1(\eta_1,\eta_2,t) = 0,\ x_2(\eta_1,\eta_2,t) = 0 & \Theta \end{cases} \tag{22}$$

wherever $\mho = [0,\ 1[\times]0,1[$ through the region $\omega =]\alpha_1,\beta_1[\times]\alpha_2,\beta_2[\subset \mho$. So the measurement information (2) is known via the form

$$\mathcal{Y}(t) = \mathcal{C}x_1(.,t) = \int_D x_1(\zeta,t)f(\zeta)\,d\zeta$$

Now, the issue is how to calculate $x_2(\zeta,t)$. For this step, reflect the following formula

$$\frac{\partial x}{\partial t} = \begin{bmatrix} \frac{\partial x_1}{\partial t} \\ \frac{\partial x_2}{\partial t} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{23}$$

somewhere

$$\mathcal{A}_{11} = \frac{\partial^2 x_1}{\partial \zeta^2}(\zeta_1,\zeta_2,t) + 1,\ \mathcal{A}_{22} = \frac{\partial^2 x_2}{\partial \zeta^2}(\zeta_1,\zeta_2,t) + 1 \text{ and } \mathcal{A}_{12} = \mathcal{A}_{21} = -I.$$

By the use of the previous theorem4.1, $ARGRO$-estimator can build for (22) if the measuring sensor (D,f) remains RG-strategic.

$$\begin{cases} \frac{\partial x_1}{\partial t}(\zeta_1,\zeta_2,t) = \frac{\partial^2 x_1}{\partial \zeta^2}(\zeta_1,\zeta_2,t) + (x_1(\zeta_1,\zeta_2,t) \\ -x_2(\zeta_1,\zeta_2,t)) & Q \\ x_1(\zeta_1,\zeta_2,0) = x_{01}(\zeta_1,\zeta_2) & \mho \\ x_1(\eta_1,\eta_2,t) = 0 & \Theta \end{cases} \tag{24}$$

The measuring information is achieved by

$$\mathcal{Y}(t) = \int_D x_1(\zeta,t)\, f(\zeta)\, d\,\zeta \neq 0,$$

under the detectability property, we possess the important formula

$$\lim_{n\to\infty} \|(w(.,t) + H_{ARG}x_1(.,t)) - x_2(.,t)\|_{(H^1(\omega))^n} = 0,$$

where

$$\begin{cases} \frac{\partial w}{\partial t}(\zeta_1,\zeta_2,t) = \frac{\partial^2 w}{\partial \xi^2}\ (\zeta_1,\zeta_2,t) + ((1 + H_{ARG})\, w\ (\zeta_1,\zeta,t) \\ \qquad + (1 - H_{ARG})\ \frac{\partial x_1}{\partial \xi^2}\ (\zeta_1,\zeta_2,t) + (H_{ARG}^2 - 1)\ (\zeta_1,\zeta_2,t)) & Q \\ w(\zeta_1,\zeta_2,0) = w_0\ (\zeta_1,\zeta_2) & \mho \\ w\ (\eta_1,\eta_2,t) = 0 & \Theta \end{cases} \tag{25}$$

Reflect the process system well-defined by

$$\begin{cases} \frac{\partial x_2}{\partial t}(\zeta_1,\zeta_2,t) = \frac{\partial^2 x_2}{\partial \xi^2}(\zeta_1,\zeta_2,t) + x_2(\zeta_1,\zeta_2,t) \\ \qquad -x_1(\zeta_1,\zeta_2,t) & Q \\ x_2(\zeta_1,\zeta_2,0) = x_{02}(\zeta_1,\zeta_2) & \mho \\ x_2(\eta_1,\eta_2,t) = 0 & \Theta \end{cases} \tag{26}$$

with $\mho = (0,1)\ \times\ (0,1)$ and the output function

$$\mathcal{Y}(t) = \mathcal{C}\, x_1(.,t) \tag{27}$$

Let $\omega = (\alpha_1, \beta_1) \times (\alpha_2, \beta_2)$ be studied region provided with

$$\varphi_{ij}(\zeta_1, \zeta) = \frac{2}{\sqrt{(\beta_1-\alpha_1)(\beta_2-\alpha_2)}}\, Sin\, i\pi\left(\frac{\zeta_1-\alpha_1}{\beta_1-\alpha_1}\right) Sin\, j\pi\left(\frac{\zeta_2-\alpha_2}{\beta_2-\alpha_2}\right) \tag{28}$$

and

$$\lambda_{ij} = -\left(\frac{i^2}{(\beta_1-\alpha_1)^2} + \frac{j^2}{(\beta_2-\alpha_2)^2}\right)\pi^2, \quad i, j \geq 1 \tag{29}$$

5.1. Case 1

Reflect case 1 in rectangular sensor supports which is illustrated in "Figure 3" and characterize by equations (26)-(27). So the measuring output is specified by

$$\mathcal{Y}(t) = \int_D x_2(\zeta_1, \zeta_2, t) f(\zeta_1, \zeta_2)\, d\zeta_1\, d\zeta_2, \tag{30}$$

wherever $D \subset \mho$, is the position of the sensor in zone type.

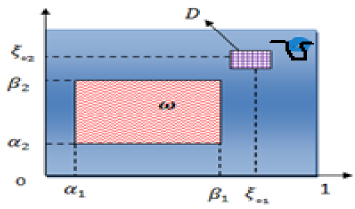

Figure 3. Case 1 of the domain, region and sensor.

So that the sensor (D, f) may be enough for deriving an $ARGRO$-observer, and $\exists\, H_{ARG} \ni (\mathcal{A}_{22} - H_{ARG}\mathcal{A}_{12})$ creates ARG-stable. Thus we have

$$\lim_{t\to\infty} \left\| \left(w\,(\zeta_1, \zeta_2, t) + H_{ARG}\, x_2(\zeta_1, \zeta_2, t) \right) - x\,(\zeta_1, \zeta_2, t) \right\|_{(H^1(\omega))^n} = 0,$$

where

$$\begin{cases} \frac{\partial w}{\partial t}\,(\zeta_1, \zeta_2, t) = \frac{\partial^2 w}{\partial \zeta^2}(\zeta_1, \zeta, t) + ((1 + H_{ARG})\, w(\zeta_1, \zeta, t) & \\ \qquad + (1 - H_{ARG})\, \frac{\partial x_2}{\partial \zeta^2}(\zeta_1, \zeta, t) + (H_{ARG}^2 - 1)(\zeta_1, \zeta_2, t)) & Q \\ w\,(\zeta_1, \xi_2, 0) = w_0(\zeta_1, \zeta_2) & \mho \\ w\,(\zeta_1, \zeta_2, t) = 0 & \Theta \end{cases} \tag{31}$$

Proposition 5.1. Supposing $D = [\zeta_{01} - l_1,\ \zeta_{01} + l_1] \times [\zeta_{02} - l_2, \zeta_{02} + l_2] \subset \mho$ as in Fig.3. Thus the system (31) is not $ARGRO$-observer for the systems (26)-(30), if for any i_0, $i_0\,(\zeta_{01} - \alpha_1)/(\beta_1 - \alpha_1)$ and $i_0\,(\zeta_{02} - \alpha_2)/(\beta_2 - \alpha_2)$ is rational number and f stays symmetric around the $x_{01} = \xi_{01}$.

Proof. Supposing $i_0 = 1, (\zeta_{01} - \alpha_1)/(\beta_1 - \alpha_1)$ and $i_0(\zeta_{02} - \alpha_2)/(\beta_2 - \alpha_2) \in Q$, then there exists $j_0 \geq 1$ such that $Sin\,(j_0\pi c_1/\beta_1\, \alpha_1) = 0$. But

$$\mathcal{Y}(t) = \langle f_1, \varphi_{i_0 j_0} \rangle = \left(\frac{4}{(\beta_1-\alpha_1)\,(\beta_2-\alpha_2)}\right)^{1/2} \int_{\alpha_2-\zeta_2}^{\alpha_2+\zeta_2} \int_{\alpha_1-\zeta_1}^{\alpha_1+\zeta_1} f_1(\zeta_1,\zeta_2)$$

$$Sin\left[\frac{j_0\,\pi\,\zeta_1}{(\beta_1-\alpha_1)}\right] Sin\left[\frac{j_0\pi\,\zeta_2}{(\beta_2-\alpha_2)}\right] d\zeta_1\, d\zeta_2$$

If f satisfy symmetry property for $x_{01} = \zeta$, this implies

$$\mathcal{Y}(t) = \langle f_1, \varphi_{i_0 j_0} \rangle = 0.$$

Therefore, the dynamic system (31) is not $ARGRO$-observer for the systems (26)-(30).□

5.2. Case 2

Deliberate case 2 in circular supports which is shown in Fig.4., and described by equations (26)-(32). So the measuring output is stated by

$$\mathcal{Y}(t) = \int_D x_1(r,\theta,t) f(r,\theta) d\theta, \tag{32}$$

where $D = (r,\theta) \subset \mho$, is the position of the using sensor " Figure 4".

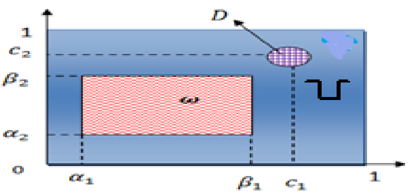

Figure 4. Case 2 of the domain, region and sensor.

So that (D, f) may be enough to ensure $ARGRO$-observer [8], and by employing the same way in case 1 implies the following

$$\lim_{t\to\infty} \|(w(r,\theta,t) + H_{ARG} x_2(r,\theta,t)) - x(r,\theta,t)\|_{(H^1(\omega))^n} = 0,$$

where

$$\begin{cases} \frac{\partial w}{\partial t}(r,\theta,t) = \frac{\partial^2 w}{\partial \zeta^2}(r,\theta,t) + ((1+H_{ARG})\, w\,(r,\theta,t) & \\ \quad +(1-H_{ARG})\frac{\partial x_2}{\partial \zeta^2}(r,\theta,t) + (H_{ARG}^2 - 1)\,(r,\theta,t)) & Q \\ w\,(r,\theta,0) = w_0\,(r,\theta) & \mho \\ w\,(r,\theta,t) = 0 & \Theta \end{cases} \tag{33}$$

Then, we have the following result:

Proposition 5.2. Supposing $D = D(c,r) \subset \mho$, $c = (c_1, c_2)$. So the process (33) is not $ARGRO$-observer for (26)-(32), if $\forall i_0, i_0(c_{01}-\alpha_1)/(\beta_1-\alpha_1)$ and $i_0(c_{02}-\alpha_2)/(\beta_2-\alpha_2)$ is rational number in order f satisfy symmetry property around the $x_{01} = c_{01}$.

Proof. Assuming $i_0 = 1$, then and $\exists\, j_0 \geq 1$, $\ni\, Cos\,(j_0\,\pi c_1/\beta_1\,\alpha_1) = 0$. Thus the measuring information (32) with some modification variables can be characterized by $x_2 = c_1 + \hat{r}\,cos\,\theta$, and $x_2 = c_2 + \hat{r}\,sin\,\theta$. Thus we have

$$\mathcal{Y}(t) = \langle f_1, \varphi_{i_0 j_0} \rangle = \left(\frac{4}{(\beta_1-\alpha_1)(\beta_2-\alpha_2)}\right)^{1/2} \int_0^{2\pi}\int_0^{r} f_1(c_1 + \hat{r}\,\mathrm{Cos}\,\theta, c_2 + \hat{r}\,\mathrm{Sin}\,\theta)\, Sin\left[\frac{j_0\pi(c_1+\hat{r}\,\mathrm{Sin}\,\theta)}{(\beta_1-\alpha_1)}\right] Sin\left[\frac{j_0\pi(c_1+\hat{r}\,\mathrm{Sin}\,\theta)}{(\beta_2-\alpha_2)}\right] \hat{r}\, d\hat{r}\, d\theta$$

Since f_1 is symmetric about $x_2 = c_1$, the function

$$(\hat{r},\theta) \to f_1(c_1 + \hat{r}\,\mathrm{Cos}\,\theta\,, c_2 + \hat{r}\,\mathrm{Sin}\,\theta) cos\left[\frac{j_0\pi(c_1+\hat{r}\,\mathrm{Sin}\,\theta)}{(\beta_2-\alpha_2)}\right]$$

is symmetric on $[0,\pi]$ about $\theta = \pi/2$ for all $\hat{r}$. But the function

$$(\hat{r},\theta) \to Sin\left[\frac{j_0\pi + \hat{r}\mathrm{Sin\,s}\,\theta)}{(\beta_2 - \alpha_2)}\right]$$

verify anti-symmetry property on $[0,\pi]$ around $\pi/2$. Therefore,

$$\mathcal{Y}(t) = \langle f_1, \varphi_{i_0 j_0} \rangle = 0$$

and hence, the process (33) is not $ARGRO$-observer for (26)-(32).□

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6. Conclusion

The existing of an $ARGRO$-observer for a class of DPS have been introduced and characterized. Precisely, we have specified an approach for construct an $ARGRO$-estimator which rebuild a gradient of the state in the reflected region ω. For the future work, may be study the possibility to extend these results to the case of semi-linear in distributed parameter systems.

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