Pairwise Completely Regular in Double Topological Spaces

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Abstract

The concept of intuitionistic topological space was introduced by Çoker. The aim of this paper is to generalize notions between bitopological spaces and double topological spaces and also give a notion of pairwise completely regular for double-topological spaces.

1-Introduction

The concept of a fuzzy topology introduced by Çhange [2], after the introduction of fuzzy sets by Zadeh. Later this concept was extended to intuitionistic fuzzy topological spaces by Çoker in [4]. In [5] Coker studied continuity, connectedness, compactness and separation axioms in intuitionistic fuzzy topological spaces. In this paper, we follow the suggestion of J.G. Garcia and S.E. Rodabaugh [7] that (double fuzzy set) is a more appropriate name than (intuitionistic fuzzy set) ,and therefore adopt the term (double-set) for the intuitionistic set, and (double-topology) for the intuitionistic topology of Dogan Çoker, (this issue), we denote by **Dbl-Top** the construct (concrete texture over set) whose objects are pairs (X,τ) where τ is a double-topology on X. In section three, we discuss making use of this relation between bitopological spaces and double- topological spaces, we generalize a notion of completely regular for double- topological space in section four.

2-Preliminaries

Throughout the paper by X we denote a non-empty set. In this section we shall present various fundamental definitions and propositions. The following definition is obviously inspired by Atanassov [1].

2.1. Definition. [3] A double-set (Ds in brief) A is an object having the form A= <x, A_1, A_2>,
Where A_1 and A_2 are subsets of X satisfying $A_1 \cap A_2 = \phi$. The set A_1 is called the set of members of A, while A_2 is called the set of non- members of A.
Throughout the remainder of this paper we use the simpler A = (A_1, A_2) for a double-set.

2.2. Remark. Every subset A of X is may obviously be regarded as a double-set having the form $A' = (A, A^c)$,
Where $A^c = X \backslash A$ is the complement of A in X.
We recall several relations and operations between DS'^{s} as follows:

2.3. Definition. [3] Let the DS's A and B on X be the form A = (A_1, A_2), B= (B_1, B_2), respectively. Furthermore, let $\{A_j : j \in J\}$ be an arbitrary family of DS's in X, where $A_j = (A_j^{(1)}, A_j^{(2)})$. Then

(a) $A \subseteq B$ *if and only if* $A_1 \subseteq B_1$ *and* $A_2 \supseteq B_2$;

(b) $A = B$ *if and only if* $A \subseteq B$ *and* $B \subseteq A$;

(c) $\overline{A} = (A_2, A_1)$ denotes the complement of A ;

(d) $\bigcap A_j = (\bigcap A_j^{(1)}, \bigcup A_j^{(2)})$;

(e) $\bigcup A_j = (\bigcup A_j^{(1)}, \bigcap A_j^{(2)})$;

(f) $[]A = (A_1, A_1^c)$;

(g) $\langle\rangle A = (A_2^c, A_2)$;

(h) $\underset{\sim}{\phi} = (\phi, X)$ and $\underset{\sim}{X} = (X, \phi)$.

In this paper we require the following:

(i) $()A = (A_1, \phi)$ and $)(A = (\phi, A_2)$.

Now we recall the image and preimage of DS's under a function.

2.4. Definition. [3, 9] Let $x \in X$ be a fixed element in X. Then:

(a)The DS given by $\underset{\sim}{x} = (\{x\}, \{x\}^c)$ is called a double–point (DP in brief X).

(b)The DS $\underset{\approx}{x} = (\phi, \{x\}^c)$ is called a vanishing double-point (VDP in brief X).

2.5. Definition. [3, 9]

(a) Let $\underset{\sim}{x}$ be a DP in X and A= (A_1, A_2) be a DS in X. Then $\underset{\sim}{x} \in A$ iff $x \in A_1$.

(b) Let $\underset{\approx}{x}$ be a VDP in X and A = (A_1, A_2) a DS in X. Then $\underset{\approx}{x} \in A$ iff $x \notin A_2$.

It is clear that $\underset{\sim}{x} \in A \Leftrightarrow \underset{\sim}{x} \subseteq A$ and that $\underset{\approx}{x} \in A \Leftrightarrow \underset{\approx}{x} \subseteq A$.

2.6. Definition. [5] A double-topology (DT in brief) on a set X is a family τ of DS's in X satisfying the following axioms:

T1: $\underset{\sim}{\phi}, \underset{\sim}{X} \in \tau$,

T2: $G_1 \bigcap G_2 \in \tau$ for any $G_1, G_2 \in \tau$,

T3: $\bigcup G_j \in \tau$ for any arbitrary family $\{G_j : j \in J\} \subseteq \tau$.

In this case the pair (X, τ) is called a double-topological space (DTS in brief), and any DS in τ is known as a double open set (DOS in brief). The complement $\overline{A}$ of a DOS A in a DTS is called a double closed set (DCS in brief) in X.

2.7. Definition. [5] Let (X, τ) be a DTS and A = (A_1, A_2) be a DS in X.

Then the interior and closure of A are defined by:

$\text{int}(A) = \bigcup\{G : G\, is\, a\, DOS\, in\, X\, and\, G \subseteq A\}$,

$cl(A) = \bigcap(H : H\, is\, a\, DCS\, in\, X\, and\, A \subseteq H\}$, respectively.

It is clear that cl (A) is a DCS in and int(A) a DOS in X . Moreover, A is a DCS in X iff cl(A) = A , and A is a DOS in X iff int(A) = A.

2.8. Example. [5] Any topological space (X, τ_0) gives rise to a DT of the form $\tau = \{A' : A \in \tau_0\}$ by identifying a subset A in X with its counterpart A'= (A, A^c), as in Remark 2.2.

3- The Constructs Dbl-Top and Bitop:

We begin recalling the following result which associates a bitopology with a double topology.

3.1. Proposition. [5] Let (X, τ) be a DTS.

(a) $\tau_1 = \{A_1 : \exists A_2 \subseteq X\, with\, A = (A_1, A_2) \in \tau\}$ is a topology on X.

(b) $\tau_2^* = \{A_2 : \exists A_1 \subseteq X \text{ with } A = (A_1, A_2) \in \tau\}$ is the family of closed sets of the topology $\tau_2 = \{A_2^c : \exists A_1 \subseteq X \text{ with } A = (A_1, A_2) \in \tau\}$ on X.

(c) Using (a) and (b) we may conclude that (X, τ_1, τ_2) is a bitopological space.

3.2. Proposition. Let (X,u,v) be a bitopological space. Then the family

$$\{(U, V^c) : U \in u,\ V \in v, U \subseteq V\}$$

Is a double topology on X.

Proof. The condition $U \subseteq V$ ensures that $U \cap V^c = \phi$, while the given family contains $\underset{\sim}{\phi}$ because $\phi \in u, v$, and it contains $\underset{\sim}{X}$ because $X \in u, v$. Finally this family is closed under finite intersections and arbitrary unions by Definition 2.3 (d, e) and the corresponding properties of the topologies u and v.

3.3. Definition. Let (X,u,v) be a bitopological space. Then we set

$$\tau_{uv} = \{(U, V^c) : U \in u,\ V \in v, U \subseteq V\}$$

and call this the double topology on X associated with (X,u,v).

3.4. Proposition. If (X,u,v) is a bitopological space and τ_{uv} the corresponding DT on X, then

$$(\tau_{uv})_1 = u \text{ and } (\tau_{uv})_2 = v.$$

Proof. $U \in u$ implies $(U, \phi) \in \tau_{uv}$ since $U \subseteq X \in v$, so $u \subseteq (\tau_{uv})_1$. Conversely, take $U \in (\tau_{uv})_1$. Then $(U, B) \in \tau_{uv}$ for some $B \subseteq X$, and now $U \in u$. Hence $(\tau_{uv})_1 \subseteq u$, and the first equality is proved. □

The proof of the second equality may be obtained in a similar way, and we omit the details.

4- Pairwise Regular and Pairwise Completely regular Double-Topological Spaces:

In this section we shall investigated the concept of pairwise regular and pairwise completely regular in double topological spaces.

4.1. Definition. In a bitopological space (X,u,v), u is said to be regular with respect to v if for each x in X there is a u-neighborhood base of v-closed sets or equivalently if for x in X and each u-closed set F with $x \notin F$ there are u-open set G and v-open set H such that $x \in G$ and $F \subset H$ and that $G \cap H = \phi$. (X,u,v) is said to be pairwise regular if u is regular w.r.t v and v is regular w.r.t u.

4. 2. Proposition. If (X, τ) is pairwise regular DTS then for every point $a \in X$ and every neighborhood N_0 of a, there exists a neighborhood M of a such that $cl\, M \subseteq)(L$ for $M, L \in \tau$.

Proof:

Let (X, τ) pairwise regular, let N be any neighborhood of $\underset{\sim}{a}$,

$\exists$ DOS G such that $\underset{\sim}{a} \in G \subset N$, since $\overline{G}$ is DCS $\underset{\sim}{a} \notin \overline{G}$, by regularity

$\exists\, M, L \in \tau$, $\underset{\sim}{a} \in M, \underset{\approx}{a} \notin L, \overline{G} \subseteq L, M \subseteq ()\overline{L}$.

To prove $cl\, M \subseteq)(L$,

Let $M = (B, C), L = (F, D)$

$\because M \subseteq ()\overline{L} \rightarrow M \cap \overline{(()\overline{L})} = \phi$

$\therefore cl(B, C) \subset (\phi, F)$ and so $cl\, M \subset)(L$.

□

4.3. Definition. In a bitopological space *(X,u,v)*, u is say to be completely regular w.r.t. v if for each point x and each u- open neighborhood U of x , there is a u-lower semi-continuous (l.s.c) and v-upper semi-continuous (u.s.c) function $f: X \to [0,1]$ such that f(x) = 1 and f (X\U) = 0. *(X,u,v)* is say to be pairwise completely regular if u is completely regular , if u is completely regular w.r.t v and v is completely regular w.r.t u.

4.4. Proposition. If *(X,u,v)* is pairwise completely regular then (X,τ_{uv}) is pairwise completely regular.

Proof:

Let u be completely regular w.r.t v then for each x in X and each u-open neighborhood U of x $\exists$ u-(l.s.c) and v-(u.s.c) function $f: X \to [0,1]$ such that f(x) = 1 and f (X\U) = 0.

Take $G = (U,\phi) \in \tau_{uv}$ then $x \in U$ is means that $\underset{\sim}{x} \in G$ then G is open neighborhood $\underset{\sim}{x}$ then there exist l.s.c and u.s.c function such that f ($\underset{\sim}{x}$) = 1 and f (X\G) = 0.

Now let v be completely regular w.r.t u then for each $y \in X$ and each v-open neighborhood V of y $\exists$ v-(l.s.c) and u-(u.s.c) function $f: X \to [0,1]$ such that f(y) = 1 and f (X\V) = 0.

Take $H = (\phi, V^c) \in \tau_{uv}$ hence $y \notin V^c \to \underset{\approx}{y} \in H$ then H is open neighborhood of $\underset{\approx}{y}$ then there exist l.s.c and u.s.c function $f: X \to [0,1]$ such that f ($\underset{\approx}{y}$) = 1 and f (X\H) = 0.

□

4.5. Proposition. If (X,τ) is pairwise completely regular then (X,τ_1,τ_2) is pairwise completely regular.

Proof:

Let (X,τ) be pairwise completely regular, to prove τ_1 is completely regular w.r.t τ_2.

For each x in X and $G = (A,B) \in \tau$ *with* $\underset{\sim}{x} \in G$ then $x \in A \in \tau_1$, there exist τ_1-(l.s.c) and τ_2-(u.s.c) function $f: X \to [0,1]$ such that f(x) = 1 and f (X\A) = 0. (now to prove τ_2 is completely regular w.r.t τ_1), take y in x and let $H = (C,D) \in \tau$ an open neighborhood of $\underset{\approx}{y}$, this means that $y \notin D, y \in D^c \in \tau_2$, there is τ_2-(l.s.c) and τ_1-(u.s.c) function $f: X \to [0,1]$ such that f(y) = 1 and f (X\$D^c$) = 0 then (X,τ_1,τ_2) is pairwise completely regular.

□

4.6. Corollary. If *(X,u,v)* is pairwise completely regular iff (X,τ_{uv}) is pairwise completely regular.

Proof: Necessity follow from proposition 4.4 and Sufficiency from proposition 4.5 and proposition 3.6. □

4.7. Theorem. Product of pairwise completely regular bitopological spaces is pairwise completely regular.

4.8. Proposition. Product of pairwise completely regular (X,τ_{uv}) space is pairwise completely regular.

Proof:

Let $(X_\alpha, \tau_{u_\alpha v_\alpha})_{\alpha\in\Delta}$ be a family spaces. Let $X = X\{X_\alpha : \alpha \in \Delta\}$ and $\tau_{uv} = X\{\tau_{u_\alpha v_\alpha} : \alpha \in \Delta\}$. For $x \in X$ *and* $G = (U,\phi) \in \tau_{uv}$, *with* $x \in U$ *then* $\underset{\sim}{x} \in G$, let us call a $\tau_{uv} - l.s.c$, $\tau_{uv} - u.s.c$ function $f: X \to [0,1]$ meant for ($\underset{\sim}{x}$, G) whenever f($\underset{\sim}{x}$) = 1 and f (X\G) = 0. If $f_1, f_2, ..., f_n$ are functions meant for ($\underset{\sim}{x}$, G_1), ($\underset{\sim}{x}$, G_2),...,($\underset{\sim}{x}$, G_n) and if g($\underset{\sim}{x}$) = sup{f_i($\underset{\sim}{x}$) : i =1,..., n} then g is a function meant for $(\underset{\sim}{x},\bigcap\{G_i : i=1,...,n\})$. Consequently in (X,τ_{uv}). τ_{uv} is completely regular if for each $x\in X$ and τ_{uv}-open neighborhood G of $\underset{\sim}{x}$ belonging to some sub-base for the topology τ_{uv}, there is a function f meant for ($\underset{\sim}{x}$, G). Let $\underset{\sim}{x}\in G$ *and* $G_\alpha \in \tau_{u_\alpha v_\alpha}$ be neighborhood of x_α *in* X_α , and let f be the function meant for $(\underset{\sim\alpha}{x},G_\alpha)$, then f_0P_α *where* P_α *is the projection* mapping onto X_α becomes a function meant for $(\underset{\sim}{x},P_\alpha^{-1}(G_\alpha))$. Since $\{P_\alpha^{-1}(G_\alpha)\}$ from a sub-base for τ_{uv}. Now for y *in* X and $H=(\phi,V^c)\in\tau_{uv}$ *with* $y\notin V^c \to \underset{\approx}{y}\in H$, similarly as above we show that P_α is the projection mapping onto X_α becomes a function meant for $(\underset{\approx}{y},P_\alpha^{-1}(H_\alpha))$. Since $\{P_\alpha^{-1}(H_\alpha)\}$ form a sub-base for τ_{uv}.

□

4. 8. Proposition. Product of pairwise completely regular double topological space (X,τ) is pairwise completely regular.

4. 9. Proposition. Product of pairwise completely regular (X,τ_1,τ_2) space is pairwise completely regular.

Proof:

Let $(X_\alpha,\tau_{1\alpha},\tau_{2\alpha})_{\alpha\in\Delta}$ be a family space. Let $X=X\{X_\alpha : \alpha\in\Delta\}$ and $\tau_1 = X\{\tau_{1\alpha} : \alpha\in\Delta\}$ and $\tau_2 = X\{\tau_{2\alpha} : \alpha\in\Delta\}$. For $x\in X$ and $G=(A,B)\in\tau$, *with* $\underset{\sim}{x}\in G$, $A\in\tau_1$ *with* $x\in A$, let us call a τ_1-(l.s.c) and τ_2-(u.s.c) function $f : X\to[0,1]$ meant for (x, A) whenever f(x) = 1 and f (X\A) = 0. If $f_1, f_2,..., f_n$ are functions meant for (x, A1), (x, A2),..., (x, An) and if $g(x)=\sup\{f_i(x) : i=1,..., n\}$ then g is a function meant for $(x,\bigcap\{A_i : i=1,...,n\})$. Consequently in (X,τ_1,τ_2) , τ_1 is completely regular w.r.t τ_2, if for each $x\in X$ and each τ_1-open neighborhood A of x belonging to some sub-base for the topology τ_1 , there is a function f meant for (x, A) .Let $x\in A$ *and* $A_\alpha\in\tau_{1\alpha}$ be a neighborhood of x_α *in* X_α and let f be the function meant for (x_α, A_α), then f_0P_α *where* P_α *is the projection* mapping onto X_α becomes a function meant for $(x,P_\alpha^{-1}(A_\alpha))$. Since $\{P_\alpha^{-1}(A_\alpha)\}$ from a sub-base for τ_1, we have shown that τ_1 is completely regular w.r.t τ_2, now to prove that τ_2 is completely regular w.r.t τ_1. For $y\in X$ and let $H=(C,D)\in\tau$ $\underset{\approx}{y}\in H$, this means that $y\notin D, y\in D^c\in\tau_2$, let us call a τ_2-(l.s.c) and τ_1-(u.s.c) function $f : X\to[0,1]$ meant for (y,D^c) f(y) = 1 and f (X\ D^c) = 0 and we complete as above .

□

References

[1] Atanassov, K. Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20(1), (1986) 84-96.
[2] Chang, C. L. Fuzzy Topological Spaces, J. Math. Appl. 24, (1968) 182-190.
[3] Coker, D. A note on Intuitionistic Sets and Intuitionistic Points, Turkish J. Math. 20(3), (1996) 343-351.
[4] Coker, D. and Demirci , M. on Intuitionistic Fuzzy Points, Notes IFS 1(2), (1995) 79-84.
[5] Coker, D. and Es, A. H. on Fuzzy Compactness in Intuitionistic Fuzzy Topological Spaces, J. Fuzzy Mathematics 3(4), (1995) 899-909.
[6] Fletcher, P. H. B. Holye and C. W. Patty, The Comparison of Topologies, Duke Math. J.36, (1965) 325-331.
[7] Garia, J.G. and Rodabaugh, S.E. Order-Theoretic, Topological, Categorical Redundancies of Interval-Valued Sets, Grey Sets, Vague Sets, Interval-Valued (Intuitionistic) Sets, (Intuitionistic) Fuzzy Sets and Topologies, Fuzzy Sets and Systems 156(3), (2005) 445-484.
[8] Kelly, J. G. Bitopological Space , Proc. London Math. Soc.13, (1963) 71-89.
[9] Ozcag, S. and Coker, D. on Connectedness in Intuitionistic Fuzzy Special Topological Spaces, Int. J. Math. Math. Sci. 21(1),33-40, 1998.
[10] Zadeh, L. A. Fuzzy Sets, Inform. and Control 8, (1965) 338-353.