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RESEARCH ARTICLE

On the Solution of Multi-Dimensional Nonlinear Integral Equation with Modified Newton Method

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In this note, multi-dimensional nonlinear integral equations (multi-dimensional NIEs) in R^n is considered. Implementing modified Newton method (Modified NM) reduces multi-dimensional NIEs into multi-dimensional linear integral equations (multi-dimensional LIEs), which can be solved by discritization method. Quadrature method together with collocation are used to find values of unknown functions. Existence and uniqueness solution of the problems are shown. The rate of convergence of the proposed method is proved using the principle of majorant function. Finally, numerical examples are provided and it reveals that proposed method is both accurate and effective as well as comparisons with other methods are also presented.

Keywords: The Modified Newton Method, Nonlinear Multi-Dimensional Integral Equations, Quadrature Formula, The Majorant Function.

1. INTRODUCTION

It is known that much work has been done on developing and analyzing numerical methods for solving one-dimensional integral equations of the second kind. The Adomain decomposition method employed to solve the system of nonlinear Volterra-Fredholm integral equations and Volterra integro-differential equations as revealed in Refs. [1 and 2]. The Newton-Kantorovich method is used in Refs. [3 and 4] respectively, to solve the systems of 2×2 and $n \times n$ nonlinear Volterra integral equations. In Ref. [5], Volterra integral equations with separable kerenels are solved utilizing the differential transform method. Volterra integral and integro-differential equations are solved by Iterated collocation method and Implicitly linear collocation method as shown in Refs. [6 and 7]. Indeed, many methods are utilized to solve one dimension integral equations, including but not limited to projection methods, Nystrom method, iteration method, quadrature methods and expansion methods, (see Refs. [8–11]). In two dimensional integral equations, there are several literature which attempted to introduce the analytical and numerical solutions. For instance, in Ref. [12] the numerical method based on interpolation by Gaussian radial basis function has been found to solve two-dimensional Fredholm integral equations. The two dimensional differential transform (TDDT) for double integrals is developed in Ref. [13], for solving a class of two-dimensional linear and nonlinear Volterra integral equations. The authors in Ref. [14] endeavored to solve two dimensional integral equations depending on the principle of wavelet based methods. Two dimensional rationalized Haar (RH) functions, as shown in Ref. [15] are applied to the numerical solution of nonlinear second kind two dimensional integral equations. The linear Fredholm integral equation is solved approximately in Ref. [16] via two-dimensional modification of hat functions, and operational matrix of integration. In Ref. [17], the piece-wise constant two-dimensional block-pulse functions and their operational matrices are applied for solving nonlinear Volterra-Fredholm integral equations of the first kind. Two dimensional orthogonal triangular functions are employed to find the numerical solution of nonlinear mixed type Volterra-Fredholm integral equations as shown in Ref. [18]. The collocation points in Ref. [19], together with operational matrices of integration are discussed for finding an approximate solution of a class of two-dimensional nonlinear Volterra integral equations. The integral mean value theorem in Ref. [20], is exploited for solving two-dimensional linear Fredholm integral equations of the second kind. In contrast, a small amount of work has been done in multi-dimensional cases, especially in R^n such as, author in Ref. [21], applied functions with

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a shifted argument to reduce the multidimensional integral equation to a finite system of linear algebraic equations. In Ref. [22] Taylor series expansion is developed for solving multidimensional integral equations. The recursive method based on approximate separation of variables, is employed in Ref. [23] to investigate the solutions of multidimensional integral equations. A computational method in Ref. [24], is proposed based on Haar wavelets trying to solve multidimensional stochastic Itô-Volterr integral equations. In Ref. [26], the convergence properties of Jacobi spectral collocation method has been introduced and used to approximate the solution of multidimensional nonlinear Volterra integral equation. The interpolation by radial basis functions (RBFs), in Ref. [27] is developed to describe a numerical scheme for solving multi-dimensional linear Fredholm integral equations of the second kind on the hypercube domains. Many application of the Nonlinear multi-dimensional integral equations of the second kind, have been applied in different fields, including nonhomogeneous elasticity and electrostatics,[28] contact problems for bodies with complex properties,[29] radio wave propagation,[30] along with many physical, mechanical and biological phenomena.

This study attempts to solve more general multi-dimensional NIEs of the second kind in the region $\Omega = \Omega_1 \times \Omega_2 \subset R^n$ where $\Omega_1 = C_{\prod_{i=1}^{n}[a_i,b_i]}$, and $\Omega_2 = C_{\prod_{i=1}^{n}[c_i,d_i]}$ of the form

$$u(\mathbf{t})-\int_{a_1}^{b_1(t)}\int_{a_2}^{b_2(t)}\dots\int_{a_n}^{b_n(t)} K(\mathbf{t},\mathbf{x})G(u(\mathbf{x}))\,dx_n\,dx_{n-1}\dots dx_1 = f(\mathbf{t}),\quad \mathbf{t}=(t_1,t_2,\dots,t_n), \mathbf{x}=(x_1,x_2,\dots,x_n) \tag{1}$$

where, $u(\mathbf{t}) \in \Omega_1$ is unknown function, $f(\mathbf{t}) \in \Omega_1$ is given function, and the kernel $K(\mathbf{t},\mathbf{x})$ is a given smooth function in Ω, the nonlinear function $G(u(\mathbf{t}))$ is continuous function defined in $(-\infty,\infty)$ and $b_i(t)$, $i=1,2,\dots,n$ are real valued continuous functions. Equation (1) refers to the multi-dimension Fredholm integral equation if $b_i(t)=b_i$, $i=1,2,\dots,n$ which are constants, otherwise $b_i(t)=t_i$, $i=1,2,\dots,n$. Equation (1) is called the multi-dimensional Volterra integral equations.

The structure of this paper is organized as follows. In Section 2 we discuss the use of modified NM for linearization of multi-dimensional NIEs. In Section 3, the approximate solution of the multi-dimensional LIEs is considered using Quadrature formula. in Section 4, the existence and uniqueness solution of the problem is proved. In Section 5, examples are provided to show the accuracy and efficiency of the method. Finally, Section 6 concludes the key ideas of the proposed approximation method.

2. LINEARIZING THE MULTI-DIMENSIONAL NONLINEAR INTEGRAL EQUATION

Let us rewrite Eq. (1) in the operator form

$$\mathbf{Q}(u(\mathbf{t}))=0,\quad \mathbf{t}=(t_1,t_2,\dots,t_n) \tag{2}$$

where

$$\mathbf{Q}(u(\mathbf{t})) = u(\mathbf{t}) - f(\mathbf{t}) - \int_{a_1}^{b_1(t)}\int_{a_2}^{b_2(t)}\dots\int_{a_n}^{b_n(t)} K(\mathbf{t},\mathbf{x}) \times G(u(\mathbf{x}))\,dx_n\,dx_{n-1}\dots dx_1 \tag{3}$$

Now we use initial iteration of the modified NM in the form

$$\mathbf{Q}'(u_0(\mathbf{t}))(u(\mathbf{t})-u_0(\mathbf{t}))+\mathbf{Q}(u_0(\mathbf{t}))=0 \tag{4}$$

where $u_0(\mathbf{t})$ is the initial guess, which is needed to be properly chosen. The Frechet derivative of $\mathbf{Q}(u(t,x))$ at the initial guess $u_0(t,x)$ is appointed as

$$\begin{aligned}\mathbf{Q}'(u_0)u &= \lim_{\rho\to 0}\frac{1}{\rho}[\mathbf{Q}(u_0+\rho u)-\mathbf{Q}(u_0)]\\ &= \lim_{\rho\to 0}\frac{1}{\rho}\left[\frac{d\mathbf{Q}(u_0)}{du}\rho u+\frac{1}{2}\frac{d^2\mathbf{Q}}{du^2}(u_0+\delta\rho)\rho^2u^2\right]\\ &= \frac{d\mathbf{Q}(u_0)}{du}u,\quad \delta\in(0,1)\end{aligned} \tag{5}$$

From Eqs. (4) and (5) we obtain

$$\left.\frac{d\mathbf{Q}}{du}\right|_{u_0}(\Delta u(t)) = -\mathbf{Q}(u_0(t)) \tag{6}$$

where $\Delta u(t,x)=u_1(t,x)-u_0(t,x)$, and $u_0(t,x)$ is the initial given function, then by establishing the solution of Eq. (6) for $\Delta u(t,x)$ the derivative is computed as

$$\begin{aligned}\frac{d\mathbf{Q}}{du}|_{u0} &= \lim_{\rho\to 0}\frac{1}{\rho}[\mathbf{Q}(u_0+\rho u)-\mathbf{Q}(u_0)]\\ &= \lim_{\rho\to 0}\frac{1}{s}\Bigg[\rho u(t,x)-\int_{a_1}^{b_1(t)}\dots\int_{a_n}^{b_n(t)} K(\mathbf{t},\mathbf{x})\\ &\quad\times[G(u_0(\mathbf{x})+\rho u(\mathbf{x}))-G(u_0(\mathbf{x}))]\,dx_n\dots dx_1\Bigg]\\ &= u(\mathbf{t})-\int_{a_1}^{b_1(t)}\int_{a_2}^{b_2(t)}\dots\int_{a_n}^{b_n(t)}[K(\mathbf{t},\mathbf{x})G'(u_0(\mathbf{x}))\\ &\quad\times u(\mathbf{x})\,dx_n\,dx_{n-1}\dots dx_1]\end{aligned} \tag{7}$$

where $G'(u_0(\mathbf{t}))$ is the derivative of $G(u(\mathbf{t}))$ for $u(\mathbf{t})$. Then Eqs. (6) and (7) yield

$$\begin{aligned}&\Delta u(\mathbf{t})-\int_{a_1}^{b_1(t)}\int_{a_2}^{b_2(t)}\dots\int_{a_n}^{b_n(t)}[K(\mathbf{t},\mathbf{x})G'(u_0(\mathbf{x}))\Delta u(\mathbf{x})]\\ &\quad\times dx_ndx_{n-1}\dots dx_1\\ &\quad= f(\mathbf{t})+\int_{a_1}^{b_t}\int_{a_2}^{b_2(t)}\dots\int_{a_n}^{b_n(t)}[K(\mathbf{t},\mathbf{x})G(u_0(\mathbf{x}))]\\ &\quad\times dx_ndx_{n-1}\dots dx_1-u_0(\mathbf{t})\end{aligned} \tag{8}$$

or

$$\Delta u(\mathbf{t})-\int_{a_1}^{b_1(t)}\int_{a_2}^{b_2(t)}\dots\int_{a_n}^{b_n(t)}\Psi_0(\mathbf{t},\mathbf{x};u_0)\Delta u(\mathbf{x}) \times dx_ndx_{n-1}\dots dx_1=\Phi_0(\mathbf{t}) \tag{9}$$

where

$$\Psi_0(\mathbf{t},\mathbf{x};u_0)=[K(\mathbf{t},\mathbf{x})G'(u_0(\mathbf{x}))] \tag{10}$$

$$\Phi_0(\mathbf{t})=f(\mathbf{t})+\int_{a_1}^{b_1(t)}\dots\int_{a_n}^{b_n(t)}[K(\mathbf{t},\mathbf{x})G(u_0(\mathbf{x}))] \times dx_n\dots dx_1-u_0(\mathbf{t}) \tag{11}$$

We observe that Eq. (9) is linear with respect to $\Delta u(\mathbf{t})$, and by solving it, we find $u_1(\mathbf{t})=\Delta u(\mathbf{t})+u_0(\mathbf{t})$. Then continuing this procedure, we get a sequence of approximate solution $u_r(\mathbf{t})$, $(r=2,3,\dots)$ from the equation;

$$\mathbf{Q}'(u_0(\mathbf{t}))\Delta u_r(\mathbf{t})+\mathbf{Q}(u_{r-1}(\mathbf{t}))=0 \tag{12}$$

which is same as the equation

$$\Delta u_r(\mathbf{t})-\int_{a_1}^{b_1(t)}\int_{a_2}^{b_2(t)}\dots\int_{a_n}^{b_n(t)}[\Psi_0(\mathbf{t},\mathbf{x};u_0)\Delta u_r(\mathbf{x})] \times dx_n\,dx_{n-1}\dots dx_1=\Phi_{r-1}(\mathbf{t}) \tag{13}$$

where

$$\Delta u_r(\mathbf{t})=u_r(\mathbf{t})-u_{r-1}(\mathbf{t}),\quad r=2,3,\dots \tag{14}$$

and

$$\Phi_{r-1}(\mathbf{t})=f(\mathbf{t})+\int_{a_1}^{b_1(t)}\dots\int_{a_n}^{b_n(t)}[K(\mathbf{t},\mathbf{x})G(u_{r-1}(\mathbf{x}))] \times dx_n\dots dx_1-u_{m-1}(\mathbf{x}),\quad \mathbf{t}=(t_1,t_2,\dots,t_n),\quad \mathbf{x}=(x_1,x_2,\dots,x_n) \tag{15}$$

Solving Eq. (13) with respect to $\Delta u_r(\mathbf{t})$ we obtain a sequence of approximate solution $u_r(\mathbf{t})$.

3. APPROXIMATE SOLUTION BY QUADRATURE METHOD

Introducing a grid

$$W=\left\{\mathbf{t}_j:(t_{1j},t_{2j},\dots t_{nj}):t_{ij}=a_i+jh_i,\ h_i=\frac{b_i-a_i}{m_i}\quad i=1,2,\dots,n,\ j=1,2,\dots,m_i\right\}$$

where m_i refers to the number of partitions in $[a_i,b_i]$, Eq. (13) becomes

$$\Delta u_r(\mathbf{t}_j)-\int_{a_1}^{b_1(t_j)}\int_{a_2}^{b_2(t_j)}\dots\int_{a_n}^{b_n(t_j)}[\Psi_0(\mathbf{t}_j,\mathbf{x};u_0)\Delta u_r(\mathbf{x})] \times dx_n\,dx_{n-1}\dots dx_1=\Phi_{r-1}(\mathbf{t}_j) \tag{16}$$

where

$$\Phi_{r-1}(\mathbf{t}_j) =f(\mathbf{t}_j)+\int_{a_1}^{b_1(t_j)}\int_{a_2}^{b_2(t_j)}\dots\int_{a_n}^{b_n(t_j)}[K(\mathbf{t}_j,\mathbf{x})G(u_{r-1}(\mathbf{x})] \times dx_n\,dx_{n-1}\dots dx_1-u_{r-1}(\mathbf{t}_j),\quad j=1,2,\dots,m_i,\ i=1,2,\dots,n \tag{17}$$

The powerful technique used to approximate the integrations of (16) is a quadrature formula. It is known that Legendre polynomials $P_n(t)$ are orthogonal on $[-1,1]$ with weight $w=1$. Consider the Gauss-Legendre quadrature formula (QF) for multi integral[8]

$$\int_{-1}^{1}\dots\int_{-1}^{1}f(x_1,x_2,\dots,x_n)\,dx_n\dots dx_1 \approx\sum_{j=1}^{m_1}\dots\sum_{j=1}^{m_n}\omega_{m_{1j}}\omega_{m_{2j}}\dots\omega_{m_{nj}}f(s_{1j},s_{2j},\dots,s_{nj})$$

where

$$\omega_{m_{ij}}=\frac{2}{(1-s_{ij}^2)[P'_{m_i}(s_{ij})]^2},\quad \sum_{j=1}^{m_i}\omega_{n_ij}=2,\quad P_{m_i}(s_{ij})\equiv 0,\ i=1,2,\dots,n,\ j=1,2,\dots,m_i \tag{18}$$

are the corresponding weights or Christoffel numbers. s_{ij}, $i=1,2,\dots,n$ and $j=1,2,\dots,m_i$ are roots of Legendre polynomials $P_{m_i}(t)$ over interval $[-1,1]$.

The Gauss-Legendre (QF) formula for arbitrary region $\prod_{i=1}^{n}[a_i,b_i]$ has the form

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\dots\int_{a_n}^{b_n}f(x_1,x_2,\dots,x_n)\,dx_n\,dx_{n-1}\dots dx_1 \approx\prod\left(\frac{b_i-a_i}{2}\right)\sum_{j=1}^{m_1}\sum_{j=1}^{m_2}\dots\sum_{j=1}^{m_n}\omega_{m_{1j}}\omega_{m_{2j}}\dots\omega_{m_{nj}} \times f(x_{1j},x_{2j},\dots,x_{nj}) \tag{19}$$

where the knots $x_{ij}=((b_i-a_i)/2)s_{ij}+((b_i+a_i)/2)$, $i=1,2,\dots,n$ and $j=1,2,\dots,m_i$.

We propose a new idea that introduces a subgrids (W_{m_i}), of ℓ_i Legendre knot points at each subinterval $[a_i,b(t_i)]$, $i=1,2,\dots,n$, which are included in the intervals $[a_i,b_i]$ that appear in the Eq. (16); such that

$$\tau_{m_ij_i}^{k_i}=\frac{b_i(t_{j_i})-a_i}{2}s_{ik_i}+\frac{b_i(t_{j_i})+a_i}{2},\quad i=1,2,\dots,n,\ j_i=1,2,\dots,m_i,\ k_i=1,2,\dots,\ell_i$$

where $\tau_{m_ij_i}^{k_i}\neq t_i$. Extending Gauss-Legendre (QF) (19) to the integral in all subintervals $[a_i,b_i(t_j)]$ in Eq. (16), we get

$$\begin{aligned}\Delta u_r(\tau_{m_1j_1}^{r_1},\tau_{m_2j_2}^{r_2},\dots,\tau_{m_nj_n}^{r_n})-\prod_{i=1}^{n}\left(\frac{b_i-a_i}{2}\right) &\times\sum_{k_1=1}^{j_1}\sum_{k_2=1}^{j_2}\dots\sum_{k_n=1}^{j_n}[\Psi_0(\tau_{m_1j_1}^{r_1},\tau_{m_2j_2}^{r_2},\dots,\tau_{m_nj_n}^{r_n}, \tau_{m_1j_1}^{k_1},\tau_{m_2j_2}^{k_2},\dots,\tau_{m_nj_n}^{k_n};u_0)\\ &\times\Delta u_r(\tau_{m_1j_1}^{k_1},\tau_{m_2j_2}^{k_2},\dots,\tau_{m_nj_n}^{k_n})\omega_{m_1k_1}\omega_{m_2k_1}\dots\omega_{m_nk_n}]\\ &=\Phi_{r-1}(\tau_{m_1j_1}^{r_1},\tau_{m_2j_2}^{r_2},\dots,\tau_{m_nj_n}^{r_n}),\quad i=1,\dots,n,\ j_i=1,\dots,m_i,\ r_i=1,\dots,\ell_i\end{aligned}$$

where

$$\Phi_{r-1}(\tau^{r_1}_{m_1 j_1}, \tau^{r_2}_{m_2 j_2}, \ldots, \tau^{r_n}_{m_n j_n})$$
$$= f(\tau^{r_1}_{m_1 j_1}, \tau^{r_2}_{m_2 j_2}, \ldots, \tau^{r_n}_{m_n j_n}) + \prod_{i=1}^{n}\left(\frac{b_i(t_j)-a_i}{2}\right)$$
$$\times \sum_{k_1=1}^{j_1}\sum_{k_2=1}^{j_2}\cdots\sum_{k_n=1}^{j_n}[K(\tau^{r_1}_{m_1 j_1}, \tau^{r_2}_{m_2 j_2}, \ldots, \tau^{r_n}_{m_n j_n}, \tau^{k_1}_{m_1 j_1}, \tau^{k_2}_{m_2 j_2}, \ldots, \tau^{k_n}_{m_n j_n})$$
$$\times G(u_{r-1}(\tau^{k_1}_{m_1 j_1}, \tau^{k_2}_{m_2 j_2}, \ldots, \tau^{k_n}_{m_n j_n}))\omega_{m_1 k_1}\omega_{m_2 k_1}\ldots\omega_{m_n} k_n]$$
$$- u_{r-1}(\tau^{r_1}_{m_1 j_1}, \tau^{r_2}_{m_2 j_2}, \ldots, \tau^{r_n}_{m_n j_n}) \quad (20)$$

Equation (20) is a linear algebraic system of $(\sum_{i=1}^n m_i) \times (\sum_{i=1}^n \ell_i)$ equations and $(\sum_{i=1}^n m_i) \times (\sum_{i=1}^n \ell_i)$ unknowns. If the non singularity of this system is achieved, then it has unique solution in terms of $\Delta u_r(\mathbf{t}, \mathbf{x})$, $(r = 2, 3, \ldots,)$. From Eq. (14) it follows that

$$u_r(\mathbf{t}, \mathbf{x}) = \Delta_r(\mathbf{t}, \mathbf{x}) + u_{r-1}(\mathbf{t}, \mathbf{x}), \quad r = 2, 3, \ldots \quad (21)$$

4. CONVERGENCE ANALYSIS

Using the general theorems of Modified NM method and their application to functional equations, we state the following theorem for successive approximations, which are characterized by Eq. (13).

First, since $f(\mathbf{t}, \mathbf{x})$, $u_0(\mathbf{t}, \mathbf{x})$, $K(\mathbf{t}, \mathbf{x})$, $G(\xi)$ and $G'(\xi)$ and $G''(\xi)$ are continuous in their domains of definitions, then they are bounded (Ref. [32], p. 33), such that

$$|f(\mathbf{t})| \le R_1, \quad |u_0(\mathbf{t})| \le R_2, \quad |K(\mathbf{t}, \mathbf{x})| \le R_3,$$
$$|G(u_0(\mathbf{t}))| \le R_4, \quad |G'(u_0(\mathbf{t}))| \le R_5,$$
$$|G''(u_0(\mathbf{t}))| \le R_6$$

Next, we use the majorant function[3]

$$Z(t) = \eta(t-t_0)^2 - (1+\eta\beta)(t-t_0) + \beta \quad (22)$$

where η and β are nonnegative real number. Let $\eta_1 = R_3 R_6 \prod_{i=1}^n (b_i - a_i)$.

THEOREM 1. *Let the operator* $\mathbf{Q}(u) = 0$ *in* (3) *is defined in*

$$\Omega_1 = \{u \in C_{\prod_{i=1}^n [a_i, b_i]} \colon \|u - u_0\| \le R\}$$

and has a continuous second derivative in

$$\Omega_0 = \{u \in C_{\prod_{i=1}^n [a_i, b_i]} \colon \|u - u_0\| \le r \le R\}$$

If
1. *The linear integral equation in Eq.* (13) *has a resolvent kernel* $\Gamma(\mathbf{t}, \mathbf{x})$ *where* $\|\Gamma\| \le R_3 R_5 e^{R_3 R_5 \prod_{i=1}^n (b_i - a_i)}$,
2. $|\Delta \mathbf{t}| \le \beta/(1+\eta\beta)$,
3. $|\mathbf{Q}''(\mathbf{t})| \le \eta_1$.

Then Eq. (1) *has a unique solution* $u^*(\mathbf{t})$ *in the closed ball* Ω_0 *and the sequence* $u_r(\mathbf{t})$, $r \ge 0$ *of successive approximation*

$$\Delta u_r(\mathbf{t}_j) - \int_{a_i}^{b_1(t_j)}\int_{a_2}^{b_2(t_j)}\cdots\int_{a_n}^{b_n(t_j}\Psi_0(\mathbf{t}_j, \mathbf{x}; u_0) \times \Delta u_r(\mathbf{x})\,dx_n\,dx_{n-1}\ldots dx_1 = \Phi_{r-1}(\mathbf{t_j}), \quad j = 1, 2, \ldots, m_i \quad (23)$$

where $\Delta u_r(\mathbf{t}) = u_r(\mathbf{t}) - u_{r-1}(\mathbf{t})$ *converges to the solution* $u^*(\mathbf{t})$. *The rate of convergence is given by*

$$\|u^* - u_r\| \le \left(\frac{2}{1+\eta\beta}\right)^r\left(\frac{1}{\eta}\right), \quad r = 1, 2, \ldots \quad (24)$$

PROOF. It is shown that Eq. (3) is reduced to Eq. (9). Therefore, we prove that Eq. (9) has unique solution $\Delta u^*(\mathbf{t})$ in term of resolvent kernel $\Gamma(\mathbf{t}, \mathbf{x})$; provided that its kernel $\Psi_0(\mathbf{t}, \mathbf{x}; u_0)$ is continuous function. Assume the integral operator $\mathbb{U}$ from $C_{\prod_{i=1}^n [a_i, b_i]} \to C_{\prod_{i=1}^n [a_i, b_i]}$ is given by

$$\mathbb{B} = \mathbb{U}(\Delta u),$$
$$\mathbb{B}(\mathbf{t}) = \int_{a_1}^{b_1(t)}\int_{a_2}^{b_2(t)}\cdots\int_{a_n}^{b_n(t)}[\Psi_0(\mathbf{t}, \mathbf{x}; u_0)\Delta u(\mathbf{x})] \times dx_n\,dx_{n-1}\ldots dx_1 \quad (25)$$

where $\Psi_0(\mathbf{t}, \mathbf{x}; u_0)$ is defined in Eq. (10). According to Eq. (9), Eq. (25) can be written as

$$\Delta u - \mathbb{U}(\Delta u) = \mathbb{F}_0(\mathbf{t}) \quad (26)$$

The solution Δu^* of Eq. (26) is written in terms of $\mathbb{F}_0$ by the formula

$$\Delta u^* = \mathbb{F}_0 + \mathbb{Z}(\mathbb{F}_0) \quad (27)$$

where $\mathbb{Z}$ is an integral operator, which can be expanded as a series in power of $\mathbb{U}$ (Ref. [31], Theorem 1, p. 378)

$$\mathbb{Z}(\mathbb{F}_0) = I + \mathbb{U}(\mathbb{F}_0) + \mathbb{U}^2(\mathbb{F}_0) + \cdots + \mathbb{U}^n(\mathbb{F}_0) + \cdots \quad (28)$$

and it is found that the powers of $\mathbb{U}$ are also integral operators. In fact

$$\mathbb{B}_n = \mathbb{U}^n,$$
$$\mathbb{Z}_n(t) = \int_{a_1}^{b_1(t)}\int_{a_2}^{b_2(t)}\cdots\int_{a_n}^{b_n(t)}[\Psi_0^n(\mathbf{t}, \mathbf{x}; u_0)\Delta u(\mathbf{x})] \times dx_n\,dx_{n-1}\ldots dx_1, \quad (n = 1, 2, \ldots) \quad (29)$$

where Ψ_0^n is the iterated kernel. Substituting Eq. (29) into Eq. (27) we obtain the solution of Eq. (26) which is of the form

$$\Delta u^*(\mathbf{t}) = \mathbb{F}_0(\mathbf{t}) + \int_{a_1}^{b_1(t)}\int_{a_2}^{b_2(t)}\cdots\int_{a_2}^{b_n(t)}[\Gamma_0(\mathbf{t}, \mathbf{x}; u_0)\mathbb{F}_0(\mathbf{x})] \times dx_n\,dx_{n-1}\ldots dx_1 \quad (30)$$

where

$$\Gamma_0(\mathbf{t},\mathbf{x};u_0)=\sum_{j=0}^{\infty}\Psi_0^{j+1}(\mathbf{t},\mathbf{x};u_0) \quad (31)$$

is the resolvent kernel. Next, we state that the series in Eq. (30) is convergent uniformly for all $\mathbf{t}=(t_1,t_2,\ldots,t_n)$ and $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, such that $t_i,x_i\in[a_i,b_i]$, $i=1,2,\ldots,n$. Since

$$|\Psi_0(\mathbf{t},\mathbf{x};u_0)|=|K(\mathbf{t},\mathbf{x})||G'(u_0(\mathbf{t}))|\le R_3R_5 \quad (32)$$

Let $R=R_3R_5$, then by mathematical induction we obtain

$$\begin{aligned}&|\Psi_0^2(\mathbf{t},\mathbf{x},u_0)|\\ &\le\int_{a_1}^{b_1(t)}\int_{a_2}^{b_2(t)}\cdots\int_{a_n}^{b_n(t)}|\Psi_0(\mathbf{t},\mathbf{x};u_0)\Psi_0(\mathbf{t},\mathbf{x};u_0)|\\ &\times dx_n\,dx_{n-1}\ldots dx_1\le\frac{R^2\prod_{i=1}^n(b_i-a_i)}{(1)!},\\ &|\Psi_0^3(\mathbf{t},\mathbf{x},u_0)|\\ &\le\int_{a_1}^{b_1(t)}\int_{a_2}^{b_2(t)}\cdots\int_{a_n}^{b_n(t)}|\Psi_0(\mathbf{t},\mathbf{x};u_0)\Psi_0^2(\mathbf{t},\mathbf{x};u_0)|\\ &\times dx_n\,dx_{n-1}\ldots dx_1\le\frac{R^3\prod_{i=1}^n(b_i-a_i)^2}{(2)!},\\ &\vdots\qquad\qquad\vdots\\ &|\Psi_0^n(\mathbf{t},\mathbf{x},u_0)|\\ &\le\int_{a_1}^{b_1(t)}\int_{a_2}^{b_2(t)}\cdots\int_{a_n}^{b_n(t)}|\Psi_0(\mathbf{t},\mathbf{x};u_0)\Psi_0^{n-1}(\mathbf{t},\mathbf{x};u_0)|\\ &\times dx_{n-1}\,dx_{n-1}\ldots dx_1\le\frac{R^n\prod_{i=1}^n(b_i-a_i)^{n-1}}{(n-1)!}\end{aligned}$$

then

$$\begin{aligned}\|\Gamma_0\|&=\|\mathbb{Z}(\mathbb{F}_0)\|\le\sum_{j=0}^{\infty}|\Psi_0^{j+1}(\mathbf{t},\mathbf{x};u_0)|\\ &\le\sum_{j=0}^{\infty}R^{j+1}\frac{\prod_{i=1}^n(b_i-a_i)^j}{j!},\\ &=R\sum_{j=0}^{\infty}R^j\frac{\prod_{i=1}^n(b_i-a_i)^j}{j!}=Re^{R\prod_{i=1}^n(b_i-a_i)}\end{aligned}$$

Therefore, the infinite series in Eq. (31) for $\Gamma_0(\mathbf{t},\mathbf{x};u_0)$ converges uniformly for all $t_i,x_i\in[a_i,b_i]$, $i=1,2,\ldots,n$. Now, we prove $\|\mathbf{Q}''(u)\|\le\eta_1$ for all $u(\mathbf{t})\in\Omega_1$. It is shown that the second derivative $\mathbf{Q}''(u_0)(u)$ of nonlinear operator $\mathbf{Q}(u)$ at the point u_0 refers to the bilinear operator i.e., $\mathbf{Q}''(u_0)(u)=\mathbf{B}(u,u_0)$ (Ref. [31], p. 506). By definition of the second derivative, $P''(u_0)(u)$ has the form

$$\begin{aligned}\mathbf{Q}''(u_0)u&=\lim_{s\to0}\frac{1}{s}[\mathbf{Q}'(u_0+su)-\mathbf{Q}'(u_0)],\\ &=\lim_{s\to0}\frac{1}{s}\left(\frac{d^2\mathbf{Q}}{du^2}(u_0)s\overline{u}+\frac{1}{2}\frac{d^3\mathbf{Q}}{du^3}(u_0+\theta s\overline{u})s^2\overline{u}^2\right),\\ &=\frac{d^2\mathbf{Q}}{du^2}\bigg|_{u_0}\overline{u}\end{aligned}$$

then the norm of $\|d\mathbf{Q}^2/du^2\|$ has the estimate

$$\begin{aligned}\left\|\frac{d\mathbf{Q}^2}{du^2}\right\|&=\max_{\|u\|\le1}\bigg|\int_{a_1}^{b_1(t)}\int_{a_2}^{b_2(t)}\cdots\int_{a_n}^{b_n(t)}[K(\mathbf{t},\mathbf{x})G''(u_0(\mathbf{x}))\\ &\quad\times u(\mathbf{x})\overline{u}(\mathbf{x})]\,dx_n\,dx_{n-1}\ldots dx_1\bigg|\\ &\le R_3R_6\prod_{i=1}^n(b_i-a_i),\ \ i=1,2,\ldots,n\end{aligned}$$

Therefore, the second derivative exist and is bounded. This implies that $u^*(\mathbf{t})$ is the unique solution of operator Eq. (3) (Ref. [31], Theorem 6, p. 532).

The rate of convergence is given by[3]

$$\|u^*-u_r\|\le\left(\frac{2}{1+\eta\beta}\right)^r\left(\frac{1}{\eta}\right),\quad r=1,2,\ldots \quad (33)$$

5. NUMERICAL RESULTS

In this section, our aim is to show the ability of the Modified NM to solve the nonlinear integral equations by giving examples. For computing the results in each table, we use MATLAB V.Ra 2008.

Example 1. Consider the two dimensional nonlinear Volterra integral equation

$$\begin{aligned}u(t_1,t_1)-\int_0^{t_1}\int_0^{t_2}(x_1^2+e^{-2x_2})u^2(x_1,x_2)\,dx_2\,dx_1\\ =t_2^2e^{t_1}+\frac{1}{14}t_2^7-\frac{1}{14}t_2^7e^{2t_1}-\frac{1}{5}t_2^5t_1,\\ (t_1,t_2)\in[0,1]\times[0,1]\end{aligned} \quad (34)$$

The exact solution is

$$u^*(t_1,t_2)=t_2^2e^{t_1}$$

Consider the initial condition is $u_0(t_1,t_2)=t_1t_2^2$, $m_1=m_2=5$, $\ell_1=\ell_2=5$ and $h_1=h_2=0.2$. The absolute error evaluated by Modified NM are compared with the error given by two-dimensional differential transform method (TDDT);[13] as shown in Table I which shows that the results obtained by Modified NM are more accurate than the TDDT method.

Table I. Error analysis of Example 1.

(t_1,t_2)	TDDT method	Modified NM
(0.1, 0.7)	0.525896$E-11$	0.361076$E-16$
(0.2, 0.3)	0.182059$E-14$	0.183065$E-18$
(0.3, 0.9)	0.764536$E-09$	0.481935$E-12$
(0.4, 1.0)	0.437002$E-08$	0.932765$E-10$
(0.5, 0.8)	0.576196$E-09$	0.628106$E-12$
(0.6, 1.0)	0.983256$E-08$	0.269412$E-09$
(0.7, 0.6)	0.468694$E-10$	0.478234$E-11$
(0.8, 1.0)	0.174801$E-07$	0.521569$E-09$
(0.9, 0.5)	0.103376$E-10$	0.341786$E-10$
(1.0, 1.0)	0.273127$E-07$	0.651865$E-08$

Table II. Error analysis of Example 2.

$(t_1, t_2) = (1/2^s, 1/2^s)$	The method in Ref. [15]	Modified NM
$s = 1$	$7.6E-03$	$0.761092E-15$
$s = 2$	$1.8E-02$	$0.982176E-19$
$s = 3$	$3.1E-02$	$0.100653E-19$
$s = 4$	$4.2E-02$	$0.087254E-20$
$s = 5$	$5.0E-02$	$0.009127E-20$
$s = 6$	$4.3E-04$	$0.269412E-22$

EXAMPLE 2. Consider the two dimensional nonlinear Fredholm integral equation

$$u(t_1, t_1) - \int_0^1 \int_0^1 \frac{t_1}{1+t_2}(1+x_1+x_2)u^2(x_1, x_2)\,dx_2\,dx_1 = \frac{1}{(1+t_1+t_2)^2} - \frac{t_1}{6(1+t_2)^2}, \quad (t_1, t_2) \in [0, 1] \times [0, 1] \tag{35}$$

The exact solution is

$$u^*(t_1, t_2) = \frac{1}{(1+t_1+t_2)^2}$$

Consider the initial condition is $u_0(t_1, t_2) = t_1 t_2$, $m_1 = m_2 = 5$, $\ell_1 = \ell_2 = 5$ and $h_1 = h_2 = 0.2$. The absolute error established by Modified NM are compared with the error given by rationalized Haar functions;[15] as illustrated in Table II which reveals that the results obtained by Modified NM are more accurate than the method in Ref. [15].

6. CONCLUSION

In this article, the Modified NM is presented to solve the nonlinear multi dimensional integral equation. We proposed an idea by introducing a subgrid of collocation points $\tau^{k_i}_{m_i j_i}$, $i = 1, 2, \ldots, n$, $j_i = 1, 2, \ldots, m_i$ and $k_i = 1, 2, \ldots, \ell_i$ which are included in $[a, b_i(t)]$. The theorem of existence and uniqueness of approximate solution is established based on the general theorems of Modified NM. Numerical examples are given to show the efficiency of the method.

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