Results Math (2023) 78:3
Online First

https://doi.org/10.1007/s00025-022-01775-5

Results in Mathematics

E-Univex Sets, E-Univex Functions and E-Differentiable E-Univex Programming

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Abstract. In this paper, we introduce a new concept of sets and a new class of functions called E-univex sets and E-univex functions, respectively. For an E-differentiable function, the concept of E-univexity is introduced by generalizing several concepts of generalized convexity earlier defined into optimization theory. In addition, some properties of E-differentiable E-univex functions are investigated. Further, also concepts of E-differentiable generalized E-univexity are introduced. Then, the sufficiency of the so-called E-Karush–Kuhn–Tucker necessary optimality conditions are proved for an E-differentiable nonlinear optimization problem in which the involved functions are E-univex and/or generalized E-univex.

Mathematics Subject Classification. 90C26, 90C30, 26B25.

Keywords. E-differentiable function, E-univex sets, E-univex functions, E-optimality conditions.

1. Introduction

Convexity and its various generalizations have played an essential role in the development of various fields of applied and pure sciences. In the recent past, convexity and its generalizations have piqued interest and used in establishing optimality conditions of optimization problems (see, for example, [1,2,5,6,9,10,12,15–20,25–29,32], and others).

One of the generalizations of convexity is the concept of univexity introduced by Bector et al. [11] which unifies various concepts of generalized convexity established in literature. In [33], Youness introduced the concept of

Published online: 10 November 2022

E-convexity. This kind of generalized convexity is based on the effect of an operator $E : R^n \to R^n$ on the sets and the domain of functions. In recent years, this concept has attracted great interest from various researchers, and some properties of this concept were developed (see, for example, [3,4,6–8,13,14,21–24,30,31,34], and others). Later, Megahed et al. [23] introduced the definition of an E-differentiable function.

In this paper, the class of pre-E-univexity functions and classes of generalized pre-E-univexity functions are defined for not necessarily E-differentiable functions. Also, a new class of E-differentiable E-univex functions and new concept of E-univex sets are introduced. The concept of E-univexity is introduced by generalizing the concepts of convexity, univexity, B-vexity, invexity, E-B-invexity, E-convexity and E-invexity. Further, some properties of E-univex functions are studied and they are illustrated by some examples of such generalized convex functions. In addition, E-univex functions are extended to pseudo-E-univex, strictly pseudo-E-univex and quasi-E-univex functions. In order to show their applications, the sufficiency of the so-called E-Karush–Kuhn–Tucker necessary optimality conditions are proved for an E-differentiable nonlinear optimization problem in which the involved functions are E-univex and/or generalized E-univex. The aforesaid results are illustrated by a suitable example of a nonconvex optimization problem with (generalized) E-univex functions.

2. Preliminaries

Let R^n be the n-dimensional Euclidean space and R^n_+ be its nonnegative orthant.

Definition 1. A function $\varphi : M \to R$ is said to be increasing on a nonempty set $M \subset R^n$ if and only if for all $x, y \in M$,

$$x \leq y \Longrightarrow \varphi(x) \leq \varphi(y).$$

Definition 2. A function $\varphi : M \to R$ is said to be strictly increasing on a nonempty set $M \subset R^n$ if and only if for all $x, y \in M$,

$$x < y \Longrightarrow \varphi(x) < \varphi(y).$$

Definition 3 [5]**.** Let $E : R^n \to R^n$. A set $M \subseteq R^n$ is said to be an E-invex set with respect to $\eta : M \times M \to R^n$ if and only if the relation

$$E\left(u\right) + \lambda\eta\left(E\left(x\right), E\left(u\right)\right) \in M$$

holds for all $x, u \in M$ and $\lambda \in [0, 1]$.

We recall the definition of a pre-univex function introduced by Bector et al. [11].

Definition 4 [11]**.** Let M be a nonempty invex subset of R^n. A real-valued function $f : M \to R$ is said to be pre-univex on M with respect to $\eta : M \times M \to R^n$, $b : M \times M \times [0,1] \to R_+$ and $\Phi : R \to R$ if and only if the following inequality

$$f\left(u + \lambda\eta\left(x, u\right)\right) \leq f\left(u\right) + \lambda b(x, u, \lambda)\Phi[f\left(x\right) - f\left(u\right)] \tag{1}$$

holds for all $x, u \in M$ and any $\lambda \in [0,1]$.

Now, we introduce the concept of an E-pre-univex function. Let $M \subseteq R^n$ be a nonempty E-invex set.

Definition 5. Let $E : R^n \to R^n$. A real-valued function $f : M \to R$ is said to be E-pre-univex on M with respect to $\eta : M \times M \to R^n$, $b : M \times M \times [0,1] \to R_+$ and $\Phi : R \to R$ if and only if the following inequality

$$f\left(E\left(u\right) + \lambda\eta\left(E\left(x\right), E\left(u\right)\right)\right) - f\left(E\left(u\right)\right) \leq \lambda b(E(x), E(u), \lambda)\Phi[f\left(E\left(x\right)\right) - f\left(E\left(u\right)\right)] \tag{2}$$

holds for all $x, u \in M$ and any $\lambda \in [0,1]$.

It is clear that every pre-univex function is E-pre-univex (if E is the identity map).

Definition 6. Let $E : R^n \to R^n$. A real-valued function $f : M \to R$ is said to be strictly E-pre-univex on M with respect to $\eta : M \times M \to R^n$, $b : M \times M \times [0,1] \to R_+$ and $\Phi : R \to R$ if and only if the following inequality

$$f\left(E\left(u\right) + \lambda\eta\left(E\left(x\right), E\left(u\right)\right)\right) - f\left(E\left(u\right)\right) < \lambda b(E(x), E(u), \lambda)\Phi[f\left(E\left(x\right)\right) - f\left(E\left(u\right)\right)] \tag{3}$$

holds for all $x, u \in M$, $E(x) \neq E(u)$, and any $\lambda \in (0,1)$.

Now, we present an example of such an E-pre-univex function which is not pre-univex.

Example 1. Let $E : R \to R$, $f : R \to R$ and $\Phi : R \to R$ be defined by $f(x) = \cos(x)$, $\Phi(x) = 2x$, $E(x) = \frac{\pi}{2} - x$, where $\eta(E(x), E(u)) = \begin{cases} \frac{\sin(x) - \sin(u)}{\cos(u)} & \text{if } x \geq u \wedge x, u \in (0, \frac{\pi}{2}) \\ 0 & \text{otherewise} \end{cases}$, $b(E(x), E(u), \lambda) = \begin{cases} 1 & \text{if } x \geq u \wedge x, u \in (0, \frac{\pi}{2}) \\ 0 & \text{otherewise} \end{cases}$. Then, by Definition 5, f is an E-pre-univex function on R, while f is not pre-univex, because for $x = \frac{\pi}{3}$, $u = \frac{\pi}{6}$, and $\lambda = \frac{1}{2}$, we have

$$f\left(u + \lambda\eta\left(x, u\right)\right) \approx 0.98 > f\left(u\right) + \lambda b(x, u, \lambda)\Phi[f\left(x\right) - f\left(u\right)] = 0.5.$$

Hence, f is not pre-univex with respect to η, b, and Φ defined above. In other words, the class of E-pre-univex functions is larger than the class of pre-univex functions.

Definition 7. Let $E : R^n \to R^n$. A real-valued function $f : M \to R$ is said to be E-pre-pseudounivex on M with respect to $\eta : M \times M \to R^n$, $b : M \times M \times [0,1] \to R_+$ and $\Phi : R \to R$ if and only if the following relation

$$\begin{aligned}&\Phi[f(E(x)) - f(E(u))] < 0\\&\quad \Longrightarrow b(E(x), E(u), \lambda) f(E(u) + \lambda\eta(E(x), E(u))) \leq b(E(x), E(u), \lambda) f(E(u))\end{aligned} \tag{4}$$

holds for all $x, u \in M$ and any $\lambda \in [0,1]$.

Definition 8. Let $E : R^n \to R^n$. A real-valued function $f : M \to R$ is said to be E-pre-quasiunivex on M with respect to $\eta : M \times M \to R^n$, $b : M \times M \times [0,1] \to R_+$ and $\Phi : R \to R$ if and only if the following relation

$$\begin{aligned}&\Phi[f(E(x)) - f(E(u))] \leq 0\\&\quad \Longrightarrow b(E(x), E(u), \lambda) f(E(u) + \lambda\eta(E(x), E(u))) \leq b(E(x), E(u), \lambda) f(E(u))\end{aligned} \tag{5}$$

holds for all $x, u \in M$ and any $\lambda \in [0,1]$.

We now give the definition of an E-differentiable function introduced by Megahed et al. [23].

Definition 9. [23] Let $E : R^n \to R^n$ and $f : M \to R$ be a (not necessarily) differentiable function at a given point $u \in M$. It is said that f is an E-differentiable function at u if and only if $f \circ E$ is a differentiable function at u (in the usual sense), that is,

$$(f \circ E)(x) = (f \circ E)(u) + \nabla (f \circ E)(u)(x - u) + \theta(u, x - u)\|x - u\|, \tag{6}$$

where $\theta(u, x - u) \to 0$ as $x \to u$.

Now, we introduce new concepts of generalized convexity for E-differentiable functions. Let $M \subseteq R^n$ be a nonempty E-invex set with respect to η.

Definition 10. Let $E : R^n \to R^n$ and $f : M \to R$ be an E-differentiable function at u on M. It is said that f is E-univex at $u \in M$ on M with respect to η, b and Φ if, there exist $\eta : M \times M \to R^n$, $b : M \times M \to R_+$ and $\Phi : R \to R$ such that, for all $x \in M$, the inequality

$$b(E(x), E(u))\Phi[f(E(x)) - f(E(u))] \geq \nabla f(E(u))\eta(E(x), E(u)) \tag{7}$$

holds. If inequality (7) holds for any $u \in M$, then f is E-univex with respect to η, b and Φ on M.

Remark 1. Note that the Definition 10 generalizes and extends several generalized convexity notions, previously introduced in the literature. Indeed, there are the following special cases:

a) If $b(x, u) = 1$ and $\Phi(x) = x$, then the definition of an E-univex function reduces to the definition of an E-invex function introduced by Abdulaleem [5].

b) If f is differentiable and $E(x) \equiv x$ (E is an identity map), then the definition of an E-univex function reduces to the definition of an univex function introduced by Bector et al. [11].
c) If f is differentiable, $E(x) \equiv x$ (E is an identity map), $b(x, u) = 1$ and $\Phi(x) = x$, then the definition of an E-univex function reduces to the definition of an invex function introduced by Hanson [15].
d) If f is differentiable, $E(x) \equiv x$ (E is an identity map) and $\Phi(x) = x$, then the definition of an E-univex function reduces to the definition of a B-invex function introduced by Bector et al. [9] and Suneja et al. [32].
e) If $\Phi(x) = x$, then the definition of an E-univex function reduces to the definition of an E-B-invex function introduced by Abdulaleem [8].
f) If $\eta(x, u) = x - u$, $b(x, u) = 1$ and $\Phi(x) = x$, then we obtain the definition of an E-differentiable E-convex vector-valued function introduced by Megahed et al. [23].
g) If f is differentiable, $E(x) = x$ and $\eta(x, u) = x - u$, $b(x, u) = 1$ and $\Phi(x) = x$, then the definition of an E-univex function reduces to the definition of a differentiable convex vector-valued function.
h) If f is differentiable and $\eta(x, u) = x - u$, $b(x, u) = 1$ and $\Phi(x) = x$, then we obtain the definition of a differentiable E-convex function introduced by Youness [33].

Definition 11. Let $E : R^n \to R^n$ and $f : M \to R$ be an E-differentiable function at u on M. It is said that f is strictly E-univex at $u \in M$ on M with respect to η, b and Φ if, there exist $\eta : M \times M \to R^n$, $b : M \times M \to R_+$ and $\Phi : R \to R$ such that, for all $x \in M$ $(E(x) \neq E(u))$, the inequality

$$b(E(x), E(u))\Phi\left[f(E(x)) - f(E(u))\right] > \nabla f(E(u))\eta(E(x), E(u)) \tag{8}$$

holds. If inequality (8) holds for any $u \in M$ $(E(x) \neq E(u))$, then f is strictly E-univex with respect to η, b and Φ on M.

Now, we present an example of such an E-univex function (with respect to η, b and Φ) which is not univex (with respect to the same given η, b and Φ), B-invex (with respect to the same given η and b), invex (with respect to the same given η) or convex.

Example 2. Let $E : R \to R$ and $f, \Phi : R \to R$ be defined by $f(x) = -x^2$, $\Phi(x) = 2x$, $E(x) = \begin{cases} \sqrt{-x} & \text{if } x \leq 0 \\ 0 & \text{otherwise,} \end{cases}$ $\eta(x, u) = \begin{cases} 2u^2 - 2x^2 & \text{if } x \leq 0,\ u \leq 0 \\ x - u & \text{otherwise} \end{cases}$, $b(x, u) = \begin{cases} 1 & \text{if } x \leq 0,\ u \leq 0 \\ 0 & \text{otherwise} \end{cases}$. Then f is an E-univex function on R with respect to η, b and Φ defined above, but it is not univex with respect to η, b and Φ defined above as can be seen by taking $x = 2$, $u = 4$, since the inequality

$$b(x, u)\Phi\left[f(x) - f(u)\right] < \nabla f(u)\eta(x, u)$$

holds. Moreover, by the definition of an univex function [11], it follows that f is not univex with respect to η, b and Φ given above. Also, f is not B-invex with respect to η, b defined above as can be seen by taking $x = 1$, $u = 2$, since the inequality

$$b(x,u)\,[f(x) - f(u)] < \nabla f(u)\eta(x,u)$$

holds. Note that, by the definition of a B-invex function (see, [9,32]), it follows that f is not B-invex with respect to η, b given above. Further, f is not invex with respect to η defined above as can be seen by taking $x = 1$, $u = 0$, since the inequality

$$f(x) - f(u) < \nabla f(u)\eta(x,u)$$

holds. Further, by the definition of an invex function [15], it follows that f is not invex on R with respect to η given above. It is not difficult to see that f is not a convex function.

Now, we present an example of such an E-differentiable E-univex function (with respect to η, b and Φ) which is not E-invex (with respect to the same given η), E-B-invex (with respect to the same given η and b), or E-convex.

Example 3. Let $E : R \to R$ and $f, \Phi : R \to R$ be defined by $f(x) = \sqrt[3]{x},\ \Phi(x) = 3x,\ E(x) = x^9$, where

$$\eta(E(x),E(u)) = \begin{cases} x^2+xu+u^2 & \text{if } x > u \\ x^2 - u^2 & \text{otherwise,} \end{cases} \quad b(E(x),E(u)) = \begin{cases} \frac{u^2}{x-u} & \text{if } x > u \\ 0 & \text{otherwise} \end{cases}.$$

Note that f is nondifferentiable at $x = 0$, but $f \circ E$ is a differentiable function at $x = 0$. Then, by Definition 10, f is an E-univex function on R with respect to η, b and Φ given above. Moreover, as it follows from the definition of an E-B-invex function [8], f is not an E-B-invex function on R with respect to η and b given above as can be seen by taking $x = 2$, $u = 1$, since the inequality

$$b(E(x),E(u))\,[f(E(x)) - f(E(u))] < \nabla f(E(u))\eta(E(x),E(u))$$

holds. Also, f is not E-invex on R with respect to η defined above as can be seen by taking $x = 0$, $u = -2$, since the inequality

$$f(E(x)) - f(E(u)) < \nabla f(E(u))\eta(E(x),E(u))$$

holds. Note that, by the definition of an E-invex function [5], it follows that f is not E-invex on R with respect to η given above. Further, it is not difficult to see that, by the definition of an E-convex function [33], f is not E-convex at $x = 4$, $u = 1$, since the inequality

$$f(E(x)) - f(E(u)) < \nabla f(E(u))(E(x) - E(u))$$

holds.

Definition 12. Let $E : R^n \to R^n$ and $f : M \to R$ be an E-differentiable function at u on M. It is said that f is pseudo-E-univex at $u \in M$ on M with respect to η, b and Φ if, there exist $\eta : M \times M \to R^n$, $b : M \times M \to R_+$ and $\Phi : R \to R$ such that, for all $x \in M$, the following relation

$$\nabla f(E(u))\eta(E(x), E(u)) \geq 0 \Longrightarrow b(E(x), E(u))\Phi\left[f(E(x)) - f(E(u))\right] \geq 0 \tag{9}$$

holds for all $x \in M$. If relation (9) holds for any $u \in M$, then f is pseudo E-univex with respect to η, b and Φ on M.

Definition 13. Let $E : R^n \to R^n$ and $f : M \to R$ be an E-differentiable function at u on M. It is said that f is strictly pseudo-E-univex at $u \in M$ on M with respect to η, b and Φ if, there exist $\eta : M \times M \to R^n$, $b : M \times M \to R_+$ and $\Phi : R \to R$ such that, for all $x \in M$ $(E(x) \neq E(u))$, the following relation

$$\nabla f(E(u))\eta(E(x), E(u)) \geq 0 \Longrightarrow b(E(x), E(u))\Phi\left[f(E(x)) - f(E(u))\right] > 0 \tag{10}$$

holds for all $x \in M$. If relation (10) holds for any $u \in M$ $(E(x) \neq E(u))$, then f is strictly pseudo-E-univex with respect to η, b and Φ on M.

Example 4. The function defined in Example 1 is E-univex on R but not strictly pseudo-E-univex, as can be seen by taking $x = \frac{\pi}{4}$, $u = \frac{\pi}{3}$, since $\nabla f(E(u))\eta(E(x), E(u)) = 0$, but $b(E(x), E(u))\Phi\left[f(E(x)) - f(E(u))\right] = 0$. Hence, by the definition of a strictly pseudo-E-univex function, it follows that f is not strictly pseudo-E-univex on R.

Definition 14. Let $E : R^n \to R^n$ and $f : M \to R$ be an E-differentiable function at u on M. It is said that f is quasi-E-univex at $u \in M$ on M with respect to η, b and Φ if, there exist $\eta : M \times M \to R^n$, $b : M \times M \to R_+$ and $\Phi : R \to R$ such that, for all $x \in M$, the following relation

$$b(E(x), E(u))\Phi\left[f(E(x)) - f(E(u))\right] \leq 0 \Longrightarrow \nabla f(E(u))\eta(E(x), E(u)) \leq 0 \tag{11}$$

holds for all $x \in M$. If relation (11) holds for any $u \in M$, then f is quasi E-univex with respect to η, b and Φ on M.

Remark 2. Every E-univex function is quasi-E-univex. However, the converse is not true.

Now, we present an example of such a quasi-E-univex function which is not E-univex.

Example 5. Let $E : R \to R$ and $f, \Phi : R \to R$ be defined by $f(x) = -\sqrt[3]{x^2}$, $\Phi(x) = 2x$, $E(x) = x^3$, $\eta(E(x), E(u)) = \begin{cases} x^2u + u & \text{if } x \geq u, \\ x + u & \text{if } x < u. \end{cases}$ $b(E(x), E(u)) = \begin{cases} 1 & \text{if } x \geq u, \\ 0 & \text{if } x < u. \end{cases}$. Then, f is a quasi-E-univex function on R

but not E-univex with respect to η, b and Φ defined above. In fact, if we take, for example, $x = 1$, $u = 2$, then Definition 10 is not satisfied because

$$b(E(x), E(u))\Phi\left[f(E(x)) - f(E(u))\right] < \nabla f(E(u))\eta(E(x), E(u))$$

holds. Hence, by the definition of an E-univex function, it follows that f is not E-univex on R with respect to η, b and Φ given above.

Now, we introduce the concept of an E-univex set.

Definition 15. Let $E : R^n \to R^n$. A set $S \subseteq R^n \times R$ is said to be an E-univex set with resect to $\eta : M \times M \to R^n$, $b : M \times M \times [0, 1] \to R_+$ and $\Phi : R \to R$, if for any $(E(x), \beta) \in S$, $(E(u), \gamma) \in S$ the relation

$$\left(E\left(u\right) + \lambda\eta\left(E\left(x\right), E\left(u\right)\right), \gamma + \lambda b(E(x), E(u), \lambda)\Phi(\beta - \gamma)\right) \in S$$

holds for any $\lambda \in [0, 1]$.

Remark 3. If $E(x) = x$, then the definition of an E-univex set with respect to η, b, and Φ reduces to the definition of an univex set with respect to η, b, and Φ (see Bector et al. [11]).

Now, we present an example of such an E-univex set with respect to η, b, and Φ which is not an univex set with respect to η, b, and Φ.

Example 6. Let $S = ([1, 16] \cup [-16, -1]) \times [1, 4]$ and $E, \Phi : R \to R$, be defined by $\Phi(x) = 2x$, $E(x) = \begin{cases} x^2 & \text{if } -4 \leq x \leq 4 \\ -1 & \text{otherwise} \end{cases}$, $\eta(x, u) = \begin{cases} x - u & \text{if } x \leq u \\ -16 - u & \text{otherwise} \end{cases}$, $b(x, u, \lambda) = \begin{cases} 1 & \text{if } x \leq u \\ 0 & \text{otherwise} \end{cases}$. Then, by Definition 15, S is an E-univex set with respect to η, b, and Φ given above. However, as it follows from the definition of an univex set [11], S is not an univex set with respect to η, b, and Φ given above as can be seen by taking $x = -1$, $u = 1$, $\gamma = 1$, $\beta = 1$ and $\lambda = \frac{1}{2}$, we obtain

$$\left(u + \lambda\eta\left(x, u\right), \gamma + \lambda b(x, u, \lambda)\Phi(\beta - \gamma)\right) = (0, 1) \notin S.$$

Theorem 1. *Let $E : R^n \to R^n$. If $(S_i)_{i\in I}$, $i \in I = \{1, ..., k\}$ is a family of E-univex sets in $R^n \times R$ with respect to $\eta : M \times M \to R^n$, $b : M \times M \times [0, 1] \to R_+$ and $\Phi : R \to R$, then their intersection $\bigcap_{i\in I} S_i$ is an E-univex set.*

Proof. Let $E : R^n \to R^n$ and $(E(x), \beta) \in \bigcap_{i\in I} S_i$, $(E(u), \gamma) \in \bigcap_{i\in I} S_i$, $\lambda \in [0, 1]$. Since each S_i is an E-univex set therefore, for each $i \in I$, $(E(x), \beta) \in S_i$, $(E(u), \gamma) \in S_i$, and, moreover,

$$\left(E\left(u\right) + \lambda\eta\left(E\left(x\right), E\left(u\right)\right), \gamma + \lambda b(E(x), E(u), \lambda)\Phi(\beta - \gamma)\right) \in S_i.$$

Thus,

$$\left(E\left(u\right) + \lambda\eta\left(E\left(x\right), E\left(u\right)\right), \gamma + \lambda b(E(x), E(u), \lambda)\Phi(\beta - \gamma)\right) \in \bigcap_{i\in I} S_i \quad \text{for } \lambda \in [0, 1]$$

which completes the proof of this theorem. $\square$

Now, we give the definition of E-epigraph and we discuss a characterization of an E-univex function in terms of its E-epigraph.

Definition 16. Let $E : M \to M$. We define the E-epigraph of $f : M \to R$ as follows

$$\mathrm{epi}_E(f) = \{(E(x), \beta) \in M \times R : f(E(x)) \leq \beta\}.$$

Now, we give an example that sets of the E-epigraph of f and the epigraph of f are not equal.

Example 7. Let $S = [0, 4] \times R$, $E : R \to R$ and $f, \Phi : R \to R$ be defined by $f(x) = |x|$, $E(x) = x^2$, $\Phi(x) = 2x$, $\eta(E(x), E(u)) = \begin{cases} \frac{2x^2-2u^2}{2u} & \text{if } xu \neq 0 \\ 0 & otherwise \end{cases}$, $b(E(x), E(u), \lambda) = \begin{cases} 1 & \text{if } xu \neq 0 \\ 0 & otherwise \end{cases}$. Then, by Definition 10, f is an E-univex function on R with respect to η, b and Φ given above. Further, the E-epigraph of f

$$\mathrm{epi}_E(f) = \{(E(x), \beta) \in [0, 4] \times R : x^2 \leq \beta\} \tag{12}$$

is E-univex on $[0, 4] \times R$. This means that, for any $(E(x), \beta) \in S$, $(E(u), \gamma) \in S$ the following inequality

$$f(E(u) + \lambda\eta(E(x), E(u)) \leq \gamma + \lambda b(E(x), E(u), \lambda)\Phi(\beta - \gamma) \tag{13}$$

holds for any $\lambda \in [0, 1]$.

Note that function f in Example 7 show that sets $\mathrm{epi}_E(f)$ and epi (f) given in [11] are not equal. Thus,

$$\mathrm{epi}_E(f) = \{(E(x), \beta) \in [0, 4] \times R : x^2 \leq \beta\}, \tag{14}$$

$$\mathrm{epi}(f) = \{(x, \beta) \in [0, 4] \times R : |x| \leq \beta\} \tag{15}$$

hence, $\mathrm{epi}_E(f) \neq$ epi (f).

Now, we give a sufficient condition for the function f to be an E-univex function.

Theorem 2. *Let $E : M \to M$ and $f : M \to R$ be a function defined on an E-invex set with respect to η. If the E-epigraph epi_E (f) of f is an E-univex set in $M \times R$ with respect to η, b and Φ, then f is an E-univex function on M.*

Proof. Assume that $\mathrm{epi}_E(f)$ is an E-univex set. Let $x, u \in M$, then $(E(x), f(E(x))) \in \mathrm{epi}_E(f)$, $(E(u), f(E(u))) \in \mathrm{epi}_E(f)$. By E-univexity of the set $\mathrm{epi}_E(f)$ in $M \times R$ with respect to η, b and Φ, we have

$$\begin{aligned}&(E(u) + \lambda\eta(E(x), E(u)), f(E(u)) + \lambda b(E(x), E(u), \lambda)\Phi[f(E(x)) \\ &\quad - f(E(u))]) \in \mathrm{epi}_E(f)\end{aligned}$$

for $\lambda b(E(x), E(u), \lambda) \in [0,1]$, from which it follows that

$$f\left[E\left(u\right)+\lambda \eta\left(E\left(x\right), E\left(u\right)\right)\right] \leq f\left(E\left(u\right)\right)+\lambda b(E(x), E(u), \lambda) \Phi[f\left(E\left(x\right)\right)-f\left(E\left(u\right)\right)]$$

for $\lambda b(E(x), E(u), \lambda) \in [0,1]$. Hence f is an E-univex function. $\square$

Definition 17. Let $E : R^n \to R^n$. It said that $\overline{x} \in R^n$ is a global E-minimizer of $f : M \to R$ if the inequality

$$f\left(E\left(\overline{x}\right)\right) \leq f\left(E\left(x\right)\right)$$

holds for all $x \in M$.

Proposition 1. *Let $E : R^n \to R^n$ and $f : M \to R$ be an E-differentiable function on M. If $\overline{x} \in M$ is a global E-minimizer of f, then $\nabla f\left(E(\overline{x})\right) = 0$.*

Proof. We assume that $\overline{x} \in M$ is a global E-minimizer of f and let $d = -\nabla f\left(E(\overline{x})\right)$. Now, we prove that there does not exist $d = -\nabla f\left(E(\overline{x})\right) \in R^n$, $d \neq 0$, satisfying the inequality

$$\nabla f\left(E(\overline{x})\right) d = - \parallel \nabla f\left(E(\overline{x})\right) \parallel^2 < 0, \quad d \in R^n. \tag{16}$$

By assumption, f is E-differentiable at $\overline{x}$. Thus, by Definition 9, we have

$$\left(f \circ E\right)\left(x\right) = \left(f \circ E\right)\left(\overline{x}\right) + \nabla\left(f \circ E\right)\left(\overline{x}\right)\left(x - \overline{x}\right) + \theta_f\left(\overline{x}, x - \overline{x}\right)\|x - \overline{x}\| \tag{17}$$

where $\theta_f\left(\overline{x}, x - \overline{x}\right) \to 0$ and $\frac{x - \overline{x}}{\|x - \overline{x}\|} \to d$ as $x \to \overline{x}$ together with (16), we get that the following inequality

$$\left(f \circ E\right)\left(x\right) < \left(f \circ E\right)\left(\overline{x}\right)$$

holds, which is a contradiction to the assumption that $\overline{x} \in M$ is a global E-minimizer of f. This means that (16) is not satisfied. Thus, $\nabla f\left(E(\overline{x})\right) d = 0$, for all $d \in R^n$. This implies that $\nabla f\left(E(\overline{x})\right) = 0$. $\square$

Proposition 2. *Let $E : R^n \to R^n$ be an operator, $\Phi : R \to R$ be strictly increasing with $\Phi(0) = 0$, $b(E(x), E(\overline{x})) > 0$ and $f : R^n \to R$ be an E-differentiable E-univex function on M with respect to η, Φ and b. If $\nabla f\left(E(\overline{x})\right) = 0$, then $\overline{x}$ is an E-minimizer of f.*

Proof. Let $E : R^n \to R^n$ be an operator. Further, assume that $f : M \to R$ is an E-differentiable E-univex function on M with respect to η, Φ and b. Hence, by Definition 10, the inequality

$$b(E(x), E(\overline{x}))\Phi\left[f(E(x)) - f(E(\overline{x}))\right] \geq \nabla f(E(\overline{x}))\eta(E(x), E(\overline{x})) \tag{18}$$

holds for all $x \in M$. Since $\Phi : R \to R$ is strictly increasing with $\Phi(0) = 0$, $b(E(x), E(\overline{x})) > 0$, $\nabla f\left(E(\overline{x})\right) = 0$ and (18), therefore, we have that the relation

$$\Phi\left[f(E(x)) - f(E(\overline{x}))\right] \geq \Phi(0) = 0 \tag{19}$$

implies that the inequality

$$f\left(E\left(\overline{x}\right)\right) \leq f\left(E\left(x\right)\right)$$

holds for all $x \in M$. This means, by Definition 17, that $\overline{x}$ is an E-minimizer of f. □

Proposition 3. *Let $E : R^n \to R^n$ be an operator, $\Phi : R \to R$ be strictly increasing with $\Phi(0) = 0$, $b(E(x), E(\overline{x})) > 0$ and $f : R^n \to R$ be an E-differentiable pseudo-E-univex function on M with respect to η, Φ and b. If $\nabla f(E(\overline{x})) = 0$, then $\overline{x}$ is an E-minimizer of f.*

Proof. The proof of this proposition follows from Definitions 12 and 17 . □

3. Optimality Conditions for *E*-Differentiable Optimization Problem

In the paper, we consider the following constrained optimization problem:

$$\begin{array}{c} f(x) \to \min \\ \text{subject to } g_i(x) \leq 0,\ i \in I = \{1, ..., k\} \end{array} \tag{P}$$

where $f : R^n \to R$ and $g_i : R^n \to R$, $i \in I$, are E-differentiable functions on R^n. We will write $g := (g_1, ..., g_k) : R^n \to R^k$ for convenience. Let

$$D := \{x \in R^n : g_i(x) \leq 0,\ i \in I\}$$

be the set of all feasible solutions of (P). Further, $I(\overline{x})$ is the set of all active inequality constraints at point $\overline{x} \in D$, that is, $I(\overline{x}) = \{i \in I : g_i(\overline{x}) = 0\}$.

Definition 18. A point $\overline{x}$ is said to be an optimal solution of (P) if and only if there exists no other feasible point x such that

$$f(x) < f(\overline{x}).$$

Now, for the considered E-differentiable optimization problem (P), we define its associated differentiable E-optimization problem (P_E) as follows:

$$\begin{array}{c} f(E(x)) \to \min \\ \text{subject to } g_i(E(x)) \leq 0,\ i \in I = \{1, ..., k\} \end{array} \tag{P_E}$$

We call the problem (P_E) an E-optimization problem. Let

$$D_E := \{x \in R^n : g_i(E(x)) \leq 0,\ i \in I\}$$

be the set of all feasible solutions of (P_E).

Definition 19. A point $E(\overline{x})$ is said to be an E-optimal solution of (P) if and only if there exists no other feasible point $E(x)$ such that

$$f(E(x)) < f(E(\overline{x})).$$

Lemma 1 [3]**.** *Let $E : R^n \to R^n$ be a one-to-one and onto operator. Then $E(D_E) = D$.*

Lemma 2. *Let $\overline{y} \in D$ be an E-optimal solution of (P). Then, there exists $\overline{x} \in D_E$ such that $\overline{y} = E(\overline{x})$ and $\overline{x}$ is an optimal solution of (P_E).*

Proof. Let $\overline{y} \in D$ be an E-optimal solution of (P). By Lemma 1, it follows that there exists $\overline{x} \in D_E$ such that $\overline{y} = E(\overline{x})$. Now, we prove that $\overline{x}$ is an optimal solution of (P_E). By means of contradiction, suppose that $\overline{x}$ is not an optimal solution of (P_E). Then, by Definition 19, there exists $\widehat{x} \in D_E$ such that $f(E(\widehat{x})) < f(E(\overline{x}))$. By Lemma 1, we have that there exists $\widehat{y} \in D$ such that $\widehat{y} = E(\widehat{x})$. Hence, the inequality above implies that $f(\widehat{y}) < f(\overline{y})$, which is a contradiction to the optimal solution of $\overline{y}$ for (P). The proof in the case when $\overline{y} \in D$ is an E-optimal solution of (P) is similar. □

Theorem 3 [5,23] **.** (E-Karush–Kuhn–Tucker necessary optimality conditions). *Let the objective function f, the constraint functions g_i, $i \in I$, be E -differentiable at $\overline{x} \in D_E$. Further, let $\overline{x}$ be an optimal solution of (P_E) (and, thus, $E(\overline{x})$ be an E-optimal solution of (P)) and the E-Guignard constraint qualification* [5] *be satisfied at $\overline{x}$. Then, there exist Lagrange multiplier $\overline{\lambda} \in R^k$ such that*

$$\nabla f(E(\overline{x})) + \sum_{i=1}^{k} \overline{\lambda}_i \nabla g_i(E(\overline{x})) = 0, \tag{20}$$

$$\overline{\lambda}_i g_i(E(\overline{x})) = 0,\ i \in I, \tag{21}$$

$$\overline{\lambda} \geq 0. \tag{22}$$

Definition 20. It is said that $\left(E(\overline{x}), \overline{\lambda}\right) \in D \times R^k$ is an E-Karush–Kuhn–Tucker point (E-KKT-point, in short) for the considered optimization problem (P) if the E-Karush–Kuhn–Tucker necessary optimality conditions (20)–(22) are satisfied at $E(\overline{x})$ with Lagrange multiplier $\overline{\lambda}$.

Now, we prove the sufficiency of the E-KKT necessary optimality conditions for the considered E-differentiable optimization problem (P) under E-univexity hypotheses.

Theorem 4. *Let $\left(\overline{x}, \overline{\lambda}\right) \in D_E \times R^k$ be a KKT-point of the E-optimization problem (P_E). Further, assume the following hypotheses are fulfilled:*

a) the objective function f, is E-univex at $\overline{x}$ on D_E with respect to η, Φ_f and b_f,
b) each inequality constraint g_i, $i \in I_E(\overline{x})$, is E-univex at $\overline{x}$ on D_E with respect to η, Φ_{g_i} and b_{g_i},
c) Φ_f is strictly increasing and Φ_{g_i}, $i \in I$ is increasing with $\Phi_f(0) = 0$ and $\Phi_{g_i}(0) = 0$,
d) $b_f(E(x), E(\overline{x})) > 0$ and $b_{g_i}(E(x), E(\overline{x})) \geq 0$, $i \in I_E(\overline{x})$ for all $x \in D_E$.

Then $\overline{x}$ is an optimal solution of the problem (P_E) and, thus, $E(\overline{x})$ is an E-optimal solution of the problem (P).

Proof. By assumption, $\left(\overline{x}, \overline{\lambda}\right) \in D_E \times R^k$ is a KKT-point of the problem (P_E). Then, by Definition 20, the E-KKT necessary optimality conditions

(20)–(22) are satisfied at $\overline{x}$ with Lagrange multiplier $\overline{\lambda} \in R^k$. We proceed by contradiction. Suppose, contrary to the result, that $\overline{x}$ is not an optimal solution of (P_E). Hence, by Definition 19, there exists another $x \in D_E$ such that

$$f(E(x)) < f(E(\overline{x})) \tag{23}$$

that is

$$f(E(x)) - f(E(\overline{x})) < 0. \tag{24}$$

Since Φ_f is strictly increasing with $\Phi_f(0) = 0$ and $b_f(E(x), E(\overline{x})) > 0$, therefore, we have

$$b_f(E(x), E(\overline{x}))\, \Phi\left[f(E(x)) - f(E(\overline{x}))\right] < 0. \tag{25}$$

From hypothesis a), by Definition 10, the following inequality

$$b_f(E(x), E(\overline{x}))\, \Phi_f\left[f(E(x)) - f(E(\overline{x}))\right] \geq \nabla f(E(\overline{x}))\, \eta(E(x), E(\overline{x})) \tag{26}$$

holds. Combining (25) and (26), we get

$$\nabla f(E(\overline{x}))\, \eta(E(x), E(\overline{x})) < 0. \tag{27}$$

From hypothesis b), by Definition 10, the following inequalities

$$b_{g_i}(E(x), E(\overline{x}))\, \Phi_{g_i}\left[g_i(E(x)) - g_i(E(\overline{x}))\right] \geq \nabla g_i(E(\overline{x}))\, \eta(E(x), E(\overline{x})), \quad i \in I(E(\overline{x})) \tag{28}$$

hold. Multiplying inequalities (28) by the corresponding Lagrange multipliers, we obtain

$$b_{g_i}(E(x), E(\overline{x}))\, \overline{\lambda}_i \Phi_{g_i}\left[g_i(E(x)) - g_i(E(\overline{x}))\right] \geq \overline{\lambda}_i \nabla g_i(E(\overline{x}))\, \eta(E(x), E(\overline{x})), \quad i \in I(E(\overline{x})) \tag{29}$$

and, moreover, by $g(E(\overline{x})) = 0$,

$$b_{g_i}(E(x), E(\overline{x}))\, \overline{\lambda}_i \Phi_{g_i}\left[g_i(E(x))\right] \geq \overline{\lambda}_i \nabla g_i(E(\overline{x}))\, \eta(E(x), E(\overline{x})), \ i \in I(E(\overline{x})) \tag{30}$$

Since Φ_{g_i}, $i \in I$, is increasing and satisfies $\Phi_{g_i}(0) = 0$, then, using (30) together with $x \in D_E$ and $\overline{x} \in D_E$, we obtain

$$\overline{\lambda}_i \nabla g_i(E(\overline{x}))\, \eta(E(x), E(\overline{x})) \leq 0, \, i \in I(E(\overline{x})). \tag{31}$$

Thus,

$$\sum_{i=1}^{k} \overline{\lambda}_i \nabla g_i(E(\overline{x})) \eta(E(x), E(\overline{x})) \leq 0. \tag{32}$$

Combining (27) and (32), we obtain that the following inequality

$$\left[\nabla(f \circ E)(\overline{x}) + \sum_{i=1}^{k} \overline{\lambda}_i \nabla g_i(E(\overline{x}))\right] \eta(E(x), E(\overline{x})) < 0$$

holds, which is a contradiction to the E-KKT necessary optimality condition (20). Since $\overline{x}$ is an optimal solution of the problem (P_E), by Lemma 2, $E(\overline{x})$ is an E-optimal solution of the problem (P). Thus, the proof of this theorem is completed. □

Theorem 5. *Let $(\overline{x}, \overline{\lambda}) \in D_E \times R^k$ be a KKT-point of the E-optimization problem (P_E). Further, assume the following hypotheses are fulfilled:*

a) the function f, is strictly E-univex at $\overline{x}$ on D_E with respect to η, Φ_f and b_f,

b) each constraint function g_i, $i \in I_E(\overline{x})$, is E-univex at $\overline{x}$ on D_E with respect to η, Φ_{g_i} and b_{g_i},

c) Φ_f is strictly increasing and Φ_{g_i}, $i \in I$ is increasing with $\Phi_f(0) = 0$ and $\Phi_{g_i}(0) = 0$,

d) $b_f(E(x), E(\overline{x})) > 0$ and $b_{g_i}(E(x), E(\overline{x})) \geq 0$, $i \in I_E(\overline{x})$ for all $x \in D_E$.

Then $\overline{x}$ is an optimal solution of the problem (P_E) and, thus, $E(\overline{x})$ is an E-optimal solution of the problem (P).

In order to illustrate the sufficient optimality conditions established in Theorem 4, we now present an example of an E-differentiable optimization problem in which the involved functions are E-univex.

Example 8. Consider the following nondifferentiable optimization problem

$$\begin{aligned} f(x) = \sin(\sqrt[3]{x_1}) + \sin(\sqrt[3]{x_2}) &\rightarrow \min \\ \text{subject to } g_1(x) = (\sin \sqrt[3]{x_1} - 1)^2 + \sin \sqrt[3]{x_2} - 1 &\leq 0, \\ g_2(x) = 2\sqrt[3]{x_1} + 3\sqrt[3]{x_2} - \tfrac{9}{2} &\leq 0, \\ g_3(x) = \sqrt[3]{x_1^2} + \sqrt[3]{x_2^2} - 3 &\leq 0, \\ g_4(x) = -\sin \sqrt[3]{x_1} &\leq 0, \\ g_5(x) = -\sin \sqrt[3]{x_2} &\leq 0. \end{aligned} \tag{P1}$$

Note that the set of all feasible solutions of (P1) is $D = \{(x_1, x_2) \in R^2 : (\sin \sqrt[3]{x_1} - 1)^2 + \sin \sqrt[3]{x_2} - 1 \leq 0,\ 2\sqrt[3]{x_1} + 3\sqrt[3]{x_2} - \frac{9}{2} \leq 0,\ \sqrt[3]{x_1^2} + \sqrt[3]{x_2^2} - 3 \leq 0,\ -\sin \sqrt[3]{x_1} \leq 0,\ -\sin \sqrt[3]{x_2} \leq 0\}$. Let

$$\eta(x, u) = \begin{cases} \left(\frac{\sin(\sqrt[3]{x_1}) - \sin(\sqrt[3]{u_1})}{\sin(\sqrt[3]{u_1}) + 1}, \frac{\sin(\sqrt[3]{x_2}) - \sin(\sqrt[3]{u_2})}{\sin(\sqrt[3]{u_2}) + 1} \right) & \text{if } x > u, \\ 0 & \text{otherwise.} \end{cases}$$

$$b_f(x, u) = b_{g_i}(x, u) = \begin{cases} \frac{1}{4} \text{ if } x > u, \\ 0 \text{ otherwise,} \end{cases}$$

and $\Phi_f(x) = \Phi_{g_i}(x) = 2x$. The functions constituting problem (P1) are non-differentiable at $(0, 0)$. Let $E : R^2 \rightarrow R^2$ be defined as follows $E(x_1, x_2) =$

$\left(x_1^3, x_2^3\right)$. Now, for the considered E-differentiable problem (P1), we define its associated differentiable problem (P1$_E$) as follows

$$\begin{aligned} f(E(x)) = \sin(x_1) + \sin(x_2) \to \min \\ \text{subject to } g_1(E(x)) = (\sin x_1 - 1)^2 + \sin x_2 - 1 \leq 0, \\ g_2(E(x)) = 2x_1 + 3x_2 - \tfrac{9}{2} \leq 0, \\ g_3(E(x)) = x_1^2 + x_2^2 - 3 \leq 0, \\ g_4(E(x)) = -\sin x_1 \leq 0, \\ g_5(E(x)) = -\sin x_2 \leq 0. \end{aligned} \tag{P1$_E$}$$

Note that the set of all feasible solutions of the problem (P1$_E$) is $D_E = \{(x_1, x_2) \in R^2 : (\sin x_1 - 1)^2 + \sin x_2 - 1 \leq 0,\ 2x_1 + 3x_2 - \frac{9}{2} \leq 0,\ x_1^2 + x_2^2 - 3 \leq 0,\ -\sin x_1 \leq 0,\ -\sin x_2 \leq 0\}$. Note that all functions constituting the problem (P1$_E$) are differentiable at $(0,0)$. Then, it can also be shown that the E-Karush–Kuhn–Tucker necessary optimality conditions (20)–(22) are fulfilled at $(0,0)$ with Lagrange multipliers $\overline{\lambda} = (\frac{1}{4}, 0, 0, \frac{1}{2}, \frac{5}{4})$. Further, all hypotheses of Theorem 4 are fulfilled, it can be proved that f, is E-univex at $\overline{x}$ on D_E with respect to η, b_f and Φ_f given above, each function g_i, $i = 1, 2, ..., 5$, is E-univex at $\overline{x}$ on D_E with respect to η, b_{g_i} and Φ_{g_i} given above. Hence, $\overline{x} = (0,0)$ is an optimal solution of the problem (P1$_E$) and, thus, $E\left(\overline{x}\right) = (0,0)$ is an E-optimal solution of the problem (P). Further, that the sufficient optimality conditions under univexity are not applicable since not all functions constituting problem (P1) are differentiable univex with respect to η, b given above (see, Bector et al. [11]). Moreover, that the sufficient optimality conditions under E-invexity are not applicable since not all functions constituting problem (P1) are E-invex with respect to η given above.

Now, under generalized E-univexity notions, we prove the sufficient optimality conditions for the problem (P$_E$).

Theorem 6. *Let $\left(\overline{x}, \overline{\lambda}\right) \in D_E \times R^k$ be a KKT-point of the E-optimization problem (P$_E$). Further, assume the following hypotheses are fulfilled:*

a) the function f, is pseudo-E-univex at $\overline{x}$ on D_E with respect to η, Φ_f and b_f,
b) each constraint function g_i, $i \in I_E\left(\overline{x}\right)$, is quasi-$E$-univex at $\overline{x}$ on D_E with respect to η, Φ_{g_i} and b_{g_i},
c) Φ_f is strictly increasing and Φ_{g_i}, $i \in I$ is increasing with $\Phi_f(0) = 0$ and $\Phi_{g_i}(0) = 0$,
d) $b_f\left(E\left(x\right), E\left(\overline{x}\right)\right) > 0$ and $b_{g_i}\left(E\left(x\right), E\left(\overline{x}\right)\right) \geq 0$, $i \in I_E\left(\overline{x}\right)$ for all $x \in D_E$.

Then $\overline{x}$ is an optimal solution of the problem (P$_E$) and, thus, $E\left(\overline{x}\right)$ is an E-optimal solution of the problem (P).

Proof. By assumption, $(\overline{x}, \overline{\lambda}) \in D_E \times R^k$ be a KKT-point of the E-optimization problem (P$_E$). Then, by Definition 20, the KKT necessary optimality conditions (20)–(22) are satisfied at $\overline{x}$ with Lagrange multiplier $\overline{\lambda} \in R^k$. We proceed by contradiction. Suppose, contrary to the result, that $\overline{x}$ is not an optimal solution of (P$_E$). Hence, by Definition 19, there exists another $x \in D_E$ such that

$$f(E(x)) < f(E(\overline{x})). \tag{33}$$

Thus,

$$f(E(x)) - f(E(\overline{x})) < 0. \tag{34}$$

Since Φ_f is strictly increasing, $\Phi_f(0) = 0$ and $b_f(E(x), E(\overline{x})) > 0$, we obtain

$$b_f(E(x), E(\overline{x}))\, \Phi_f[f(E(x)) - f(E(\overline{x}))] < 0. \tag{35}$$

By assumption, f is pseudo-E-univex at $\overline{x}$ on D_E with respect to η, Φ_f and b_f. Then, by Definition 12, we get

$$\nabla(f \circ E)(\overline{x})\, \eta(E(x), E(\overline{x})) < 0. \tag{36}$$

From $x \in D_E$, $x \in D_E$, the E-KKT necessary optimality conditions (21) and (22) imply

$$g_i(E(x)) - g_i(E(\overline{x})) \leq 0,\ i \in I(E(\overline{x})).$$

Since the function Φ_{g_i} is increasing and $\Phi_{g_i}(0) = 0$ and $b_{g_i}(E(x), E(\overline{x})) \geq 0$, by Definition 14, we have

$$b_{g_i}(E(x), E(\overline{x}))\, \Phi_{g_i}[g_i(E(x)) - g_i(E(\overline{x}))] \leq 0,\ i \in I(E(\overline{x})).$$

By assumption, g_i is quasi-E-univex at $\overline{x}$ on D_E with respect to η, Φ_{g_i} and b_{g_i}. Then, by Definition 14, we get

$$\nabla g_i(E(\overline{x}))\, \eta(E(x), E(\overline{x})) \leq 0,\ i \in I(E(\overline{x})). \tag{37}$$

Thus, by $\overline{\lambda}_i \geq 0,\ i \in I(E(\overline{x}))$, we obtain

$$\sum_{i \in I(E(\overline{x}))} \overline{\lambda}_i \nabla g_i(E(\overline{x}))\, \eta(E(x), E(\overline{x})) \leq 0.$$

Hence, taking into account $\overline{\lambda}_i = 0$, $i \notin I(E(\overline{x}))$, we have

$$\sum_{i=1}^{k} \overline{\lambda}_i \nabla g_i(E(\overline{x}))\, \eta(E(x), E(\overline{x})) \leq 0. \tag{38}$$

Combining (36) and (38), we get that the following inequality

$$\left[\nabla(f \circ E)(\overline{x}) + \sum_{i=1}^{k} \overline{\lambda}_i \nabla g_i(E(\overline{x}))\right] \eta(E(x), E(\overline{x})) < 0$$

which is a contradiction to the E-KKT necessary optimality condition (21). Since $\overline{x}$ is an optimal solution of the problem (P$_E$), by Lemma 2, $E(\overline{x})$ is an E-optimal solution of the problem (P). Thus, the proof of this theorem is completed. □

Theorem 7. *Let $\left(\overline{x}, \overline{\lambda}\right) \in D_E \times R^k$ be a KKT-point of the E-optimization problem (P_E). Further, assume the following hypotheses are fulfilled:*

a) the function f, is strictly pseudo-E-univex at $\overline{x}$ on D_E with respect to η, Φ_f and b_f,

b) each constraint function g_i, $i \in I_E\left(\overline{x}\right)$, is quasi-E-univex at $\overline{x}$ on D_E with respect to η, Φ_{g_i} and b_{g_i},

c) Φ_f is strictly increasing and Φ_{g_i}, $i \in I$ is increasing with $\Phi_f(0) = 0$ and $\Phi_{g_i}(0) = 0$,

d) $b_f\left(E\left(x\right), E\left(\overline{x}\right)\right) > 0$ and $b_{g_i}\left(E\left(x\right), E\left(\overline{x}\right)\right) \geq 0$, $i \in I_E\left(\overline{x}\right)$ for all $x \in D_E$.

Then $\overline{x}$ is an optimal solution of the problem (P_E) and, thus, $E\left(\overline{x}\right)$ is an E-optimal solution of the problem (P).

Example 9. Consider the following nonconvex nondifferentiable optimization problem

$$\begin{gathered} f(x) \to \min \\ \text{subject to } g_1(x) \leq 0, \\ g_2(x) \leq 0, \end{gathered} \tag{P2}$$

where,

$$f(x) = \begin{cases} x & \text{if } x < 0, \\ \sqrt[3]{x} & \text{if } x \geq 0. \end{cases}$$

$$g_1(x) = \begin{cases} -\sqrt[3]{x} & \text{if } x < 0, \\ 1 - e^{\sqrt[3]{x}} & \text{if } x \geq 0. \end{cases} \qquad g_2(x) = \begin{cases} \sqrt[3]{x^2} & \text{if } x < 0, \\ -\sqrt[3]{x} & \text{if } x \geq 0. \end{cases}$$

Let $\eta(x,u) = (\sqrt[3]{x} - u)^5$, $\Phi_f(x) = \Phi_{g_i}(x) = 3x$, $b_f(x,u) = \frac{1}{3}x^{\frac{4}{3}} + u + 1$, $b_{g_1}(x,u) = b_{g_2}(x,u) = \begin{cases} 1 \text{ if } x < u, \\ 0 \text{ if } x \geq u. \end{cases}$. The functions constituting problem (P2) are nondifferentiable at 0. Let $E : R \to R$ be defined as follows $E\left(x\right) = x^3$. Now, for the considered E-differentiable problem (P2), we define its associated differentiable problem ($P2_E$) as follows

$$\begin{gathered} f(E(x)) \to \min \\ \text{subject to } g_1(E(x)) \leq 0, \\ g_2(E(x)) \leq 0, \end{gathered} \tag{$P2_E$}$$

where,

$$f(E(x)) = \begin{cases} x^3 & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$$

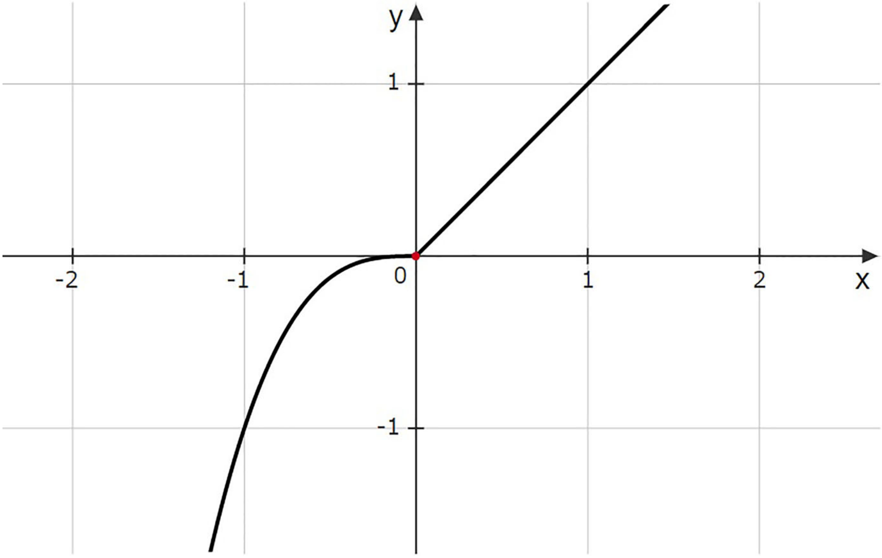

FIGURE 1. $\overline{x} = 0$ is an optimal solution of the problem (P2$_E$)

$g_1(E(x)) = \begin{cases} -x & \text{if } x < 0, \\ 1 - e^x & \text{if } x \geq 0. \end{cases}$ $\quad g_2(E(x)) = \begin{cases} x^2 & \text{if } x < 0, \\ -x & \text{if } x \geq 0. \end{cases}$. Note that the set of all feasible solutions of the problem (P2$_E$) is $D_E = \{x \in R : x \geq 0\}$. It can be shown that the E-Karush–Kuhn–Tucker necessary optimality conditions (20)–(22) are fulfilled at $\overline{x} = 0$ with Lagrange multipliers $\overline{\lambda}_1 = \overline{\lambda}_2 = \frac{1}{2}$. Further, all hypotheses of Theorem 6 are fulfilled, it can be show that f, is pseudo-E-univex at $\overline{x}$ on D_E with respect to η, Φ_f and b_f given above, functions g_1 and g_2 are quasi-E-univex at $\overline{x}$ on D_E with respect to η, Φ_{g_i} and b_{g_i} given above. Hence, $\overline{x} = 0$ is an optimal solution of the problem (P2$_E$) and, thus, $E(\overline{x}) = 0$ is an E-optimal solution of the problem (P2) (Fig. 1).

Example 10. Consider the following nonconvex nondifferentiable optimization problem

$$\begin{aligned} &\text{minimize } f(x) = x + \cos(\sqrt[3]{x^2}) \\ &\text{subject to } \ g_1(x) = 1 - e^{\sqrt[3]{x}} \leq 0, \\ &\qquad\qquad g_2(x) = -\arctan \sqrt[3]{x} \leq 0. \end{aligned} \tag{P3}$$

Note that $D = \left\{x \in R : 1 - e^{\sqrt[3]{x}} \leq 0, -\arctan \sqrt[3]{x} \leq 0\right\}$ is the feasible solution set of the considered optimization problem (P3). Let

$$\eta(x,u)=\begin{cases}\sqrt[3]{x^2}+\sqrt[3]{u^2}+xu & \text{if } x>u,\\ \\ \sqrt[3]{x}-\sqrt[3]{u} & \text{if } x\le u.\end{cases}$$

$$b_f(x,u)=b_{g_1}(x,u)=\begin{cases}\frac{\sqrt[3]{u^2}}{\sqrt[3]{x}-\sqrt[3]{u}} & \text{if } x>u\\ 0 & \text{if } x\le u\end{cases},\ b_{g_2}(x,u=\begin{cases}u+1 & \text{if } x>u\\ 0 & \text{if } x\le u\end{cases},$$

and $\Phi(x)=3x$. Let $E:R\to R$ be a mapping defined by $E(x)=x^3$. Then, we define the following differentiable E-optimization problem (P3$_E$) as follows

$$\begin{aligned}&\text{minimize } (f\circ E)(x)=x^3+\cos(x^2)\\ &\text{subject to } (g_1\circ E)(x)=1-e^x\le 0,\\ &\qquad (g_2\circ E)(x)=-\arctan x\le 0.\end{aligned}\tag{P3$_E$}$$

Note that $D_E=\{x\in R: x\ge 0\}$ is the feasible solution set of the problem (P3$_E$) and $\overline{x}=0$ is a feasible solution of the nonlinear optimization problem (P3$_E$). Then, all hypotheses of Theorem 7 are fulfilled, it can be show that f, is strictly pseudo-E-univex at $\overline{x}$ on D_E with respect to η, Φ_f and b_f given above, functions g_1 and g_2 are quasi-E-univex at $\overline{x}$ on D_E with respect to η, Φ_{g_i} and b_{g_i} given above. Hence, $\overline{x}=0$ is an optimal solution of the problem (P3$_E$) and, thus, $E(\overline{x})=0$ is an E-optimal solution of the problem (P3).

4. Concluding Remarks

In the paper, new concepts of nondifferentiable generalized convexity notions have been introduced. The so-called E-differentiable E-univexity unify the concepts of convexity, univexity defined by Bector et al. [11], B-invexity defined by Bector et al. [9], Suneja et al. [32], invexity defined by Hanson [15], E-B-invexity defined by Abdulaleem [8], E-convexity defined by Youness [33] and E-invexity defined by Abdulaleem [5]. Further, some properties of E-univex functions have been studied. Also E-univex functions have been extended to pseudo-E-univex, strictly pseudo-E-univex and quasi-E-univex functions. In order to show their applications, the sufficiency of the so-called E-Karush–Kuhn–Tucker necessary optimality conditions have been proved for a nonlinear E-differentiable optimization problem in which the involved functions are E-univex and/or generalized E-univex. The aforesaid results have been illustrated by an example of nonconvex E-differentiable optimization problem with E-univex functions. Hence, the optimality results have been generalized and extended in the paper to new classes of E-differentiable optimization problems in comparison to those ones actually existing in the optimization literature.

However, some interesting topics for further research remain. It would be of interest to investigate whether it is possible to prove similar results for other classes of E-differentiable optimization problems. We shall investigate these questions in subsequent papers.

Funding Not applicable.

Data availability Not applicable.

Declarations

Conflict of interest The author declares to have no competing interests.

Code availability Not applicable.

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Received: September 6, 2021.
Accepted: October 28, 2022.