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MAJORIZATION PROPERTIES FOR SUBCLASS OF ANALYTIC P-VALENT FUNCTIONS ASSOCIATED WITH GENERALIZED DIFFERENTIAL OPERATOR INVOLVING MITTAG-LEFFLER FUNCTION

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Abstract. A new class in the open unit disc of analytic p-valent functions is introduced in this paper. This subclass $\mathcal{S}_{\lambda,p}^{m,j}(a_1, b_1, \alpha, \beta, A, B, \gamma)$ is mainly defined by the generalized hypergeometric function. The majorization properties for the functions in this class are introduced. Moreover, we investigate the coefficient estimates for this class.

1. Introduction and Preliminaries

We begin by letting $U = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk of the complex plane and $\mathcal{A}_p$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \qquad (p \in \mathbb{N} = \{1, 2, 3, ...\}), \tag{1.1}$$

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which are analytic and p-valent in U. For simplicity, we write $\mathcal{A}_1 = \mathcal{A}$. The Hadamard product (or convolution) $f * g$ for two analytic functions f defined in (1.1) and

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}$$

is given by

$$f(z) * g(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k}.$$

Let f and g be two analytic functions in U. Then we say that f is majorized by g in U (see [9]) and write

$$f(z) \ll g(z) \qquad (z \in U), \tag{1.2}$$

if there exists an analytic function $\phi(z)$ in U such that

$$|\phi(z)| \leq 1, \quad f(z) = \phi(z) g(z) \qquad (z \in U). \tag{1.3}$$

It may be noted here that (1.2) is closely related to the concept of quasi-subordination between analytic functions.

Given two analytic functions f and g in U, the subordination between them is written as $f \prec g$ or $f(z) \prec g(z)$, that is, we say $f(z)$ is subordinate to $g(z)$ if there is a Schwarz function w with $w(z) = 0$, $|w(z)| < 1$, $(z \in U)$ such that $f(z) = g(w(z))$ for all $z \in U$. Furthermore, if $g(z)$ is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

El-Ashwah [6] studied the p-valent function $\mathcal{H}_p(a_1, b_1; z)$, which defined by generalized hypergeometric function as follows:

$$\mathcal{H}_p(a_1, b_1; z) = z^p + \sum_{k=1}^{\infty} \frac{(a_1)_k \ldots (a_r)_k}{(b_1)_k \ldots (b_s)_k} \frac{z^{p+k}}{k!}, \quad p \in \mathbb{N} \tag{1.4}$$

where $a_i \in \mathbb{C}, b_n \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}, (i = 1, \ldots, r, n = 1, \ldots, s)$, and $r \leq s + 1; r, s \in \mathbb{N}_0$, and $(\upsilon)_k$ is the Pochhammer symbol defined by

$$(\upsilon)_k = \frac{\Gamma(\upsilon + k)}{\Gamma(\upsilon)} = \begin{cases} \upsilon(\upsilon+1)\ldots(\upsilon+k-1), & k = 1, 2, 3, \ldots, \\ 1, & k = 0. \end{cases}$$

The following defines the familiar Mittag-Leffler function $E_\alpha(z)$ which is introduced by Mittag-Leffler [10] and [11] and its generalization $E_{\alpha,\beta}(z)$ is introduced by Wiman [21]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},$$

and

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},$$

where $\alpha,\beta\in\mathbb{C}$, $Re(\alpha)>0$ and $Re(\beta)>0$.

As a result, a lot of useful work have been made by many researchers in attempt to explain Mittag-Leffler function and its generalization, for examples, see [3], [14], [18], [19] and [20].

Corresponding to $E_{\alpha,\beta}(z)$, we define the function $Q_{\alpha,\beta}(z)$ by

$$\begin{aligned} Q_{\alpha,\beta}(z) &= z\Gamma(\beta)E_{\alpha,\beta}(z) \\ &= z+\sum_{k=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)}z^k. \end{aligned}$$

Now, for $f\in\mathcal{A}$ we define the following differential operator: $D_\lambda^m(\alpha,\beta)f:\mathcal{A}\longrightarrow\mathcal{A}$ by

$$D_\lambda^0(\alpha,\beta)f(z)=f(z)*Q_{\alpha,\beta}(z), \tag{1.5}$$

$$D_\lambda^1(\alpha,\beta)f(z)=(1-\lambda)(f(z)*Q_{\alpha,\beta}(z))+\lambda z(f(z)*Q_{\alpha,\beta}(z))' \tag{1.6}$$

$$\vdots$$

$$D_\lambda^m(\alpha,\beta)f(z)=D_\lambda^1(D_\lambda^{m-1}(\alpha,\beta)f(z)) \tag{1.7}$$

If f in $\mathcal{A}$, then from (1.6) and (1.7) we see that

$$D_\lambda^m(\alpha,\beta)f(z)=z+\sum_{k=2}^{\infty}[1+(k-1)\lambda]^m\frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)}a_kz^k. \tag{1.8}$$

Now, we define the operator $D_\lambda^m(\alpha,\beta)f(z)$ in (1.8) of a function $f\in\mathcal{A}_p$ given by (1.1) as

$$D_{\lambda,p}^m(\alpha,\beta)f(z)=z^p+\sum_{k=1}^{\infty}\left[\frac{p+k\lambda}{p}\right]^m\frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)}a_{p+k}z^{k+p},\qquad p\in\mathbb{N}, \tag{1.9}$$

where $m\in\mathbb{N}_0,\lambda\geq 0$.

Corresponding to $\mathcal{H}_p(a_1,b_1;z)$ which defined in (1.4), $D_{\lambda,p}^m(\alpha,\beta)f(z)$ defined in (1.9) and using Hadamard product, we define a new generalized derivative operator $\widetilde{\mathfrak{D}}_{\lambda,p}^m(\alpha,\beta,a_1,b_1)f(z)$ as follows:

Definition 1.1. Let $f \in \mathcal{A}_p$. Then the generalized derivative operator $\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\alpha,\beta,a_1,b_1)f(z) : \mathcal{A}_p \to \mathcal{A}_p$ is given by

$$\begin{aligned} &\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\alpha,\beta,a_1,b_1)f(z) \\ &= \mathcal{H}_p(a_1,b_1;z) * D^{m}_{\lambda,p}(\alpha,\beta)f(z) \\ &= z^p + \sum_{k=1}^{\infty}\left[\frac{p+k\lambda}{p}\right]^m \frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)} \frac{(a_1)_k ... (a_r)_k}{(b_1)_k ... (b_s)_k} \frac{a_{p+k} z^{p+k}}{k!}. \end{aligned} \tag{1.10}$$

We can easily verify from (1.10) that

$$\begin{aligned} p\widetilde{\mathfrak{D}}^{m+1}_{\lambda,p}(\alpha,\beta,a_1,b_1)f(z) &= (p-p\lambda)\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\alpha,\beta,a_1,b_1)f(z) \\ &\quad + \lambda z(\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\alpha,\beta,a_1,b_1)f(z))'. \end{aligned} \tag{1.11}$$

Remark 1.2. It can be seen that

- For $r = 1, s = 0, a_1 = 1, \alpha = 0, \beta = 1$ and $p = 1$, we get Al-Oboudi operator [1].
- For $r = 1, s = 0, a_1 = 1, \alpha = 0, \beta = 1$, $\lambda = 1$ and $p = 1$, we get Sălăgean operator [17].
- For $r = 1, s = 0, a_1 = 1, m = 0$ and $p = 1$, we get $\mathbb{E}_{\alpha,\beta}(z)$ [19].
- For $m = 0, \alpha = 0$ and $\beta = 1$, we get the operator studied by El-Ashwah [6].
- For $m = 0, \alpha = 0, \beta = 1, r = 1, s = 0, a_1 = \delta + 1$ and $p = 1$, we obtain the operator introduced by Ruscheweyh [16].
- For $m = 0, \alpha = 0, \beta = 1, r = 2, s = 1$ and $p = 1$, we obtain the operator which was given by Hohlov [8].
- For $m = 0, \alpha = 0, \beta = 1, r = 2, s = 1, a_{2=1}$ and $p = 1$, we obtain the operator was given by Carlson and Shaffer [4].
- For $m = 0, \alpha = 0, \beta = 1$ and $p = 1$ we obtain the operator studied by Dziok and Srivastava [5].

Next, as a result of full utilization of differential operator $\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\alpha,\beta,a_1,b_1)f(z)$, we define and study the class $\mathcal{S}^{m,j}_{\lambda,p}(a_1,b_1,\alpha,\beta,A,B,\gamma)$ as follows:

Definition 1.3. Let $f \in \mathcal{A}_p$. Then $f \in \mathcal{S}^{m,j}_{\lambda,p}(a_1,b_1,\alpha,\beta,A,B,\gamma)$ of p-valent functions of complex order $\gamma \neq 0$ in U, if it satisfies the condition

$$\left\{1 + \frac{1}{\gamma}\left(\frac{z\left(\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\alpha,\beta,a_1,b_1)f(z)\right)^{(j+1)}}{\left(\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\alpha,\beta,a_1,b_1)f(z)\right)^{j}} - p + j\right)\right\} \prec \frac{1+Az}{1+Bz}, \quad (z \in U) \tag{1.12}$$

where $p \in \mathbb{N}, m, j \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\}, \gamma \in \mathbb{C} \setminus \{0\}, \lambda \geq 0, -1 \leq B < A \leq 1, a_i \in \mathbb{C}, b_n \in \mathbb{C} \setminus \{0, -1, -2, ...\}, (i = 1, ..., r, n = 1, ..., s)$, and $r \leq s + 1; r, s \in \mathbb{N}_0$.

Remark 1.4. It can be seen that, by specializing the parameters, the class $\mathcal{S}_{\lambda,p}^{m,j}(a_1, b_1, \alpha, \beta, A, B, \gamma)$ is reduced to numerous known subclasses of analytic functions, for examples:

- when $m = 0, \alpha = 0, \beta = 1, p = 1, j = 0, r = 2, s = 1, a_1 = b_1, a_2 = 1, A = 1$ and $B = -1$, then the class $\mathcal{S}_{\lambda,p}^{m,j}(a_1, b_1, \alpha, \beta, A, B, \gamma)$ reduces to the class $\mathcal{S}_\gamma$.
- when $m = 0, \alpha = 0, \beta = 1, p = 1, j = 1, r = 2, s = 1, a_1 = b_1, a_2 = 1, A = 1$ and $B = -1$, then the class $\mathcal{S}_{\lambda,p}^{m,j}(a_1, b_1, \alpha, \beta, A, B, \gamma)$ reduces to the class $\mathcal{C}_\gamma$.
- when $m = 0, \alpha = 0, \beta = 1, p = 1, j = 0, r = 2, s = 1, a_1 = b_1, a_2 = 1, A = 1, B = -1$ and $\gamma = 1 - \delta$ then the class $\mathcal{S}_{\lambda,p}^{m,j}(a_1, b_1, \alpha, \beta, A, B, \gamma)$ reduces to the class $\mathcal{S}^*(\delta)$.

The classes $\mathcal{S}_\gamma$ and $\mathcal{C}_\gamma$ are the classes of starlike and convex of complex of order $\gamma \neq 0$ in U introduced by Nasr and Aouf [12] and the class $\mathcal{S}^*(\delta)$ denote the class of starlike functions of order δ in U (see [15]).

2. Main Results

In our first theorem, we begin with majorization problem for functions belonging to the class $\mathcal{S}_{\lambda,p}^{m,j}(a_1, b_1, \alpha, \beta, A, B, \gamma)$.

Theorem 2.1. *Let $f \in \mathcal{A}_p$ and suppose that $g \in \mathcal{S}_{\lambda,p}^{m,j}(a_1, b_1, \alpha, \beta, A, B, \gamma)$. If $\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(\alpha, \beta, a_1, b_1)f(z)\right)^{(j)}$ is majorized by $\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(\alpha, \beta, a_1, b_1)g(z)\right)^{(j)}$ in U, then*

$$\left|\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m+1}(\alpha, \beta, a_1, b_1)f(z)\right)^{(j)}\right| \leq \left|\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m+1}(\alpha, \beta, a_1, b_1)g(z)\right)^{(j)}\right| \quad for \quad |z| \leq r_0, \tag{2.1}$$

where $r_0 = r_0(p, \lambda, \gamma)$ is the smallest positive root of the equation

$$\begin{aligned} &r^3 \left|\gamma(A - B) + \left(\frac{p}{\lambda}\right) B\right| - \left|\left(\frac{p}{\lambda}\right) + 2|B|\right| r^2 \\ &- \left[\left|\gamma(A - B) - \left(\frac{p}{\lambda}\right) B\right| + 2\right] r + \left(\frac{p}{\lambda}\right) = 0, \end{aligned} \tag{2.2}$$

for $-1 \leq B < A \leq 1$; $\lambda \geq 0$; $p \in \mathbb{N}$; $\gamma \in \mathbb{C} \setminus \{0\}$.

Proof. Since $g \in \mathcal{S}^{m,j}_{\lambda,p}(a_1, b_1, \alpha, \beta, A, B, \gamma)$ we find from (1.12) that

$$1 + \frac{1}{\gamma}\left(\frac{z\left(\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\alpha,\beta,a_1,b_1)g(z)\right)^{(j+1)}}{\left(\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\alpha,\beta,a_1,b_1)g(z)\right)^{(j)}} - p + j\right) = \frac{1+Aw(z)}{1+Bw(z)}, \tag{2.3}$$

where $\gamma \in \mathbb{C}\setminus\{0\}, j, p \in \mathbb{N}$ and $w(z) = d_1 z + d_2 z^2 + ..., w \in \mathcal{P}$, $\mathcal{P}$ denotes the well-known class of bounded analytic functions in U (see Goodman [7]) with

$$w(0) = 0, \quad |w(z)|<1 \quad (z \in U).$$

From (2.3), we get

$$\frac{z\left(\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\alpha,\beta,a_1,b_1)g(z)\right)^{(j+1)}}{\left(\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\alpha,\beta,a_1,b_1)g(z)\right)^{(j)}} = \frac{(p-j) + \left[\gamma(A-B) + (p-j)B\right]w(z)}{1+Bw(z)}. \tag{2.4}$$

It follows from (1.11) that

$$\begin{aligned} z(\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\alpha,\beta,a_1,b_1)f(z))^{(j+1)} &= \frac{p}{\lambda}\left(\widetilde{\mathfrak{D}}^{m+1}_{\lambda,p}(\alpha,\beta,a_1,b_1)f(z)\right)^{(j)} \\ &\quad + \left(p - j - \frac{p}{\lambda}\right)\left(\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\alpha,\beta,a_1,b_1)f(z)\right)^{(j)}. \end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5), we can get

$$\begin{aligned} &\left|\left(\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\alpha,\beta,a_1,b_1)g(z)\right)^{(j)}\right| \\ &\qquad \le \frac{\left(\frac{p}{\lambda}\right)\left[1+|B||z|\right]}{\frac{p}{\lambda} - \left|\gamma(A-B) + \left(\frac{p}{\lambda}\right)|B|\right||z|}\left|\left(\widetilde{\mathfrak{D}}^{m+1}_{\lambda,p}(\alpha,\beta,a_1,b_1)g(z)\right)^{(j)}\right|. \end{aligned} \tag{2.6}$$

Next, since $\left(\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\alpha,\beta,a_1,b_1)f(z)\right)^{(j)}$ is majorized by $\left(\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\alpha,\beta,a_1,b_1)g(z)\right)^{(j)}$ in U, it follows from (1.3) that

$$\left(\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\alpha,\beta,a_1,b_1)f(z)\right)^{(j)} = \phi(z)\left(\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\alpha,\beta,a_1,b_1)g(z)\right)^{(j)}. \tag{2.7}$$

Differentiating (2.7) with respect to z and multiplying by z, we get

$$\begin{aligned} z\left(\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\alpha,\beta,a_1,b_1)f(z)\right)^{(j+1)} &= z\phi'(z)\left(\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\alpha,\beta,a_1,b_1)g(z)\right)^{(j)} \\ &\quad + z\phi(z)\left(\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\alpha,\beta,a_1,b_1)g(z)\right)^{(j+1)}. \end{aligned} \tag{2.8}$$

Now, using (2.5) in (2.8), it yields

$$\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m+1}(\alpha,\beta,a_1,b_1)f(z)\right)^{(j)} = \frac{z\phi^{'}(z)\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(\alpha,\beta,a_1,b_1)g(z)\right)^{(j)}}{p/\lambda} + \phi(z)\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m+1}(\alpha,\beta,a_1,b_1)g(z)\right)^{(j)}. \tag{2.9}$$

Noting that $\phi(z) \in \mathcal{P}$ satisfies the inequality (see [13])

$$|\phi^{'}(z)| \leq \frac{1-|\phi(z)|^2}{1-|z|^2} \quad (z \in U), \tag{2.10}$$

and making use of (2.6) and (2.10) in (2.9), we get

$$\begin{aligned}&\left|\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m+1}(\alpha,\beta,a_1,b_1)f(z)\right)^{(j)}\right| \\ &\leq \left[|\phi(z)| + \frac{1-|\phi(z)|^2}{1-|z|^2}\frac{|z|(1+|B||z|)}{\left(\frac{p}{\lambda}\right) - \left|\gamma(A-B)+\left(\frac{p}{\lambda}\right)|B|\right||z|}\right] \\ &\quad\times\left|\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m+1}(\alpha,\beta,a_1,b_1)g(z)\right)^{(j)}\right|.\end{aligned} \tag{2.11}$$

Let $|z| = r$ and $|\phi(z)| = \rho$, $(0 \leq \rho \leq 1)$. Then we have

$$\begin{aligned}&\left|\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m+1}(\alpha,\beta,a_1,b_1)f(z)\right)^{(j)}\right| \\ &\leq \frac{\psi(\rho)}{(1-r^2)\left[\left(\frac{p}{\lambda}\right) - \left|\gamma(A-B)+\left(\frac{p}{\lambda}\right)B\right|r\right]}\left|\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m+1}(\alpha,\beta,a_1,b_1)g(z)\right)^{(j)}\right|,\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}\psi(p) = &-r(1+|B|r)\rho^2 + (1-r^2)\left[\left(\frac{p}{\lambda}\right) - \left|\gamma(A-B)+\left(\frac{p}{\lambda}\right)B\right|r\right]\rho \\ &+ r(1+|B|r)\end{aligned} \tag{2.13}$$

takes its maximum value at $\rho = 1$, with $r_0 = r_0(p,\lambda,\gamma)$ and r_0 is the smallest positive root of (2.2). Moreover, if $0 \leq \vartheta \leq r_0$, then the function $\chi(\rho)$ defined by

$$\begin{aligned}\chi(\rho) = &-\vartheta(1+|B|\vartheta)\rho^2 + (1-\vartheta^2)\times\left[\left(\frac{p}{\lambda}\right) - \left|\gamma(A-B)+\left(\frac{p}{\lambda}\right)B\right|\vartheta\right]\rho \\ &+ \vartheta(1+|B|\vartheta)\end{aligned} \tag{2.14}$$

is an increasing function on the interval $0 \leq \rho \leq 1$, so that

$$\chi(\rho) \leq \chi(1) = (1-\vartheta^2)\left[\left(\frac{p}{\lambda}\right) - \left|\gamma(A-B) + \left(\frac{p}{\lambda}\right)B\right|\vartheta\right]\rho,$$

$0 \leq \vartheta \leq r_0$, $0 \leq \rho \leq 1$. Hence, setting $\rho = 1$ in (2.12), we conclude that (2.1) of Theorem 2.1 holds true for

$$|z| \leq r_0 = r_0(p,\lambda,\gamma),$$

where $r_0(p,\lambda,\gamma)$ is the smallest positive root of (2.2). This completes the proof of Theorem 2.1. □

Putting $A = 1$ and $B = -1$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.2. *Let $f \in \mathcal{A}_p$ and suppose that $g \in \mathcal{S}^{m,j}_{\lambda,p}(a_1, b_1, \alpha, \beta, \gamma)$. If $\left(\widetilde{\mathfrak{D}}^m_{\lambda,p}(\alpha,\beta,a_1,b_1)f(z)\right)^{(j)}$ is majorized by $\left(\widetilde{\mathfrak{D}}^m_{\lambda,p}(\alpha,\beta,a_1,b_1)g(z)\right)^{(j)}$ in U, then*

$$\left|\left(\widetilde{\mathfrak{D}}^{m+1}_{\lambda,p}(\alpha,\beta,a_1,b_1)f(z)\right)^{(j)}\right| \leq \left|\left(\widetilde{\mathfrak{D}}^{m+1}_{\lambda,p}(\alpha,\beta,a_1,b_1)g(z)\right)^{(j)}\right| \quad for \quad |z| \leq r_0,$$

where

$$r_0 = r_0(p,\lambda,\gamma) = \frac{l - \sqrt{l^2 - 4\left(\frac{p}{\lambda}\right)\left|2\gamma - \frac{p}{\lambda}\right|}}{2\left|2\gamma - \frac{p}{\lambda}\right|}$$

and

$$l = 2 + \left(\frac{p}{\lambda}\right) + \left|2\gamma - \frac{p}{\lambda}\right|, (\lambda \geq 0; p \in \mathbb{N}; \gamma \in \mathbb{C} \setminus \{0\}).$$

Putting $p = 1, m = 0, j = 0, \lambda = 1, \alpha = 0, \beta = 1, r = 2, s = 1, a_1 = b_1$ and $a_2 = 1$ in Corollary 2.2, we get the following corollary:

Corollary 2.3. ([2]) *Let $f \in \mathcal{A}_p$ and suppose that $g \in \mathcal{S}_\gamma$. If $f(z)$ is majorized by $g(z)$ in U, then we have*

$$\left|f^{'}(z)\right| \leq \left|g^{'}(z)\right| \quad for \quad |z| \leq r_0,$$

where

$$r_0 = r_0(\gamma) = \frac{3 + |2\gamma - 1| - \sqrt{9 + 2|2\gamma-1| + |2\gamma-1|^2}}{2|2\gamma - 1|}.$$

For $\gamma = 1$, Corollary 2.3 reduces to the following result:

Corollary 2.4. ([9]) *Let $f \in \mathcal{A}_p$ and suppose that $g \in \mathcal{S}^* = \mathcal{S}^*(0)$. If $f(z)$ is majorized by $g(z)$ in U, then we have*

$$\left|f^{'}(z)\right| \leq \left|g^{'}(z)\right| \quad for \quad |z| \leq 2 - \sqrt{3}.$$

Now, we obtain the coefficient estimate for a function belongs to the class $\mathcal{S}_{\lambda,p}^{m,j}(a_1,b_1,\alpha,\beta,A,B,\gamma)$ when $j=0$.

Definition 2.5. Let $f\in\mathcal{A}_p$, then $f\in\mathcal{S}_{\lambda,p}^{m}(a_1,b_1,\alpha,\beta,A,B,\gamma)$ of p-valent functions of complex order $\gamma\neq 0$ in U, if it satisfies the condition

$$1+\frac{1}{\gamma}\left(\frac{z\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(\alpha,\beta,a_1,b_1)f(z)\right)'}{\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(\alpha,\beta,a_1,b_1)f(z)\right)}-p\right)\prec\frac{1+Az}{1+Bz}, \tag{2.15}$$

where $p\in\mathbb{N}, m\in\mathbb{N}_0=\mathbb{N}\bigcup\{0\}, \gamma\in\mathbb{C}\setminus\{0\}, \lambda\geq 0, -1\leq B<A\leq 1, a_i\in\mathbb{C}, b_n\in\mathbb{C}\setminus\{0,-1,-2,...\}, (i=1,...,r, n=1,...,s)$, and $r\leq s+1; r,s\in\mathbb{N}_0$.

Theorem 2.6. *Let $f\in\mathcal{A}_p$. if f satisfies the condition*

$$\frac{\sum_{k=1}^{\infty}\left[k+|\gamma(A-B)-kB|\right]\left[\frac{p+k\lambda}{p}\right]^m\frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)}\frac{(a_1)_k...(a_r)_k}{(b_1)_k...(b_s)_k k!}|a_{p+k}|}{|\gamma|(A-B)}\leq 1, \tag{2.16}$$

then $f\in\mathcal{S}_{\lambda,p}^{m}(a_1,b_1,\alpha,\beta,A,B,\gamma)$.

Proof. Let $f\in\mathcal{S}_{\lambda,p}^{m}(a_1,b_1,\alpha,\beta,A,B,\gamma)$. Then we can write (2.15) as follows:

$$1+\frac{1}{\gamma}\left(\frac{z\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(\alpha,\beta,a_1,b_1)f(z)\right)'}{\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(\alpha,\beta,a_1,b_1)f(z)\right)}-p\right)=\frac{1+Aw(z)}{1+Bw(z)},$$

which implies

$$\begin{aligned}&\frac{z\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(\alpha,\beta,a_1,b_1)f(z)\right)'}{\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(\alpha,\beta,a_1,b_1)f(z)\right)}-p\\&=\left[\gamma(A-B)-B\left(\frac{z\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(\alpha,\beta,a_1,b_1)f(z)\right)'}{\left(\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(\alpha,\beta,a_1,b_1)f(z)\right)}-p\right)\right]w(z).\end{aligned} \tag{2.17}$$

From (2.17), we obtain

$$\begin{aligned}&\frac{pz^p+\sum_{k=1}^{\infty}\left[\frac{p+k\lambda}{p}\right]^m\frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)}\frac{(a_1)_k...(a_r)_k}{(b_1)_k...(b_s)_k k!}(p+k)a_{p+k}z^{p+k}}{z^p+\sum_{k=1}^{\infty}\left[\frac{p+k\lambda}{p}\right]^m\frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)}\frac{(a_1)_k...(a_r)_k}{(b_1)_k...(b_s)_k k!}a_{p+k}z^{p+k}}-p\\&=\left\{\gamma(A-B)-B\left[\frac{pz^p+\sum_{k=1}^{\infty}\left[\frac{p+k\lambda}{p}\right]^m\frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)}\frac{(a_1)_k...(a_r)_k}{(b_1)_k...(b_s)_k k!}(p+k)a_{p+k}z^{p+k}}{z^p+\sum_{k=1}^{\infty}\left[\frac{p+k\lambda}{p}\right]^m\frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)}\frac{(a_1)_k...(a_r)_k}{(b_1)_k...(b_s)_k k!}a_{p+k}z^{p+k}}-p\right]\right\}w(z),\end{aligned}$$

which yields

$$\frac{\sum_{k=1}^{\infty} k\left[\frac{p+k\lambda}{p}\right]^m \frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)} \frac{(a_1)_k...(a_r)_k}{(b_1)_k...(b_s)_k k!} a_{p+k} z^k}{1+\sum_{k=1}^{\infty} \left[\frac{p+k\lambda}{p}\right]^m \frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)} \frac{(a_1)_k...(a_r)_k}{(b_1)_k...(b_s)_k k!} a_{p+k} z^k}$$

$$= \left\{ \gamma(A-B) - B\left[\frac{\sum_{k=1}^{\infty} k\left[\frac{p+k\lambda}{p}\right]^m \frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)} \frac{(a_1)_k...(a_r)_k}{(b_1)_k...(b_s)_k k!} a_{p+k} z^k}{1+\sum_{k=1}^{\infty} \left[\frac{p+k\lambda}{p}\right]^m \frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)} \frac{(a_1)_k...(a_r)_k}{(b_1)_k...(b_s)_k k!} a_{p+k} z^k}\right]\right\} w(z).$$

Since $|w(z)| \leq 1$,

$$\left|\sum_{k=1}^{\infty} k\left[\frac{p+k\lambda}{p}\right]^m \frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)} \frac{(a_1)_k...(a_r)_k}{(b_1)_k...(b_s)_k k!} a_{p+k} z^k\right|$$

$$\leq \left|\gamma(A-B) - \sum_{k=1}^{\infty} [Bk-\gamma(A-B)] \left[\frac{p+k\lambda}{p}\right]^m \frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)} \frac{(a_1)_k...(a_r)_k}{(b_1)_k...(b_s)_k k!} a_{p+k} z^k\right|.$$

Letting $|z| \to 1^-$ through real values, we have

$$\sum_{k=1}^{\infty} [k + |\gamma(A-B) - kB|] \left[\frac{p+k\lambda}{p}\right]^m \frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)} \frac{(a_1)_k...(a_r)_k}{(b_1)_k...(b_s)_k k!} |a_{p+k}| \leq |\gamma|(A-B).$$

Therefore, we have

$$\frac{\sum_{k=1}^{\infty} [k + |\gamma(A-B) - kB|] \left[\frac{p+k\lambda}{p}\right]^m \frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)} \frac{(a_1)_k...(a_r)_k}{(b_1)_k...(b_s)_k k!} |a_{p+k}|}{|\gamma|(A-B)} \leq 1.$$

This completes the proof of Theorem 2.6. □

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References

[1] F.M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Internat. J. Math. Math. Sci., **27** (2004), 1429–1436.

[2] O. Altintaş, Ö. Özkan and H.M. Srivastava, *Majorization by starlike functions of complex order,* Complex Variables Theory Appl., **46**(3) (2001), 207–218.

[3] A.A. Attiya, *Some applications of Mittag-Leffler function in the unit disk,* Filomat, **30** (2016), 2075–2081.

[4] B.C. Carlson and D.B. Shaffer, *Starlike and prestarlike hypergeometric functions,* SIAM J. Math. Anal., **15** (1984), 737–745.

[5] J. Dziok and H.M. Srivastava, *Classes of analytic functions associated with the generalized hypergeometric function,* Appl. Math. Comput., **103** (1999), 1–13.

[6] R.M. El-Ashwah, *Majorization properties for subclass of analytic p-valent functions defined by the generalized hypergeometric function,* Tamsui Oxf. J. Math. Sci., **28**(4) (2012), 395–405.
[7] A.W. Goodman, *Univalent Functions*, Mariner Publishing Company, Tampa, Florida, 1983.
[8] J.E. Hohlov, O*perators and operations on the class of univalent functions,* Izvestiya Vysshikh Uchebnykh Zavedenii Matematika, **10** (1978), 83–89.
[9] T.H. MacGregor, *Majorization by univalent functions,* Duke Math. J., **34** (1967), 95–102.
[10] G.M. Mittag-Leffler, *Sur la nouvelle fonction* $E_\alpha(x)$*,* CR Acad. Sci. Paris, **137**(2) (1903), 554–558.
[11] G.M. Mittag-Leffler, *Sur la representation analytique d'une branche uniforme d'une fonction monogene,* Acta Mathematica, **29**(1) (1905), 101–181.
[12] M.A. Nasr and M.K. Aouf, *Starlike function of complex order,* J. Nature. Sci. Math., **25** (1985), 1–12.
[13] Z. Nehari, *Conformal Mappings*, McGraw-Hill Book Company, New York, Toronto, London, 1952.
[14] H. Rehman, M. Darus, and J. Salah, *Coefficient properties involving the generalized K-Mittag-Leffler functio*ns, Transyl. Jour. Math. Mech. (TJMM), **9**(2) (2017), 155–164.
[15] M.S. Robertson, *On the theory of univalent functions*, Ann. Math., **37** (1936), 374–408.
[16] S. Ruscheweyh, *New criteria for univalent functions,* Proc. Amer. Math. Soc., **49** (1975), 109–115.
[17] G.S. Sălăgean, *Subclasses of univalent functions,* Lecture Notes in Math., Springer-Verlag, Heidelberg, **1013** (1983), 362–372.
[18] J. Salah and M. Darus, *A note on generalized Mittag-Leffler function and application,* Far East Jour. Math. Sci., **48**(1) (2011), 33–46.
[19] H.M. Srivastava, B.A. Frasin and V. Pescar, U*nivalence of integral operators involving Mittag-Leffler functions*, Appl. Math. Inf. Sci., **11**(3) (2017), 635–641.
[20] H.M. Srivastava and Ž. Tomovski, *Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel*, Appl. Math. Comput., **211** (2009), 198–210.
[21] A. Wiman, *Über den fundamentalsatz in der teorie der funktionen* $E_\alpha(x)$*,* Acta Mathematica, **29**(1) (1905), 191–201.