Quasi-Banach Space for the Sequence Space ℓ_p, where $0 < p < 1$

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المستخلص:

الهدف من هذا البحث هو إدخال مفهوم فضاء بناخ إلى حد ما لفضاء المتتابعات $0 < p < 1$,ℓ_p. هذا المفهوم يعتمد على توسيع مهم لمفهوم فضاء القياس إلى حد ما كما معرف في المصدر [3]. حيث ندرس فضاء المتتابعات $0 < p < 1$,ℓ_p ونبرهن هذا الفضاء هو فضاء قياس إلى حد ما ولكن ليست فضاء قياس. ثم نكتشف العديد من النتائج المهمة والمتصلة بتقارب المتتابعات في فضاء القياس إلى حد ما. ومن ثم نثبت بان فضاء القياس إلى حد ما تحت أي شرط يكون فضاء القياس إلى حد ما تام أو فضاء بناخ إلى حد ما. كذلك نثبت كل فضاء بناخ هو فضاء بناخ إلى حد ما ولكن العكس غير صحيح .

Abstract:

The aim of this paper is to introduce the concept of a quasi-Banach space for the sequence space ℓ_p, $0 < p < 1$. This concept is based on the important extension of a quasi-normed space concept as defined in [3]. We consider the space of sequence ℓ_p, $0 < p < 1$ and we prove this space is a quasi-normed space but it is not normed space. Thus, we explore many interesting results connected with convergent sequence in a quasi-normed space. We show that, the quasi-normed space under which condition is a complete quasi-normed space or a quasi Banach space. We also show that every Banach space is a quasi Banach space and the converse is not true.

Key words: Sequence space ℓ_p, $0 < p < 1$, Quasi-normed space, Quasi Banach space.

1. Introduction

Functional analysis is a scientific discipline of fairly recent origin. It provides a power full tool to discover solution to problems occurring in pure, applied social sciences, for instance physics, engineering, medicine, agro-industries, ecology, economics and bio-mathematics [2, 7].

One of the important notions in functional analysis is the concept of Banach space. This concept was introduced by Polish mathematician Stefan Banach in 1922 and has received much attention in the literatures ([4, 6] and references therein).

The purpose of this paper is to introduce the notion of a quasi-Banach space for sequence space ℓ_p where $0 < p < 1$. This paper is organised as follows. Section 2 devotes an introduction of the sequence space ℓ_p, $0 < p < 1$.We give the definition of sequence space ℓ_p, where $0 < p < 1$, and we show that it is not normed space. In section 3 we prove that, this space is a quasi-normed space and we give some interesting results concerning this notion. In the last section, we study the convergence and completeness sequence in a quasi-normed space ℓ_p, $0 < p < 1$ in order to show that it is a quasi-Banach space.

2. Sequence space ℓ_p, $0 < p < 1$

In this section, we mention the definition of sequence space ℓ_p, $0 < p < 1$, as well as, some concepts and results related to this space.

Definition 2.1 [1]: The sequence space ℓ_p, $0 < p < 1$, is the space of all sequences $x = \{x_i\}$ in *R* or *C* such that $\sum_{i=1}^{\infty} |x_i|^p < \infty$.

Remark 2.2: The sequence space $\ell_p,\ 0<p<1,$ with the function

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$$

is not a normed space, because the condition (3) of the norm definition [4] is not satisfied. To explain this remark, we consider the following example:

Example 2.3: Let $p=1/2$, and suppose x and y are two sequences, where

$$x = \{x_i\{ = \{0.1, 0, 0, 0, \ldots\{ \text{ and } y = \{y_i\} = \{0, 0.2, 0, 0, \ldots\{.$$

Then we have:

$$\|x+y\|_{1/2} = \left(\sum_{i=1}^{\infty} |x_i+y_i|^{1/2}\right)^2 = 0.5828424, \text{ and}$$

$$\|x\|_{1/2} + \|y\|_{1/2} = \left(\sum_{i=1}^{\infty} |x_i|^{1/2}\right)^2 + \left(\sum_{i=1}^{\infty} |y_i|^{1/2}\right)^2 = 0.3$$

It is clear that:

$$\|x+y\|_{1/2} > \|x\|_{1/2} + \|y\|_{1/2}$$

Thus, the space $\ell_p,\ 0<p<1,$ is not a normed space.

Lemma 2.4 [5]: Let $\{x_i\{$ be any (real or complex) sequence and $0<p\le 1$.

Then,

$$\left|\sum_{i=1}^{\infty} x_i\right|^p \le \sum_{i=1}^{\infty} |x_i|^p$$

3. Quasi-normed space for the space $\ell_p,\ 0<p<1$

In this section, we introduce the concept of a quasi-normed space for ℓ_p, $0<p<1$, i.e., we extend the concept of a quasi-normed space as in **[**3**]** to the case of sequence space $\ell_p,\ 0<p<1$**.** In this paper, we usually consider the space $X = \ell_p,\ 0<p<1$.

Definition 3.1: Let $X = \ell_p,\ 0 < p < 1.$ is a vector space over a field F. A quasi-norm on * is a function $_q\|.\| : X \longrightarrow R_{+0}$ such that:

(1) $_q\| x \| \geq 0 \qquad \forall x \in X,\ _q\| x \| = 0$ if and only if $x = 0$,

(2) $_q\| \lambda x \| = | \lambda |_q\| x \| \qquad \forall x \in X, \forall \lambda \in F$

(3) There exists a constant $\gamma \geq 1$ such that,

$$_q\| x + y \| \leq \gamma \left(_q\| x \| + {}_q\| y \| \right) \qquad \forall x, y \in X$$

The pair $\left(X, {}_q\| . \|\right)$ is called a quasi-normed space. We say simply that $_q X$ is a quasi-normed space.

Proposition 3.2: The sequence space $\ell_p,\ 0 < p < 1,$ with:

$$_q\| x \|_p = \left(\sum_{i=1}^{\infty} | x_i |^p \right)^{1/p} \quad \text{is a quasi-normed space.}$$

Proof: We must satisfy the three conditions of definition 3.1

(1) Since $|\ x_i\ | \geq 0 \ \ \forall i \Rightarrow {}_q\| x \|_p = \left(\sum_{i=1}^{\infty} | x_i |^p \right)^{1/p} \geq 0 \qquad \forall x \in \ell_p$

and

$$_q\| x \|_p = 0 \Leftrightarrow x_i = 0 \ \ \forall i \ \Leftrightarrow x = 0$$

(2) Now, we have:

$$_q\| \lambda x \|_p = \left(\sum_{i=1}^{\infty} | \lambda x_i |^p \right)^{1/p} = \left(\sum_{i=1}^{\infty} | \lambda |^p | x_i |^p \right)^{1/p}$$

$$= | \lambda |_q\| x \|_p \quad , \forall x \in \ell_{\mathbf{p}}, \quad \lambda \in F$$

(3) Let $x, y \in \ell_p$, where $x = \{ x_i \{$ and $y = \} y_i \{$. Then by using lemma 2.4, we have:

$$_q\| x + y \|_p^p = \sum_{i=1}^{\infty} | x_i + y_i |^p \leq \sum_{i=1}^{\infty} (| x_i |^p + | y_i |^p)$$

$$= {}_q\| x \|_p^p + {}_q\| y \|_p^p$$

This implies that:

$$ {}_q\| x+y \|_p \le \left({}_q\| x \|_p^p + {}_q\| y \|_p^p \right)^{1/p} \tag{3.1} $$

Now,

$$ \left({}_q\| x \|_p^p + {}_q\| y \|_p^p \right)^{1/p} \le \left(2 \max \left\{ {}_q\| x \|_p^p , {}_q\| y \|_p^p \right\} \right)^{1/p} $$

$$ \le 2^{1/p} \left({}_q\| x \|_p + {}_q\| y \|_p \right) $$

Thus,

$$ \left({}_q\| x \|_p^p + {}_q\| y \|_p^p \right)^{1/p} \le 2^{1/p} \left({}_q\| x \|_p + {}_q\| y \|_p \right) \tag{3.2} $$

From the inequality (3.1) and (3.2), we obtain:

$$ {}_q\| x+y \|_p \le 2^{1/p} \left({}_q\| x \|_p + {}_q\| y \|_p \right) $$

Hence the space $\ell_p ,\ 0 < p < 1$ is a quasi-normed space.

Remark 3.3: From the proposition 3.2, we note that a constant γ in the definition 3.1 with $\ell_p ,\ 0 < p < 1$, can be taken to be $2^{1/p}$ **.**

Remark 3.4: According to the definition 3.1 and the definition of the norm as in [4], a normed space $\ell_p , 1 \le p < \infty$, with the norm

$$ \| x \|_p = \left(\sum_{i=1}^{\infty} | x_i |^p \right)^{1/p} $$

is a quasi-normed space. Conversely, in general, is not true and is true only if $\gamma = 1$, as it is shown in the following example:

Example 3.5: Consider the sequence space $\ell_p ,\ 0 < p < 1$, then we can deduce that, from the remark 2.2 and the proposition 3.2, this space is not normed space, but it may be a quasi-normed space.

4. Quasi-Banach space for the space $\ell_p ,\ 0 < p < 1$

In this section, we introduce the notion of a quasi-Banach space for the space ℓ_p, $0<p<1$. Then, we study the convergent sequence in the quasi-normed space.

4.1 Convergent sequence in the space ℓ_p, $0<p<1$

This sub-section is to link the relation between the notion of a quasi-convergent sequence and a quasi-Cauchy sequence.

Definition 4.1: Let ${}_qX=\ell_p$, $0<p<1$ be a quasi-normed space.

(1) A sequence $\{x_n\{$ in ${}_qX$ is called a quasi-convergent to a point $x\in {}_qX$ if and only if

${}_q\| x_n - x\| \to 0$ as $n\to\infty$

(2) A sequence $\{x_n\{$ in ${}_qX$ is a quasi-Cauchy sequence if and only

${}_q\| x_n - x_m \| \to 0$ as $n,m\to\infty$.

Lemma 4.2: Let ${}_qX$ be a quasi-normed space. Then every a quasi-convergent sequence is a quasi-Cauchy sequence, but not conversely in general.

Proof: Suppose $\{x_n\{$ is a quasi-convergent to a point x in ${}_qX$, then ${}_q\| x_n - x\| \to 0$ as $n\to\infty$. By condition (3) of the quasi-norm, we have:

$$ {}_q\| x_n - x_m \| = {}_q\| x_n - x + x - x_m \| $$

$$ \le \gamma \left({}_q\| x_n - x\| + {}_q\| x_m - x\| \right) \to 0 \text{ as } n,m\to\infty. $$

Hence, $\{x_n\{$ is a quasi-Cauchy sequence. For the converse, we take the following example:

Example 4.3: Let X = C[-1,1] which is a vector space of all continuous functions defined on [-1,1] with the following a quasi-norm:

$$_q\| f \| = \left(\int_{-1}^{1} | f(x) |^2 \, dx \right)^{1/2} \qquad \left(f \in C[-1,1] \right)$$

Consider the sequence $\{ f_n \{$ in C [-1,1], defined as follows:

$$f_n(x) = \begin{cases} 0 & , \ -1 \le x \le 0 \\ nx & , \ 0 < x \le 1/n \\ 1 & , \ 1/n < x \le 1 \end{cases}$$

Then $\{ f_n(x) \}$ is a quasi-Cauchy sequence in space C [-1,1], but it is not a quasi-convergent to an element of this space [6].

Definition 4.4: A sequence $\{x_n\{$ in a quasi-normed space $_qX$ is called a quasi-bounded sequence if and only if there exists a positive real number M such that $_q\| x_n \| \le M$ for all $n \in N$.

Theorem 4.5: Let $_qX$ be a quasi-normed space, then

(1) Every a quasi-Cauchy sequence is a quasi-bounded.

(2) A quasi-convergent sequence has a unique limit.

(3) A quasi-Cauchy sequence is a quasi-convergent if and only if it has a quasi-convergent sub-sequence.

Proof: The proof of this theorem is very technical and can be found in [6] with minor different.

Definition 4.6: Let $\{ x_n \}$ be a sequence in a quasi-normed space X .

(1) A series $\sum_{n=1}^{\infty} x_n$ is called a quasi-convergent to $s \in X$ if and only if

$$_q\| S_n - s \| \to 0 \qquad (n \to \infty),$$

where s is the limit of $\{ S_n \}$. Otherwise, it is a quasi-divergent.

(3) A series $\sum_{n=1}^{\infty} x_n$ is called absolutely quasi-convergent if and only if

$$\sum_{n=1}^{\infty} {}_q\| x_n \| < \infty$$

4.2 Completeness in the space $\ell_p ,\ 0 < p < 1$

In this subsection, we prove that, a quasi-Cauchy sequence is a quasi-convergent sequence.

Definition 4.8: A quasi-normed space, in which every a quasi-Cauchy sequence is a quasi-convergent, is called a quasi-Banach space.

Remark 4.9: Now we can deduce that, the Banach space $\ell_p, 1 \le p < \infty$ is a quasi-Banach space. But the converse may not be true, as it is shown in the following example:

Example 4.10: The sequence space $\ell_p, 0 < p < 1$ is a quasi-Banach space, which will be proved in the theorem 4.12. But, it is not a Banach space by remark 2.2.

Theorem 4.11: A quasi-normed space ${}_q X$ is a complete if and only if every absolutely a quasi-convergent series in ${}_q X$ is also a quasi-convergent in ${}_q X$.

Proof: Let ${}_qX$ be a complete and $\sum_{n=1}^{\infty} {}_q\| x_n \| < \infty$. Then for $n > m$ we have

$${}_q\| S_n - S_m \| = {}_q\| x_{m+1} + \ldots + x_n \| \le \gamma \left({}_q\| x_{m+1} \| + \cdots + {}_q\| x_n \| \right), \text{ where } \gamma \ge 1.$$

This implies that ${}_q\| S_n - S_m \| \to 0$ ($m \to \infty$). Hence, $\{ S_n \{$ is a Cauchy sequence in ${}_qX$ and so a quasi-convergent, since X is a quasi-complete. Thus $\sum_{n=1}^{\infty} x_n$ is a quasi-convergent.

Conversely, let every absolutely a quasi-convergent series be a quasi-convergent, and let $\{ x_n \{$ be any a quasi-Cauchy sequence in ${}_qX$. Then we may determine natural numbers $n_1, n_2, \ldots$ with $n_1, n_2, \ldots$ such that:

$${}_q\| x_{n_{k+1}} - x_{n_k} \| < 2^{-k} \quad \text{for all } k \in N, \text{ hence } \sum_{k=1}^{\infty} {}_q\| x_{n_{k+1}} - x_{n_k} \| < \infty$$

Our assumption implies that $\sum_{k=1}^{\infty} (x_{n_{k+1}} - x_{n_k})$ converges to an element of ${}_qX$, where the partial sums of this series is equal to $x_{n_{k+1}} - x_{n_1}$. Thus we see that $\{x_{n_k}\}$ is a quasi-convergent in ${}_qX$. Therefore, the quasi-Cauchy sequence $\{ x_n \{$ has a quasi-convergent sub-sequence $\{x_{n_k}\}$. So by theorem 4.5 (3), we have $\{ x_n \{$ is a quasi-convergent in ${}_qX$. Hence, the space sequence ${}_qX$ is a complete.

Theorem 4.12: The sequence space ℓ_p, $0 < p < 1$, is a quasi-Banach space.

Proof: From the proposition 3.4, the space ℓ_p, $0 < p < 1$, is a quasi-normed space.

It remains to prove that ℓ_p, $0<p<1$, is a complete. Let $\{ x_n \{$ be a quasi-Cauchy sequence in ℓ_p, $0<p<1$, with $x_n = \left(x_1^{(n)}, x_2^{(n)}, \ldots\right)$. For each fixed k, $\left\{x_k^{(n)}\right\}$ is a quasi-Cauchy sequence, because

$$|\, x_k^{(n)} - x_k^{(m)} \,| \ \le \left(\sum_{i=1}^{\infty} |\, x_i^{(n)} - x_i^{(m)} \,|^p \right)^{1/p} =_q \| x_n - x_m \|_p \quad \text{for all } n, m \ge M$$

Let $x_k = \lim_{n\to\infty} x_k^{(n)}$. We first prove that the sequence $\{x_k\}$ is an element of the space ℓ_p. From the theorem 4.5 (1), we have $_q\| x_n \|_p \le M$ where $M > 0$. Now, for any k,

$$\left(\sum_{i=1}^{k} |\, x_i^{(n)} \,|^p \right)^{1/p} \le {}_q\| x_n \|_p \le M .$$

Now if $n \to \infty$, we obtain:

$$\left(\sum_{i=1}^{k} |\, x_i \,|^p \right)^{1/p} \le M .$$

Since k is arbitrary, this shows that$\}\ x_k \{ \in \ell_p$, $0<p<1$, and that its quasi-norm does not exceed M. Let $x = x_k$. It remains to prove that $_q\| x_n - x \|_p \to 0$. Since $\{ x_n \ \{$is a quasi-Cauchy sequence, for every $\varepsilon > 0$ there exists a positive integer N such that $_q\| x_n - x_m \|_p < \varepsilon$ for all $n, m \ge N$. Therefore, for any k,

$$\left(\sum_{i=1}^{k} |\, x_i^{(n)} - x_i^{(m)} |^p \right)^{1/p} \le {}_q\| x_n - x_m \|_p < \varepsilon \quad \text{for all } n, m \ge N$$

Keeping k and n fixed, let $m \to \infty$. This gives:

$$\left(\sum_{i=1}^{k} |\, x_i^{(n)} - x_i \,|^p \right)^{1/p} < \varepsilon \qquad \text{for all } n \ge N$$

Since this is true for all k, we can let R_{+0} and we obtain the result that

$$_q\| x_n - x \|_p < \varepsilon \qquad \text{for all} \quad n \geq N$$

Thus the space ℓ_p, $0 < p < 1$, is complete, and hence it is a quasi-Banach space.

5. Conclusion

In this paper, we have explored the notion of a quasi-Banach space for the space ℓ_p, $0 < p < 1$. Then, we prove that the sequence space ℓ_p, $0 < p < 1$, is not normed space, but may be a quasi-normed space. We present and examine many results concerning this notion. We have studied a quasi-convergent and a quasi-Cauchy sequence. Finally, we have shown that, there exists a space which is not Banach space, but may be a quasi-Banach.

References

[1] Aliprantis C. and Burkinshaw O., "Principles of Real Analysis", Elsevier North Holland, Inc. 1998.

[2] Arnold D. and Yokoyama K., "The Leslie Matrix"., Math 45-linear Algebra, David-Arnold @ Eurka. redwoods.cc.ca.us, pp. 1-11.

[3] Bastero J., Bernues J. and Pina A., "An Extension of Milmans Reverse Burn- Minkowski Inequality", Arxiv: math., FA / 9501210 V1, 1995.

[4] Curtain R. F. and Pritchard A. J., "Functional Analysis in Modern Applied Mathematics", Academic Press, Inc., London, 1977.

[5] Maddox I. J., "Elements of Functional Analysis", Cambridge University Press, New Delhi, India, 1988.

[6] Siddiqi A. H., "Functional Analysis With Applications", Tata MacGraw-Hill Publishing Company, Ltd, New Delhi, India, 1986.

[7] Sykes Z. M., "On Discrete Stable Population Theory", Biometrics, 25, PP. 285-293, 1969.