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Fibrewise Topological Group

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Abstract— *The purpose of this paper is to introduce the concepts of fibrewise topological group, fibrewise subgroup and fibrewise open and closed spaces. Also we give several results concerning it.*

I. Introduction

A fibrewise topological space over B is just a topological space X together with a continuous function $p: X \to B$ called projection. Most of the results obtained so far in this field can be found in James [2] (1984) and James [3] (1989). Our aim in this paper is to introduce the notion of fibrewise topological group. We study many properties and obtained some new results for this structure. Also we investigate important theorems and properties of fibrewise subgroup in fibrewise topological groups.

II. Preliminaries

Throughout this section we well give the basic concepts and notations which we well use in this paper:

2.1. Fibrewise topological space

Definition 2.1.1 [3]: Let B be any set. Then a fibrewise set over B consists of a set X together with a function $p: X \to B$, called the projection, where B is called a base set.

For each $b \in B$, the fibre over b is the subset $X_b = p^{-1}(b)$ of X. Also for each subset W of B, we regard $X_W = p^{-1}(W)$ is a fibrewise set over W with the projection determined by p.

Proposition 2.1.2 [3]: Let X be a fibrewise set over B, with projection p. Then Y is fibrewise set over B with projection $p\alpha$ for each set Y and function $\alpha : Y \to X$.

In particular X' is a fibrewise set over B with projection $p/_{X'}$ for each subset X' of X. Also X is fibrewise set over B' with projection βp for each superset B' of B and function $\beta : B \to B'$.

Definition 2.1.3 [3]: If X and Y are fibrewise sets over B, with projections p and q respectively, a function $\varphi : X \to Y$ is said to be fibrewise function if $q\varphi = p$ in other words $\varphi(X_b) \subseteq Y_b$ for each $b \in B$.

Definition 2.1.4 [3]: Let $\{X_r\}$ be index family of fibrewise sets over B. Then the fibrewise product $\prod_B X_r$ is defined, as a fibrewise set over B, and comes equipped with the family of fibrewise projections

$\pi_r: \prod_B X_r \to X_r$.

Specifically the fibrewise product is defined as the subset of the ordinary product $\prod X_r$, in which the fibres are the products of the corresponding fibres of the factors X_r.

Definition 2.1.5 [3]: Let B be a topological space. Then a fibrewise topology on a fibrewise set X over B is any topology on X for which the projection p is continuous.

A fibrewise topological space over the space B is defined to be a fibrewise set over B with fibrewise topology.

The coarsest fibrewise topology on a fibrewise set X over B is the topology induced by p, in which the open sets of X are precisely the inverse images of the open sets of the B, this is called the fibrewise indiscrete topology, and the discrete topology on a fibrewise set X over B is called fibrewise discrete.

Definition 2.1.6 [3]: The fibrewise topological space X over B is fibrewise closed (fibrewise open) if the projection p is closed (open).

Definition 2.1.7 [3]: Let $\{X_r\}$ be a family of fibrewise topological spaces over B, the fibrewise topological product $\prod_B X_r$ is defined to be the fibrewise product with the fibrewise topology induced by the family of projections.

2.2. Fibrewise Group

Definition 2.2.1[1]: Let B be a group. A fibrewise group over B is a fibrewise set G with any binary operation makes G a group such that the projection $p : G \to B$ is homomorphism.

Definition 2.2.2 [1]: Let G be a fibrewise group over B. Then any subgroup H of G is a fibrewise group over B with projection $p_{/H}: H \to B$, we call this group a fibrewise subgroup of G over B.

Theorem 2.2.3 [1]: Let G be a fibrewise group over B, then:

1. The fibre of the identity e_B of B, G_{e_B} is fibrewise subgroup of the fibrewise group G.
2. If B' is subgroup of a group B, then the set $G_{B'} = p^{-1}(B')$ is fibrewise subgroup of the fibrewise group G.
3. If $g \in G_b, b \in B$ then $g^{-1} \in G_{b^{-1}}$.

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Definition 2.2.4 [1]: Let G and K be two fibrewise groups over B. Then any homomorphism $\varphi : G \to K$ is called a fibrewise homomorphism if φ is a fibrewise map.

Definition 2.2.5 [1]: A bijective fibrewise homomorphism is called a fibrewise isomorphism.

Theorem 2.2.6 [1]: Let G be a fibrewise group over B with projection p and H be a fibrewise normal subgroup of G. Then G/H is fibrewise group over B, with projection $q : G/H \to B$ such that $q\pi = p$.

Theorem 2.2.7 [1]: let $\varphi : G \to K$ be a fibrewise function, where G and K are fibrewise groups over B, with p, q respectively. Then:

I. If q is injective then φ is a fibrewise homomorphism, and consequently:
   - i.) $\varphi(e_G) = e_K$, where e_G, e_K denotes the identities of G, K respective.
   - ii) $\varphi(\ker(P)) = e_K$.
   - iii) If H is fibrewise subgroup of G, then $\varphi(H)$ is fibrewise subgroup of K.
   - iv) If H' is fibrewise subgroup of K , then $\varphi^{-1}(H')$ is fibrewise subgroup of G.
   - v) If H is fibrewise normal subgroup of G, then $\varphi(H)$ is fibrewise normal subgroup of K.

1. If p is bijective and q is injective then if G is abelian then K is abelian.
2. If q is bijective and p is surjective then if G is cyclic then K is cyclic.
3. If p, q are bijective then φ is fibrewise isomorphism.

2.3. Topological Group [8]

Definition 2.3.1: A topological group G is a group which is also a topological space on G such that the maps $g \to g^{-1}$ and $(g, h) \to gh$ are continuous.

The continuity of the mappings $g \to g^{-1}$ and $(g, h) \to gh$ can be expressed as follows:

$g \to g^{-1}$ is continuous if and only if for each neighborhood W of g^{-1} there exists a neighborhood U of g such that $U^{-1} \subset W$. Similarly, $(g, h) \to gh$ is continuous if and only if for each neighborhood W of gh there exist a neighborhood U of g and a neighborhood V of h such that $UV \subset W$.

Theorem 2.3.2: A group G endowed with any topology, is a topological group if and only if, the mapping $(g, h) \to gh^{-1}$ is continuous.

Theorem 2.3.3: Let a be a fixed element of a topological group G, then $r_a : g \to ga$ and $l_a : g \to ag$ of G onto G homeomorphisms of G.

Corollary 2.3.4: Let F be a closed set, E be an open set, A be any subset of a topological group G and $a \in G$. Then aF, Fa, F^{-1} are closed sets, aE, Ea, E^{-1}, AE, EA are all open sets.

Theorem 2.3.5: Let A be an index set. For each $a \in A$, let G_a be a topological group. Then $G = \prod_{a \in A} G_a$ endowed with the product topology, is a topological group.

III. Fibrewise Topological Group

We now will introduce the concept of fibrewise topological group,

we will give and prove some new properties.

Definition 3.1: Let B be a topological group. A fibrewise topological group is a fibrewise group G over B with a fibrewise topology over B which makes the maps $g \to g^{-1}$ and $(g, h) \to gh$ continuous for any $g, h \in G$.

Examples 3.2: Let B be a topological group then:

I. Any fibrewise group over B with the fibrewise discrete topology is a fibrewise topological group over B.
II. Every fibrewise subgroup of fibrewise topological group with the subspace topology is a fibrewise topological group over B.
III. We can regard B as a fibrewise topological group over itself with the identity projection.
IV. Any topological group G, the product $B \times G$ endowed with the product topology, is fibrewise topological group over B with the first projection.
V. Let $(\mathbb{T}, .)$ be a multiplicative group of complex numbers of modulus 1 with the indiscrete topology, then the fibrewise addive group $(IR, +)$ of real numbers with the fibrewise usual topology over $\mathbb{T}$, is fibrewise topological group with projection $p: (IR, +) \to (\mathbb{T}, .)$ defined by $p(t) = e^{2\pi i t}, \forall\, t \in IR$.

Proposition 3.3: Let G be a fibrewise topological group over B. Then G_{B^*} is fibrewise topological group over B^* for each subgroup B^* of B.

Proof: Obvious.

Proposition 3.4: Let $\{H_r : r = 1,2, \dots, n\}$ be a finite family of fibrewise topological groups over B. Then the product $G = \prod_B H_r$, $r = 1,2, \dots, n$ is also fibrewise topological group.

Proof:

We have to show that the mapping $(g, h) \to gh^{-1}$ for any $g, h \in G$ is continuous. Let W be a neighborhood of gh^{-1} in G, by using Theorem 2.3.5 then $\prod H_r$, , $r = 1,2, \dots, n$ is topological group and there exist neighborhoods U of g and V of h such that $UV^{-1} \subseteq W$, where $U = \prod U_r, V = \prod V_r, \quad r = 1,2,\dots,n$ then $U' = \prod_B U_r, \ r = 1,2,\dots,n$ and $V' = \prod_B V_r, \ r = 1,2,\dots,n$ are neighborhoods of g, h respectively, also $U'V'^{-1} \subseteq UV^{-1} \subseteq W$ and since $\prod \mathrm{H}_r, \ , r = 1,2,\dots,n$ is a fibrewise topology over B, hence $\mathrm{G} = \prod_B \mathrm{H}_r \ , r = 1,2,\dots,n$ is a fibrewise topological group.

Proposition 3.5: Let G be a fibrewise topological group over B with projection p and H be a fibrewise normal subgroup of G. Then the quotient group $\mathrm{G/H}$ is a fibrewise topological group with projection $q: \mathrm{G/H} \to \mathrm{B}$ such that $q\pi = p$.

Proof:

Since the canonical mapping $\pi: \mathrm{G} \to \mathrm{G/H}$ is open and continuous and the projection $p: \mathrm{G} \to \mathrm{B}$ is continuous, then $p = q\pi \Rightarrow p^{-1} = (q\pi)^{-1} = \pi^{-1}q^{-1}$ thus $q^{-1} = \pi p^{-1}$ and let U be an open set in B then $q^{-1}(U) = \pi p^{-1}(U)$ is open set in $\mathrm{G/H}$ then the projection $q: \mathrm{G/H} \to \mathrm{B}$ is also continuous, hence $\mathrm{G/H}$ is a fibrewise topological group.

Proposition 3.6: Let $\varphi: \mathrm{G} \to \mathrm{K}$ be an open fibrewise homomorphism, where G and K are fibrewise topological groups over B, with projections p, q respectively. If G endowed with the fibrewise discrete topology then so is K.

Proof:

If G endowed with the fibrewise discrete topology then each singleton set is open, since φ is an open fibrewise homomorphism, then $\varphi(\{e_G\}) = \{e_K\}$ is open set in K and for each $k \in \mathrm{K}$, $\{k\} = k\{e_K\}$ is an open set in K. Hence each singleton set in K is open, therefor K endowed with the fibrewise discrete.

Proposition 3.7: Let G be a fibrewise topological group over B and H be a fibrewise normal subgroup of G. Then H is an open fibrewise subgroup of G if and only if $\mathrm{G/H}$ endowed with the fibrewise discrete.

Proof:

Let H be a fibrewise normal subgroup and open of G, then $g\mathrm{H}$ is an open set of G for each $g \in \mathrm{G}$. Since each singleton set $\{g\mathrm{H}\}$ is open in $\mathrm{G/H}$, this implies $\mathrm{G/H}$ endowed with the fibrewise discrete.

Conversely, if $\mathrm{G/H}$ endowed with the fibrewise discrete then each subset of $\mathrm{G/H}$ is open, thus $\mathrm{H} = g\mathrm{H}, g \in \mathrm{H}$ is open fibrewise in G.

Definition 3.8: The fibrewise topological group G is called fibrewise closed (open) if the fibrewise topology on G is fibrewise closed (fibrewise open).

Proposition 3.9: Let $\varphi: \mathrm{G} \to \mathrm{K}$ be a fibrewise function, where G and K are fibrewise topological groups over B, with projections p, q respectively. If K is fibrewise open and p is an injective function, K endowed with the fibrewise discrete then so is G.

Proof:

For any $g \in \mathrm{G}$, $\varphi(\{g\})$ is an open set in K, since the projection q is open, then $q \ \varphi(\{g\}) = p(\{g\})$ is open in B and $p^{-1}p(\{g\}) = \{g\}$ } is open in G, thus G endowed with the fibrewise discrete.

Proposition 3.10:

1- Let $\varphi: \mathrm{G} \to \mathrm{K}$ be a closed (resp. open) fibrewise function, where G and K are fibrewise topological groups over B. Then if K is fibrewise closed (resp. open) then so is G.

2- Let $\varphi: \mathrm{G} \to \mathrm{K}$ be a continuous fibrewise surjection, where G and K are fibrewise topological groups over B, if G is fibrewise closed (resp. open) then so is K.

Proof: Direct.

Proposition 3.11: Let G be a fibrewise topological group over B, if G is fibrewise closed (resp. open) over B then $\mathrm{G}_{\mathrm{B}'}$ is fibrewise closed (resp. open) over B' for each subgroup B' of B .

Proof:

Let G be a fibrewise closed over B (i.e. a continuous projection $p: \mathrm{G} \to \mathrm{B}$ is closed) and let B' be a subgroup of B, then $p^{-1}(\mathrm{B}') = \mathrm{G}_{\mathrm{B}'}$ is fibrewise subgroup of G and also $\mathrm{G}_{\mathrm{B}'}$ is fibrewise topological group over B' and the restriction $p/_{\mathrm{G}_{\mathrm{B}'}} = \mathrm{G}_{\mathrm{B}'} \to \mathrm{B}'$ is closed , since p is closed. Then $\mathrm{G}_{\mathrm{B}'}$ is fibrewise closed.

Proposition 3.12: Let B be a topological group and G be a fibrewise group over B with projection$\mathrm{p}$. If p is an isomorphism then any fibrewise open topology on G over B makes G a fibrewise topological group.

Proof:

We prove that G is a fibrewise topological group (i.e. for any $g, h, \in \mathrm{G}$; $g \to g^{-1}$ and $(g, h) \to gh$ are continuous

1- For any $g \in \mathrm{G}$ and let W be a neighborhood of $g^{-1} \Longrightarrow \mathrm{p}(W)$ is neighborhood of $p(g^{-1}) = p(g)^{-1}$, since B is a topological group then there exist neighborhood U of $p(g)$ such that $U^{-1} \subset p(W)$ and $p^{-1}(U^{-1}) \subset p^{-1}p(W) = W$, let $V = p^{-1}(U) \Rightarrow V$ is neighborhood of g and $V^{-1} = (p^{-1}(U))^{-1} = p^{-1}(U^{-1}) \subset W$. Therefore $g \to g^{-1}$ is continuous.

2- For any $(g, h) \in \mathrm{G} \times \mathrm{G}$, let W be a neighborhood of gh then $p(W)$ is neighborhood of $p(gh) = p(g)p(h)$, then there exist neighborhood U

of $p(g)$ and neighborhood V of $p(h)$ such that $UV \subset p(W)$ and $p^{-1}(UV) = p^{-1}(U)p^{-1}(V) \subset p^{-1}p(W) = W$, which $p^{-1}(U)$ is a neighborhood of g and $p^{-1}(V)$ is a neighborhood of h. This is implies $(g,h) \to gh$ is continuous.

Corollary 3.13: Any fibrewise group G with the fibrewise indiscrete topology, is fibrewise topological group with isomorphism projection.

Lemma 3.14: Let G be a fibrewise open topological group over B with projection p and H be a fibrewise subgroup of G over B. If $\mathrm{H}^{\circ} \neq \emptyset$ then H is fibrewise open topological group.

Proof:

Let $\mathrm{H}^{\circ} \neq \emptyset$ and $\mathrm{H}^{\circ}\mathrm{H}$ is open set in G and $\mathrm{H}^{\circ} \subset \mathrm{H} \Longrightarrow \mathrm{H} = \mathrm{H}^{\circ}\mathrm{H} = \bigcup_{h\in \mathrm{H}^{\circ}} \mathrm{H}$.

Then H is open in G, so $p/_{\mathrm{H}}: \mathrm{H} \to \mathrm{B}$ is open and hence H is fibrewise open topological group.

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