Sensor Structures and General Asymptotic Regional Gradient Observer in Diffusion Systems

Conference Paper · April 2017

Raheam A Mansor Al-Saphory
Tikrit University

Asmaa N. Al-Janabi
Tikrit University

Sensor Structures and General Asymptotic Regional Gradient Observer in Diffusion System

[1]Raheam A. Al-Saphory, [2]Naseif J. Al-Jawari, [3]Asmaa N. Al-Janabi

[1]Department of Mathematics, College of Education for Pure Sciences, Tikrit University, Tikrit, IRAQ.
[2]Department of Mathematics, College of Science, Al-Mustansriyah University, Baghdad, IRAQ.
[3]Department of Mathematics, College of Science, Al-Mustansriyah University, Baghdad, IRAQ.

ABSTRACT

The main idea of this paper is to give and explore original results related to the general asymptotic regional gradient observer for a diffusion system in connection with gradient strategic sensors structures. Thus, we establish a definitions and characterization of this notion and show that under what condition the general asymptotic regional gradient observer guaranteed. The obtained results are applied through the example with various gradient sensors structure.

KEYWORDS: exactly ω_G-observable, ω_{AG}-detectability, ω_{AG}-observers, ω_G-strategic sensors, Diffusion system.

INTRODUCTION

The analysis of distributed parameter system is motivated to be able to deal with many questions in real problems such as mechanic, thermic and environments as in [13-14]. Thus, the concept of observability notion has been developed and survey of these developments as in [15-16]. One of the most important notion in system theory is focused on reconstruction the state of the system from knowledge of dynamic system and output function not in whole the domain but only in the sub-region of a spatial domain as in [6,19]. An extension of theses notion is found in[7] when one may be reconstruction the gradient state observation in a critical sub-region of the spatial domain. These concepts have been extension from regional internal gradient case to the regional boundary gradient case as in [10, k[22]. Thus, it is very necessary to deal with the observer theory concept to estimate (observe) the state of dynamic system by building another dynamic system from measured input and output of the original system [17,20]. Thus, the study of a asymptotic regional behavior of the system has been introduced by Al-Saphory and El Jai [3-4]. Thus , the principal reason for introducing this concept is that, firstly it makes sense for the usual observer concept closer to real world problems, and secondly, it can be introduced and explore the original results related to the general asymptotic regional gradient observer in [11] for a diffusion system in connection with gradient strategic sensors structure. Moe precisely, we establish a definition and characterization of this notion and we show that under what condition the general asymptotic regional gradient observer is guaranteed. This is the subject of this paper which is organized as follows:

In section two is devoted to the presentation of the system under consideration. Section 3 we give some definitions related to the ω_G-strategic sensors, ω_{AG}-detectability and ω_{AG}-observers. In section 4 we gives a characterize for existing ω_{AG}-observers to provide a general asymptotic regional gradient estimator of gradient state for an original system via sensors structures and we show that there exist a general asymptotic ω_G-observers in the subregion is not general asymptotic G- observer in the whole the domain Ω.

2. Problem Statement:

Let Ω be a regular bounded open subset of R^n, with smooth boundary $\partial\Omega$ and ω be subregion of Ω, [0,T], $T > 0$ be a time measurement interval. we denoted $Q = \Omega \times]0,T[$, $\Sigma = \partial\Omega \times]0,T[$. We considered distributed parabolic system is described by the following equation:

$$\begin{cases} \frac{\partial x}{\partial t}(\xi,t) = \Delta x(\xi,t) + Bu(t) & Q \\ x(\xi,0) = x_0(\xi) & \Omega \\ x(\eta,t) = 0 & \Sigma \end{cases} \tag{1}$$

Augmented with the output function

$$y(.,t) = Cx(\ ,t) \tag{2}$$

Where Δ is a second order linear differential operator, which generator a strongly continuous semi-group $(S_A(t))_{t\geq 0}$on the Hilbert space X and is self-adjoin with compact resolvent. The operator $B \in L(R^p, X)$ and $C \in L(R^q, X)$, depend on the structure of actuators and sensors [18]. The space X, U and $\mathcal{O}$ be separable Hilbert spaces where X is the state space, $U = L^2(0,T,R^p)$ is the control space and $\mathcal{O} = L^2(0,T,R^q)$ is the observation space where p and q are the numbers of actuators and sensors (see Figure 2).

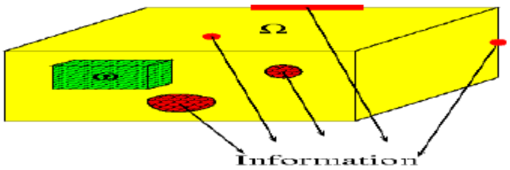

Fig. 2: The domain of $\boldsymbol{\Omega}$, the sub-region $\boldsymbol{\omega}$, various sensors locations

Under the given assumption, the system (1) has a unique solution [21]:

$$x(\xi,t) = S_A(t)x_0(\xi) + \int_0^t S_A(t-\tau)Bu(\tau)\,d\tau \tag{3}$$

The measurements are obtained through the output function by using of zone, point wise which may located in Ω (or $\partial\Omega$). [18]

$$y(.,t) = Cx(\xi,t) \tag{4}$$

- We first recall a sensors is defined by any couple (D, f), where D is its spatial support represented by a nonempty part of $\bar{\Omega}$ and f represents the distribution of the sensing measurements on D. Depending on the nature of D and f, we could have various type of sensors. A sensor may be pointwise if $D=\{b\}$ with $b \in \bar{\Omega}$ and $f = \delta(.-b)$, where δ is the Dirac mass concentrated at b. In this case the operator C is unbounded and the output function (2) can be written in the form

$$y(t) = x(b,t)$$

It may be zonal when $D \subset \bar{\Omega}$ and $f \in L^2(D)$. The output function (2) can be written in the form

$$y(t) = \int_D x(\xi,t)f(\xi)d\xi$$

In the case of boundary zone sensor, we consider $D_i = \Gamma_i \subset \partial\Omega$ and $f_i \in L^2(\Gamma_i)$, the output function (2) can be written as

$$y(.,t) = Cx(.,t) = \int_{\Gamma_i} x(\eta,t)\,f_i(\eta)\,d\eta$$

- We define the operator

$$K: x \in X \to Kx = CS_A(.)x\epsilon\,\mathcal{O}$$

We note that $K^*: \mathcal{O} \to X$ is the adjoint operator of K defined by

$$K^*y^* = \int_0^t S_A^*(s)C^*y^*(s)ds$$

- Consider the operator

$$\nabla: H^1(\Omega) \to (H^1(\Omega))^n$$

$$x \to \nabla x = \left(\frac{\partial x}{\partial \xi_1}, \dots, \frac{\partial x}{\partial \xi_n}\right)$$

and the adjoint denotes by ∇^* is given as

$$\nabla^*: (H^1(\Omega))^n \to H^1(\Omega)$$

$$x \to \nabla^* x = v$$

where v is a solution of the Dirichlet problem

$$\begin{cases} \Delta v = -\mathrm{div}(x) & in\ \Omega \\ v = 0 & on\ \ \partial\Omega \end{cases}$$

- For a nonempty subset $\omega \subset \Omega$ consider the regional gradient restriction operator

$$\chi_\omega: (H^1(\Omega))^n \to (H^1(\omega))^n$$

$$x \quad \to \quad \chi_\omega x = x\,|_\omega$$

It's adjoint is denoted by χ_ω^*.

- $$\tilde{\chi}_\omega: \begin{cases} H^1(\Omega) \to H^1(\omega) \\ x \to \tilde{\chi}_\omega x = x\,|_\omega \end{cases}$$

where $x\,|_\omega$ is the restriction of x to ω.

It's adjoint is denoted by $\tilde{\chi}_\omega^*$.

- Finally, we introduced the operator $H = \chi_\omega \nabla K^*$ from $\mathcal{O}$ into $(H^1(\omega))^n$.

The problem is how to build an approach which observe (estimates) an asymptotic regional gradient state in subregion in ω of Ω in general case only.

3. Sensor and Asymptotic ω_G-Detectability:

This concept of regional detectability is introduced and extended by Al-Saphory and El-Jai as in [1, 3-4] and then for the boundary case see [10-11]. In this section, we extend some definitions and characterizations to the concept of asymptotic regional gradient detectability in order to construct an general asymptotic ω_G-observer for the current gradient state of the original system. For this purpose we need some definitions and characterization as in [6, 9-10].

3.1 Definitions and Characterizations:

Definition 3.1:

The systems (1)-(2) are said to be exactly regionally gradient observable on ω (exactly ω_G –observable) if $Im\ H = Im\ \chi_\omega \nabla K^* = (H^1(\omega))^n$

Definition 3.2:

The systems (1)–(2) are said to be weakly regionally gradient observable on ω (weakly ω_G –observable) if $\overline{Im H} = \overline{Im\ \chi_\omega \nabla K^*} = (H^1(\omega))^n$

Remark 3.3:

The definition 3.2 is equivalent to say that the systems (1)-(2) are weakly ω_G –observable if $ker\ \ H^* = ker\ K\nabla^* {\chi_\omega}^* = \{0\}$

Definition 3.4:

A sensor (D, f) is gradient strategic on ω (ω_G-strategic) if the observed system is weakly ω_G-observable.

Definition 3.5:

The system (1)is said to be gradient stable (Ω_G-stable) if the operator A generates a semi-group which is gradient stable on the $(H^1(\Omega))^n$. It is easy to see that the system (1) is Ω_G-stable, if and only if for some positive constants M, α, we have

$\|\nabla S_A(.)\|_{(H^1(\Omega))^n} \le Me^{-\alpha t}, \forall t \ge 0$

If $(S_A(t))_{t\ge 0}$ is Ω_G-stable semi-group in $(H^1(\Omega))^n$, then for all $x_0 \in H^1(\Omega)$, the solution of associated system satisfies

$$\lim_{t\to\infty}\|\nabla x(.,t)\|_{(H^1(\Omega))^n} = \lim_{t\to\infty}\|\nabla S_A(.)x_0\|_{(H^1(\Omega))^n} = 0 \qquad (5)$$

Definition 3.6:

The systems (1)-(2) are said to be asymptotic gradient detectable (asymptotic Ω_G-detectable) if there exists an operator $H_\Omega: R^q \to (H^1(\Omega))^n$ such that $(A - H_\Omega C)$ generates a strongly continuous semi-group $(S_{H_\Omega}(t))_{t\ge 0}$ which is asymptotically G-stable on $(H^1(\Omega))^n$.

Definition 3.7:

The system (1) is said to be regional gradient stable (ω_G-stable) if the operator A generates a semi-group which is gradient stable on the $(H^1(\omega))^n$. It is easy to see that the system (1) is ω_G-stable, if and only if for some positive constants $M_{\omega_G}, \alpha_{\omega_G}$, we have

$\|\chi_\omega \nabla S_A(.)\|_{(H^1(\omega))^n} \le M_{\omega_G} e^{-\alpha_{\omega_G} t}, \forall t \ge 0$

If $(S_A(t))_{t\ge 0}$ is ω_G-stable semi-group in $(H^1(\omega))^n$, then for all $x_0 \in H^1(\Omega)$, the solution of associated system satisfies

$$\lim_{t\to\infty}\|\chi_\omega \nabla x(.,t)\|_{(H^1(\omega))^n} = \lim_{t\to\infty}\|\chi_\omega \nabla S_A(.)x_0\|_{(H^1(\omega))^n} = 0 \qquad (6)$$

Definition 3.8:

The systems (1)-(2) are said to be asymptotic regional gradient detectable (asymptotic ω_G-detectable) if there exists an operator $H_{\omega_G}: R^q \to (H^1(\omega))^n$ such that $(A - H_{\omega_G} C)$ generates a strongly continuous semi-group $(S_{H_{\omega_G}}(t))_{t\ge 0}$ which is asymptotically G-stable on $(H^1(\omega))^n$.

Remark 3.9:

In this paper, we only need the relation (6) to be true on a sub-region ω of the region Ω

$\lim_{t\to\infty}\|\nabla x(.,t)\|_{(H^1(\omega))^n} = 0$

Now, to study the relation between the asymptotic ω_G-detectable and ω_G-strategic sensors. Let us consider the systems (1)-(2) which are observed by q sensors $(D_i, f_i)_{1\le i\le q}$ with $D_i \subset \Omega$ and $f_i \in H^1(\Omega)$ for $i = 1, \dots, q$. We assume that the operator A has a complete set of eigenfunctions denoted by ψ_{mj} in $(H^1(\Omega))^n$ orthonormal in $(H^1(\omega))^n$ associated with the eigenvalues λ_m of multiplicity r_m and suppose that the system (1) has unstable modes. We have the following sufficient condition for existence of asymptotic ω_G-detectability in terms of the strategic sensors structure.

Theorem 3.10:

Suppose that there are q zone sensors $(D_i, f_i)_{1\le i\le q}$ and the spectrum of A contains J eigenvalues with non-negative real parts. The system(1) together with output function (2) is ω_G- detectable if and only if :

1. $q \ge r$
2. rank $G_m = r_m$, $\forall m, m = 1, \dots, J$ with

$$G_m = (G_m)_{ij} = \begin{cases} \psi_{mj}(b_i), f_i(.) >_{L^2(D_i)} \text{ for zone sensors} \\ \psi_{mj}(b_i) \text{for pointwise sensors} \\ < \dfrac{\partial \psi_{mj}}{\partial v}, f_i(.) >_{L^2(\Gamma_i)} \text{ for boundary zone sensors} \end{cases}$$

where sup $r_m = m < \infty$ and j $= 1, \dots, r_m$.

Proof:

The proof is limited to the case of zone sensors. Under the assumptions of section 2, the system (1) can be decomposed by the projections P and $I - P$ on two parts, unstable and stable. The state vector may be given by $x(\xi, t) = [x_1(\xi, t), x_2(\xi, t)]^{tr}$ where $x_1(\xi, t)$ is the state component of the unstable part of the system (1), may be written in the form

$$\begin{cases} \frac{\partial x_1}{\partial t}(\xi, t) = A_1 x_1(\xi, t) + PBu(t) & Q \\ x_1(\xi, 0) = x_{01}(\xi) & \Omega \\ x_1(\eta, t) = 0 & \Theta \end{cases} \tag{7}$$

and $x_2(\xi, t)$ is the component state of the stable part of the system(1) given by

$$\begin{cases} \frac{\partial x_2}{\partial t}(\xi, t) = A_2 x_2(\xi, t) + (I - P)Bu(t) & Q \\ x_2(\xi, 0) = x_{02}(\xi) & \Omega \\ x_2(\eta, t) = 0 & \Theta \end{cases} \tag{8}$$

The operator A_1 is represented by a matrix of order $(\sum_{m=1}^{J} r_m, \sum_{m=1}^{J} r_m)$ given by $A_1 = \text{diag}[\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_J, \dots, \lambda_J]$ and $PB = [G_1^{tr}, G_2^{tr}, \dots, G_J^{tr}]$. The condition (2) of this theorem, allows that the suit $(D_i, f_i)_{1\le i\le q}$ of sensors is ω_G-strategic for the unstable part of the system (1), the subsystem (7) is weakly ω_G-observable[8-9], and since it is finite dimensional, then it is exactly ω_G-observable. Therefor it is ω_G-detectable, and hence there exists an operator H_ω^1 such that $(A_1 - H_\omega^1 C)$ which is satisfied the following:
$\exists M_\omega^1, \alpha_\omega^1 > 0$ such that $\left\| e^{(A_1 - H_\omega^1 C)t} \right\|_{(H^1(\omega))^n} \le M_\omega^1 e^{-\alpha_\omega^1 (t)}$

and then we have

$$\|x_1(., t)\|_{(H^1(\omega))^n} \le M_\omega^1 e^{-\alpha_\omega^1 (t)} \|Px_0(.)\|_{(H^1(\omega))^n}.$$

Since the semi-group generated by the operator A_2 is stable on $(H^1(\omega))^n$, then there exist M_ω^2, $\alpha_\omega^2 > 0$ such that

$$\|x_2(., t)\|_{(H^1(\omega))^n} \le M_\omega^2 e^{-\alpha_\omega^2 (t)} \|(I - P)x_0(.)\|_{(H^1(\omega))^n} + \int_0^t M_\omega^2 e^{-\alpha_\omega^2 (t)} \|(I - P)x_0(.)\|_{(H^1(\omega))^n} \|u(\mathcal{T})\| \, d\mathcal{T}$$

and therefor $x(\xi, t) \to 0$ when $t \to \infty$. Finally, the system (1)-(2) are ω_G-detectable. Reciprocally, if the system (1) together with the output function (2) is ω_G -detectable, there exists an operator $H_\omega \in L(R^q, (H^1(\omega))^n$, such that $(A - H_{\omega_G} C)$ generates a ω_G-stable, strongly continuous semi-group $(S_{H_\omega}(t))_{t\ge 0}$ on the space $(H^1(\omega))^n$, which is satisfied the following:
$\exists M_\omega, \alpha_\omega > 0$ such that $\left\| \chi_\omega \nabla S_{H_\omega}(.) \right\|_{(H^1(\omega))^n} \le M_\omega e^{-\alpha_\omega (t)}$.

Thus, the unstable subsystem (7) is ω_G-detectable. We recall that a system is weakly ω_G-observable, i.e. $[K\nabla^* \chi_\omega^* x^*(., t) = 0 \Longrightarrow x^*(., t) = 0]$ [8-9]. For $x^*(., t) \in (H^1(\omega))^n$, we have

$$K\nabla^* \chi_\omega^* x^*(., t) = (\sum_{m=1}^{J} e^{\lambda_{m_j} t} < \psi_{m_j}(.), \chi_\omega^* \nabla x^*(., t) >_{(H^1(\omega))^n}, < \psi_{m_j}(.), f_j(., t) >_{H^1(\Omega)})_{1\le i\le q}$$

$$= \sum_{m=1}^{J} e^{\lambda_{m_j} t} < \nabla \psi_{m_j}(.), x^*(., t) >_{(H^1(\omega))^n} < \psi_{m_j}(.), f_j(., t) >_{H^1(\Omega)})_{1\le i\le q}$$

$$= \sum_{m=1}^{J} e^{\lambda_{m_j} t} < \psi_{m_j}(.), x^*(., t) >_{(H^1(\omega))^n} < \psi_{m_j}(.), f_j(., t) >_{(H^1(\omega))^n})_{1\le i\le q}$$

If the rank $G_m x_m \neq r_m$ for $m, m = 1, \ldots, J$, there exists $x^*(.,t) \in (H^1(\omega))^n$, such that $K\nabla^* \chi_\omega^* x^*(.,t) = 0$, this leads

$\sum_{m=1}^{J} < \psi_{m_j}(.), x^*(.,t) >_{(H^1(\omega))^n} < \psi_j(.), f_j(.,t) >_{H^1(\Omega)} = 0.$

The state vectors x_m may be given by

$x_m(.,t) = [< \psi_{1_j}(.), x^*(.,t) >_{(H^1(\omega))^n} < \psi_{Jm}(.), x^*(.,t) >_{H^1(\Omega)}]^{tr} \neq 0$

we then obtain $G_m x_m = 0$ for $m, m = 1, \ldots, J$. Consequently, the subsystem (7) is not weakly ω_G -observable and therefore the suite $(D_i, f_i)_{1\le i\le q}$ of sensors is not ω_G-strategic. Thus, the system (1)-(2) is not ω_G-detectable. Finally we have the rank $G_m \neq r_m$ for all $m, m = 1, \ldots, J$.□

Now, it is clear that:

1. A system which is G- detectable, is ω_G- detectable.
2. A system which is exponentially ω_G- detectable, is asymptotically ω_G- detectable.
3. A system which is asymptotically ω_G-detectable, is ω_G^1- detectable, for every subset ω_1 of ω but the converse is not true.

4. General Asymptotic ω_G-observers:

The purpose of this section is to introduced an approach which enable to achieve the existence of general asymptotic regional gradient observer ($GA\omega_G$-observer) in general case is derived from [1,17].This approach which allows to construct the current gradient state in ω of the systems (1)-(2) in connection of asymptotically ω_G -detectability and ω_G -strategic sensors in order to characterize the asymptotic regional gradient observability. Thus, it's necessary to approximate the dynamic characteristics of the observer in subspace spanned by a finite number of suitably chosen bases. Let us consider the operator A has a complete set of eigenfunctions φ_{nj} in $(H^1(\Omega))^n$ orthonormal to $(H^1(\omega))^n$ associated with the eigenvalues λ_n of multiplicity r_n and suppose that the system (1) has unstable modes. Then, we present sufficient conditions for existing a $GA\omega_G$-observer in order to provide an approximation to the gradient state of observed system.

4.1 Definitions:

Definition 4.1:

Suppose there exists a dynamical system with state $z(.,t) \in Z$ given by

$$\begin{cases} \frac{\partial z(,t)}{\partial t} = F_{\omega_G} z(,t) + G_{\omega_G} u(t) + H_{\omega_G} y(t) & Q \\ z(\xi, 0) = z_0(\xi) & \Omega \\ z(\eta, t) = 0 & \Sigma \end{cases} \qquad (9)$$

Where F_{ω_G} generator a strongly continuous semi-group $(S_{F_{\omega_G}}(t))_{t\ge 0}$ on separable Hilbert space Z which is asymptotically ω_G –stable. Thus, $\exists M_{F_{\omega_G}}, \alpha_{F_{\omega_G}} > 0$ such that

$\left\| S_{F_{\omega_G}}(.) \right\| \le M_{F_{\omega_G}} e^{-\alpha_{F_{\omega_G}} t}, \forall t \ge 0.$

and let $G_{\omega_G} \in L(U, Z), H_{\omega_G} \in L(\mathcal{O}, Z)$ such that the solution of (9) similar to (3)

$$z(\xi, t) = S_{F_{\omega_G}}(t) z(\xi) + \int_0^t S_{F_{\omega_G}}(t-\tau)[G_{\omega_G} u(\tau) + H_{\omega_G} y(\tau)] d\tau$$

The system (9) defines asymptotic regional gradient estimator for $\chi_\omega \nabla T x(\xi, t)$ where $x(\xi, t)$ is the solution of the systems (1)-(2) if

$\lim_{t\to\infty} \| z(.,t) - \chi_\omega \nabla T x(\xi, t) \|_{(H^1(\omega))^n} = 0$

and $\chi_\omega \nabla T$ maps $D(\Delta)$ into $D(F)$ where $z(\xi,t)$ is the solution of system (9). And (9) specifies an asymptotic ω_G-observer of the system given by (1) and (2) if the following holds:

1- There exists $M_{\omega_G} \in L(R^q, (H^1(\omega))^n)$ and $N_{\omega_G} \in L((H^1(\omega))^n)$ such that $M_{\omega_G} C + N_{\omega_G} \chi_\omega \nabla \mathrm{T} = I_{\omega_G}$.

2- $\chi_\omega \nabla T \Delta - F_{\omega_G} \chi_\omega \nabla T = H_{\omega_G} \mathrm{C}$, $G_{\omega_G} = \chi_\omega \nabla T \mathrm{B}$.

3- The system (9) defines an asymptotic ω_G-estimator for $\chi_\omega \nabla \mathrm{T} x(\xi, t)$.

The object of an asymptotic ω_G-estimator (asymptotic ω_G-observer) is to provide an approximation to the gradient state of the original system. This approximation is given by

$\hat{x}(t) = M_{\omega_G} y(t) + N_{\omega_G} z(t)$

Definition 4.2:

The systems (1)-(2) are general asymptotic regional gradient observable ($AG\omega_G$-observable) if there exists a dynamical system which is $GA\omega_G$-observer for the original system.

4.2 $GA\omega_G$-observer reconstruction method:

The problem of studying asymptotic ω_G-observability may be through the observation operator C, that means, we can the characterize the $GA\omega_G$-observer by a good choice of the sensors structure. For this objective suppose the system has q sensors $(D_i, f_i)_{1\le i\le q}$.

Consider the system

$$\begin{cases} \frac{\partial x}{\partial t}(\xi,t) = \Delta x(\xi,t) + Bu(t) & Q \\ x(\xi,0) = x_0(\xi) & \Omega \\ x(\eta,t) = 0 & \Sigma \end{cases} \tag{10}$$

In this case the output function (2)

$$y(t) = Cx(.,t) \tag{11}$$

Let ω be a given subdomain of Ω and suppose that $T \in \mathcal{L}((H^1(\Omega))^n)$, and $\chi_\omega \nabla Tx(\xi,t) = T_\omega \nabla x(\xi,t)$ there exists a system with state $z(\xi,t)$ such that

$$z(\xi,t) = \chi_\omega \nabla Tx(\xi,t) \tag{12}$$

From equation (11) and (12) we have

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} C \\ \chi_\omega \nabla T \end{bmatrix} x$$

If we assume that there exist two bounded linear operators $M_{\omega_G}: \mathcal{O} \to ((H^1(\omega))^n$ and $N_{\omega_G}: ((H^1(\omega))^n \to ((H^1(\omega))^n$, such that $M_{\omega_G}C + N_{\omega_G}T_\omega = I$, then by deriving $z(\xi,t)$ in (12) we have

$$\frac{\partial z}{\partial t}(\xi,t) = \chi_\omega \nabla T \frac{\partial x}{\partial t}(\xi,t) = \chi_\omega \nabla T\Delta x(\xi,t) + \chi_\omega \nabla TBu(t)$$

$$= \chi_\omega \nabla TAM_{\omega_G} y(\xi,t) + \chi_\omega \nabla TAN_{\omega_G} z(\xi,t) + \chi_\omega \nabla TBu(t)$$

Now, consider the system (which is destined to be the ω_G-observer)

$$\begin{cases} \frac{\partial \hat{z}}{\partial t}(\xi,t) = F_{\omega_G}\hat{z}(\xi,t) + G_{\omega_G}u(t) + H_{\omega_G}y(.,t) & Q \\ \hat{z}(\xi,0) = z_0(\xi) & \Omega \\ \hat{z}(\eta,t) = 0 & \Sigma \end{cases} \tag{13}$$

Where F_{ω_G} generates a strongly continuous semi-group $(S_{F_{\omega_G}}(t))_{t\ge 0}$ on separable Hilbert space Z which is asymptotically gradient stable thus,

$\exists M_{F_{\omega_G}}, \alpha_{F_{\omega_G}} > 0$ such that $\left\|\chi_\omega \nabla S_{F_{\omega_G}}(.)\right\| \le M_{F_{\omega_G}} e^{-\alpha_{F_{\omega_G}} t}, \forall t \ge 0.$

and let $G_{\omega_G} \in L(U,Z)$, $H_{\omega_G} \in L(\mathcal{O}, Z)$ such that the solution of (13) is given by

$$\hat{z}(\xi,t) = S_{F_{\omega_G}}(t)\hat{z}_0(\xi) + \int_0^t S_{F_{\omega_G}}(t-\tau)[G_{\omega_G}u(\tau) + H_{\omega_G}x(b_i,\tau)]d\tau$$

Now, we present the main result which enable to observe asymptotically the current gradient state in ω, by the gradient state of the system (13).

Theorem 4.3:

Suppose that the operator F_{ω_G} generates a strongly continuous semi-group which is ω_G-stable on $((H^1(\omega))^n$, then, the system (13) is $GA\omega_G$-observer for systems (1)-(2), that is,

$$\lim_{t\to\infty}[\chi_\omega \nabla Tx(\xi,t) - \hat{z}(\xi,t)] = 0$$

If the following conditions hold:

1- There exists $M_{\omega_G} \in L(R^q, (H^1(\omega))^n)$ and $N_{\omega_G} \in L((H^1(\omega))^n)$ such that $M_{\omega_G}C + N_{\omega_G}\chi_\omega \nabla T = I_\omega$.

2- $\chi_\omega \nabla T\Delta - F_{\omega_G}\chi_\omega \nabla T = H_{\omega_G}C$, $G_{\omega_G} = \chi_\omega \nabla TB$.

Proof:

Let $z(\xi,t) = \chi_\omega \nabla Tx(\xi,t)$ and $\hat{z}(\xi,t)$ be a solution of (13) and let us denote the observer error by the following form

$$e(\xi,t) = z(\xi,t) - \hat{z}(\xi,t)$$

We have

$$\frac{\partial e}{\partial t}(\xi,t) = \frac{\partial z}{\partial t}(\xi,t) - \frac{\partial \hat{z}}{\partial t}(\xi,t) = \chi_\omega \nabla T\Delta x(\xi,t) + \chi_\omega \nabla TBu(t) - F_{\omega_G}\hat{z}(\xi,t) - G_{\omega_G}u(t) - H_{\omega_G}y(\xi,t)$$

$$= F_{\omega_G}e(\xi,t) - F_{\omega_G}z(\xi,t) + \chi_\omega \nabla T\Delta x(\xi,t) - H_{\omega_G}y(\xi,t) + \chi_\omega \nabla TBu(t) - G_{\omega_G}u(t)$$

$$= F_{\omega_G}e(\xi,t) + \left[\chi_\omega \nabla T\Delta x(\xi,t) - F_{\omega_G}\chi_\omega \nabla Tx(\xi,t) - H_{\omega_G}Cx(\xi,t)\right] + [\chi_\omega \nabla TBu(t) - G_{\omega_G}u(t)]$$

$= F_{\omega_G} e(\xi, t) + [\chi_\omega \nabla T\Delta - F_{\omega_G}\chi_\omega \nabla T - H_{\omega_G} C]x(\xi, t) + [\chi_\omega \nabla TB - G_{\omega_G}]u(t) = F_{\omega_G} e(\xi, t)$

Consequently $e(\xi, t) = S_{F\omega_G}(t)[\chi_\omega \nabla T x_0(.) - \hat{z}_0(.)]$. the G-stability of the operator allows to obtain $\|e(\xi, t)\|_{(H^1(\omega))^n)} \leq M_{F_{\omega_G}} e^{-\alpha_{F\omega_G} t} \|\chi_\omega \nabla T x_0(\xi) - \hat{z}_0(\xi)\|_{(H^1(\omega))^n)}$

And therefore $\lim_{t\to\infty} e(\xi, t) = 0$.

Now, let the approximate solution to the gradient state of the original system is $\hat{x}(\xi, t) = M_{\omega_G} y(., t) + N_{\omega_G} \hat{z}(\xi, t)$, then we have

$\hat{e}(\xi, t) = x(\xi, t) - \hat{x}(\xi, t) = x(\xi, t) - M_{\omega_G} y(., t) - N_{\omega_G} \hat{z}(\xi, t)$,

$= x(\xi, t) - M_{\omega_G} Cx(\xi, t) - N_{\omega_G}\chi_\omega \nabla T x(\xi, t) + N_{\omega_G}[\chi_\omega \nabla T x(\xi, t) - \hat{z}(\xi, t)]$

$= N_{\omega_G}[\chi_\omega \nabla T x(\xi, t) - \hat{z}(\xi, t)] = N_{\omega_G}[z(\xi, t) - \hat{z}(\xi, t)] = N_{\omega_G} e(\xi, t)$

Finally, we have

$\lim_{t\to\infty} \hat{z}(\xi, t) = z(\xi, t)$. □

Thus, we can deduced a sufficient condition for existence $GA\omega_G$-observer is formulated in the following theorem.

Remark 4.4:

We can deduced that

1. The theorem 4.3 gives the conditions which guarantee that the dynamical system (9) is a $GA\omega_G$ -observer for the system(1)-(2).
2. From theorem 4.3 we get the relation between the regional gradient strategic sensor and $GA\omega_G$ -observer.
3. A system which is an AGG-observer is $AG\,\omega_G$-observer.
4. If a system is $GA\omega_G$-observer, then it is $GA\omega_G^1$-observer in every subset ω_1 of ω, but the converse is not true. This is may be explain in the following example;

Example 4.5:

Consider the system

$$\begin{cases} \frac{\partial x}{\partial t}(\xi, t) = \gamma_1 \frac{\partial^2 x}{\partial \xi^2}(\xi, t) + \gamma_2 x(\xi, t) &]0, a[, t > 0 \\ x(\xi, 0) = x_0(\xi) &]0, a[\\ x(0, t) = x(a, t) = 0 & t > 0 \end{cases} \quad (14)$$

Where $\gamma_1 > 0$, $\gamma_2 > 0$ and Ω=$]0, a[$. Let $b_i \in \Omega$ are the locations of the pointwise sensors (b_i, δ_{b_i}). Then the augmented output function is given by:

$$\boldsymbol{y}(., \boldsymbol{t}) = \int_\Omega x(\xi, t)\,\delta(\xi - b_i)d\xi. \quad (15)$$

Now, consider the dynamical system

$$\begin{cases} \frac{\partial z}{\partial t}(\xi, t) = \gamma_1 \frac{\partial^2 z}{\partial \xi^2}(\xi, t) + \gamma_2 z(\xi, t) - HC(\,z(\xi, t) - x(\xi, t)) &]0, a[, t > 0 \\ z(\xi, 0) = z_0(\xi) &]0, a[\\ z(\eta, t) = 0 & t > 0 \end{cases} \quad (16)$$

Let $\omega = (\alpha, \beta)$ be a subregion of Ω. The eigenfunctions and the eigenvalue related to the operator Δ= $(\gamma_1 \frac{\partial^2}{\partial \xi^2} + \gamma_2)$ are given by:

$\varphi_n(\xi) = \left(\frac{2}{\beta-\alpha}\right)^{\frac{1}{2}} sinn\pi\left(\frac{\xi-\alpha}{\beta-\alpha}\right)$ and $\lambda_n = 1 - \left(\frac{n\pi}{\beta-\alpha}\right)^2$.

Now, if $\frac{b_i}{a} \in Q \cap (0, a)$ then the sensor is not ω_G-strategic [8-9] for unstable subsystem of (14) and therefor the system (14)-(15) is not ω_G-detectable in Ω [4]. Then the dynamical system (16) is not ω_G – observer for the system (14)-(15). Thus, the system (14)-(15) is ω_G-detectable if $(\frac{b_i-\alpha}{\beta-\alpha}) \notin Q \cap (0, a)$. That means, the sensors (b_i, δ_{b_i}) are ω_G-strategic to unstable subsystem of (14) and therefor the systems (14)-(15) are ω_G-detectable. Therefor, the dynamical systems (16) are ω_G – observer for the system (14)-(15) [4]. □

5. Application to $GA\omega_G$-observer:

In this section, we give some results related to different types of measurements domains and boundary conditions. We consider the distributed diffusion systems defined on two dimensional domain $]0, a_1[\times]0, a_2[$ on Ω.

$$\begin{cases} \frac{\partial x}{\partial t}(\xi_1,\xi_2,t) = \frac{\partial^2 x}{\partial \xi_1^2}(\xi_1,\xi_2,t) + \frac{\partial^2 x}{\partial \xi_2^2}(\xi_1,\xi_2,t) & Q \\ x(\xi_1,\xi_2,0) = x_0(\xi_1,\xi_2) & \Omega \\ x(\xi,\eta,t) = 0 & \Sigma \end{cases} \quad (17)$$

The augmented output function is given by

$$y(.,t) = \int_{\Gamma_0} x(\eta_1,\eta_2,t) f(\eta_1,\eta_2) d\eta_1 d\eta_2 \quad (18)$$

Let $\Gamma = \{a_1\} \times]0,a_2[$ be a region on $]0,a_1[\times]0,a_2[$. The eigenfunctions of the operator $(\frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2})$ for the Dirichlet boundary condition are defined by

$$\varphi_{ij}(\xi_1,\xi_2) = \frac{2}{\sqrt{a_1 a_2}} sin\ i\pi \frac{\xi_1}{a_1} sinj\pi \frac{\xi_2}{a_2}$$

Associated with the eigenvalues

$$\lambda_{ij} = (\frac{i^2}{{a_1}^2} + \frac{j^2}{{a_2}^2})\pi^2$$

The dynamical system

$$\begin{cases} \frac{\partial z}{\partial t}(\xi_1,\xi_2,t) = \frac{\partial^2 z}{\partial \xi_1^2}(\xi_1,\xi_2,t) + \frac{\partial^2 z}{\partial \xi_2^2}(\xi_1,\xi_2,t) + Bu(\xi_1,\xi_2,t) - H_{\omega_G}(C\, z(\xi_1,\xi_2,t) - y(t)) & Q \\ z(\xi_1,\xi_2,0) = z_0(\xi_1,\xi_2) & \Omega \\ z(\eta_1,\eta_2,t) = 0 & \Sigma \end{cases} \quad (19)$$

together with the system $(17)-(18)$ are equivalent to the systems $(1)-(2)$ and (9) with the operator $A = \left(\frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2}\right)$.

Zone sensor cases:

Let the measurement support is rectangular with $]\xi_1 - l_1, \xi_1 + l_1[\times]\xi_2 - l_2, \xi_2 + l_2[\in \Omega$ (see Figure 3).

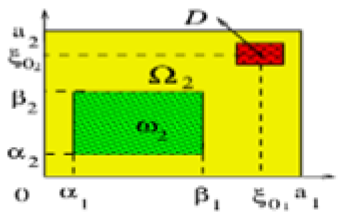

Fig. 3: Location of internal zone sensor D.

If f_1 is symmetric about $\xi_1 = \xi_{01}$ and f_2 is symmetric with respect to $\xi_2 = \xi_{02}$ then we have get the following result:

Corollary 5.1:

The dynamical system (19) is $AG\omega_G$-observer for the system (17)-(18) if $\frac{i\xi_{01}}{a_1}$ and $\frac{j\xi_{02}}{a_2} \notin N, \forall i,j = 1,\dots,J$.

In the case where $\Gamma \subset \partial\Omega$ and $f \in L^2(\Gamma)$, the sensor (D,f) may be located on the boundary in $\Gamma_0 =]\eta_{01} - l_1, \eta_{01} + l_1[\times \{a_2\}$, then we have:

Corollary 5.2:

One side case (see Figure 4):

suppose that the sensor (D,f) is located on $\Gamma_0 =]\eta_{01} - l_1, \eta_{01} + l_1[\times\{a_2\} \subset \partial\Omega$, and f is symmetric with respect to $\eta_1 = \eta_{01}$ then the dynamical systems (19) are $GA\omega_G$-observer for the system (17)-(18) if $\frac{i\xi_{01}}{a_1} \notin N, \forall i,j = 1,\dots,J$.

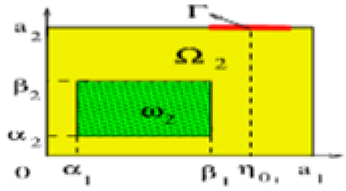

Fig. 4: Location of boundary zone sensor one side case

Two side case (see Figure 5):

suppose that the sensor (D, f) is located on $\Gamma_0 =]0, \eta_{01} + l_1[\times \{0\} \cup \{0\} \times]0, \eta_{02} + l_2[\subset \partial\Omega$, and $f|_{\Gamma_1}$ is symmetric with respect to $\eta_1 = \eta_{01}$ and the function $f|_{\Gamma_2}$ is symmetric with respect to $\eta_2 = \eta_{02}$, then the dynamical system (19) is $GA\omega_G$-observer for the system (17)-(18) if $\frac{i\xi_{01}}{a_1}$ *and* $\frac{j\xi_{02}}{a_2} \notin N, \forall i, j = 1, \dots, J$.

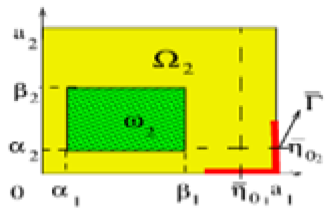

Fig. 5: Location of boundary zone sensor two side case

ii. Pointwise sensor cases:

In this cases if $b = (b_1, b_2) \in \partial\Omega$ then, we have:

Corollary 5.3:
Internal case (see Figure 6):

The dynamical system(19) is $GA\omega_G$ -observer for the systems(17)-(18) if $\frac{ib_1}{a_1}$ *and* $\frac{jb_2}{a_2} \notin N, \forall i, j = 1, \dots, J$.

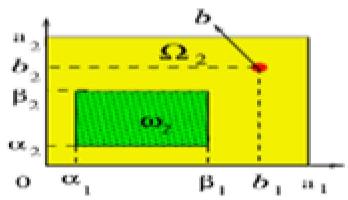

Fig. 6: Location of Internal pointwise sensors.

Filament case (see Figure 7):

Suppose that the observation is given by the filament sensor where $\sigma = Im(\gamma)$ is symmetric with respect to the line $b = (b_1, b_2)$, if $\frac{ib_1}{a_1}$ *and* $\frac{jb_2}{a_2} \notin N, \forall i, j = 1, \dots, J$. Then the dynamical system (19) is $GA\omega_G$ -observer for the systems(17)-(18).

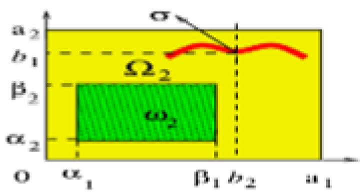

Fig. 7: Location of Internal pointwise sensors (filament case).

Boundary case (see Figure 8):

The dynamical system(19) is $GA\omega_G$-observerfor the systems(17)-(18) if $\frac{jb_2}{a_2} \notin N, \forall j = 1, \ldots, J$.

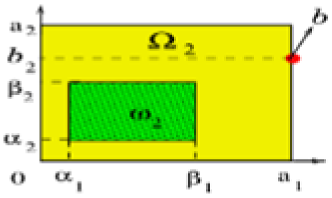

Fig. 8: Location of boundary pointwise sensor b.

Remark 5.4:

We can extend these results to the case where ω is sub-region of the boundary of the domain Ω as in[12].

Conclusion:

The concept have been studied in this paper is related to the existing and characterizing an $GA\omega_G$-observer only for a distributed parameter systems. More precisely, we have given an approach for building a $GA\omega_G$-estimator which reconstruct a gradient state in considered sub-region ω. Also, we show that there exists a dynamical system is not $GA\omega_G$-observer in usual sense but it is GAG-observer. For the future work, one can extension of these results to the problem in identity case and in reduced order case as in [8-9]

REFERENCES

[1] Al-Saphory R. and El Jai A., "Sensors Structures and Regional Detectability of Parabolic Distributed System". International Journal of Sensors and Actuators, Vol. 29, pp. 163-171, 2001.

[2] Al-Saphory R., "Asymptotic Regional Boundary Observer in Distributed Parameter Systems via Sensors Structures". Sensors, Vol. 2, pp. 137-152, 2002.

[3] Al-Saphory, R., El Jai, A., (2002),''Regional Asymptotic State Reconstruction, International Journal of System Science, 2002, Vol. 33, 1025-1037.

[4] Al-Saphory R. Al-Jawari N. and Al-Qaisi A., "Regional Gradient Detectability for Infinite Dimensional Systems". Tikrit Journal of Pure Science, Vol. 15, No. 2. pp. 1-6, 2010.

[5] Al-Saphory R., "Asymptotic Regional Analysis for a Class of Distributed Parameter Systems". Google Book, Amazon Publisher, USA, 2011.

[6] Al-Saphory R. Al-Joubory M. and Jasim M., "Regional Strategic Sensors Characterizations". Journal of Mathematical and Computational Science, Vol. 3, pp. 401-418, 2013.

[7] Al-Saphory R. Al-Jawari N. and Al-Janabi A., "Regional Gradient Strategic Sensors Characterizations". Journal of Advances in Mathematics, Vol. 10, No. 6, pp. 3605-3618, 2015.

[8] Al-Saphory, R., Al-Mullah Sh., (2015),''Regional Exponential Reduced Observability in Distributed Parameter Systems, Journal of Advance in Mathematics, 11, No.4.

[9] Al-Saphory, R., Jaafar, Z., (2015),''Regional Exponential General Observability in Distributed Parameter systems'', Journal of Advance in Mathematics, Vol. 11, No.4.

[10] Al-Saphory R. Al-Jawari N. and Al-Janabi A., "Regional Boundary Gradient Strategic Sensors Characterization". Mathematical Theory and Modeling, Vol. 6, No. 1, 2016.

[11] Al-Saphory R. Al-Jawari N. and Al-Janabi A., "General Asymptotic Regional Gradient Observer". Aust. J. Basic & Appl. Sci., Vol. 10, No. 9, pp. 8-18, 2016.

[12] Al-Saphory, R., Al-Jawari, N., Al-Janabi, A., (2016),''Regional Boundary Gradient Strategic Sensors Characterizations'', Mathematical Theory and Modeling, Vol. 5, No.14.

[13] Burns J., Jorggaard J., Cliff M. and Zietsman L., "A PDE Approach to Optimization and Control of High Performance Buildings". Workshop in Numerical Techniques for Optimization Problems with PDE Constraints, No. 04/2009, DOI: 10.4171/OWR/2009/04. 25-31 January. Houston, Denmark, 2009.

[14] Burns J., Jorggaard J., Cliff M. and Zietsman L., "Optimal Sensor Design for Estimation and Optimization of PDE Systems". American Control Conference, Marriott Waterfront, Baltimore, MD, USA. June 30-July 02, 2010.

[15] Curtain R. F. and Pritchard A. J., "Infinite Dimension Linear Theory Systems". Lecture Notes in Control and Information Sciences, Springer-Verlag, Vol. 8, 1978.

[16] Curtain R. F. and Zwart H., "An Introduction to Infinite Dimensional Linear System Theory". Springer-Verlag, New York, 1995.

[17] El Jai, A., M. Amouroux, "Sensors and Observers in Distributed Parameter Systems", International Journal of Control., 47(1): 333-347, 1988.

[18] El Jai A. and Pritchard A. J., "Sensors and Controls In the Analysis of Distributed Parameter Systems". Ellis Hardwood Series in Mathematics and Applications, Wiley, New York, 1988.

[19] El Jai A. and Hamzauoi, H., "Regional Observation and Sensors". International Journal of Applied Mathematics and Computer Sciences, Vol. 19, No. 1, pp. 5-14, 2009.

[20] Gressang, R.V., G.B. Lamont, "Observers For Characterized by semi-groups, IEEE Trans. Automat. Contr., pp: 523-528, 1975.

[21] Luenberger, D.G., ''Observers For Muli-Variables Systems", IEEE Trans. Automat. Contr., 11: 190-197, 1966.

[22] Zerrik E., Bouray H. and El Jai A., "Sensor and Regional Boundary State Reconstruction of Parabolic Systems". International Journal Sensors and Actuators Journal, Vol.75, pp. 102-117, 1999.