

# On a Subclass of Multivalent Functions Defined by Differential Operator



by

**Zain AL-abideen Abbas Nassir**

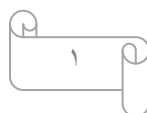
Department of Mathematics, College of Computer Science  
and Mathematics, University of AL-Qadisiya, Diwaniya, Iraq.

**E-mail:** [zain\\_alsafi67@yahoo.com](mailto:zain_alsafi67@yahoo.com)

**Abstract:** In this paper, we introduce a class of multivalent functions defined by differential operator. We obtain some geometric properties, like, coefficient inequalities, distortion and growth theorem, closure theorem, extreme points and Hadamard product (or convolution).

**Keyword:** Multivalent functions, Differential operator, Extreme points and Hadamard product.

**2019 Mathematics Subject classification:** 30C45, 30C50.



## 1. Introduction

Let  $\mathcal{A}(p)$  denote the class of functions of the form:

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n, \quad (a_n \geq 0, n \geq p+1, p \in \mathbb{N} = \{1, 2, \dots\}), \quad (1.1)$$

which are  $p$ -valent in the unit disk  $U = \{z \in \mathbb{C}: |z| < 1\}$ .

For a function  $f(z)$  in the class  $\mathcal{A}(p)$  we define

$$D_p^0 f(z) = f(z),$$

$$D_p^1 f(z) = D_p f(z) = \frac{z}{p} f'(z)$$

$$D_p^2 f(z) = D(D_p f(z)),$$

$$D_p^\mu f(z) = D(D_p^{\mu-1} f(z)) = z^p - \sum_{n=p+1}^{\infty} \binom{n}{p}^\mu a_n z^n, \quad \mu \in \mathbb{N}. \quad (1.2)$$

For  $p = 1$ , the differential operator  $D^\mu$  was introduced by Salagean [4].

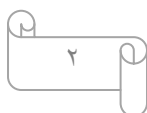
Let  $(f * g)(z)$  denote the Hadamard product of the functions  $f(z)$  and  $g(z)$ , that is, if  $f(z)$  is given by (1.1) and  $g(z)$  is given by

$$g(z) = z^p - \sum_{n=p+1}^{\infty} b_n z^n, \quad (b_n \geq 0), \quad (1.3)$$

then

$$(f * g)(z) = z^p - \sum_{n=p+1}^{\infty} a_n b_n z^n. \quad (1.4)$$

For a function  $f(z)$  of the form (1.1), we define the class  $W_p^\mu(\beta, \alpha, u, v)$  as follows:



**Definition (1.1):** Let  $f(z) \in \mathcal{A}(p)$  be given by (1.1). Then the class  $W_p^\mu(\beta, \alpha, u, v)$  is

$$W_p^\mu(\beta, \alpha, u, v) = \left\{ f \in \mathcal{A}(p) : \left| \frac{z \left( D_p^\mu f(z) \right)'' + (2-p) \left( D_p^\mu f(z) \right)'}{(u+v+2\beta) \left( D_p^\mu f(z) \right)' + \beta z \left( D_p^\mu f(z) \right)''} \right| < \alpha, 0 \leq \beta < 1, 0 < u \leq 1, 0 \leq v < 1, \mu, p \in \mathbb{N} \text{ and } 0 < \alpha < 1 \right\} \quad (1.5)$$

The same properties have been found for other classes in [2], [3] and [1].

## 2. Coefficient inequality

The following theorem has given a necessary and sufficient condition for a function  $f$  to be in the class  $W_p^\mu(\beta, \alpha, u, v)$ .

**Theorem (2.1):** Let a function  $f(z) \in \mathcal{A}(p)$ . Then  $f(z) \in W_p^\mu(\beta, \alpha, u, v)$  if and only if

$$\sum_{n=p+1}^{\infty} n[n+1+\alpha(\beta(n+1)+u+v)-p] \left( \frac{n}{p} \right)^\mu a_n \leq p[1+\alpha(\beta(p+1)+u+1)], \quad (2.1)$$

where  $0 \leq \beta < 1, 0 < u \leq 1, \mu, p \in \mathbb{N}$  and  $0 < \alpha < 1$ .

Proof: Suppose that the inequality (2.1) holds true and  $|z| = 1$ . Then, we have

$$\left| z \left( D_p^\mu f(z) \right)'' + (2-p) \left( D_p^\mu f(z) \right)' \right| - \alpha \left| (u+v+2\beta) \left( D_p^\mu f(z) \right)' + \beta z \left( D_p^\mu f(z) \right)'' \right|$$

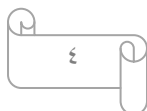
$$\begin{aligned}
&= \left| \sum_{n=p+1}^{\infty} n(n-p+1) \left(\frac{n}{p}\right)^{\mu} a_n z^{n-1} - p z^{p-1} \right| \\
&\quad - \left| \alpha p [\beta(p+1) + u + v] z^{p-1} \right. \\
&\quad \left. - \sum_{n=p+1}^{\infty} n \alpha (\beta(n+1) + u + v) \left(\frac{n}{p}\right)^{\mu} a_n z^{n-1} \right| \\
&\leq \sum_{n=p+1}^{\infty} n(n-p+1) \left(\frac{n}{p}\right)^{\mu} a_n |z|^{n-1} - p z^{p-1} \\
&\quad - \alpha p (\beta(p+1) + u + v) |z|^{p-1} \\
&\quad + \sum_{n=p+1}^{\infty} n \alpha (\beta(n+1) + u + v) \left(\frac{n}{p}\right)^{\mu} a_n |z|^{n-1} \\
&= \sum_{n=p+1}^{\infty} n [n+1 + \alpha (\beta(n+1) + u + v) - p] \left(\frac{n}{p}\right)^{\mu} a_n \\
&\quad - p [1 + \alpha (\beta(p+1) + u + v)] \leq 0,
\end{aligned}$$

by hypothesis.

Hence, by maximum modulus principle,  $f \in W_p^{\mu}(\beta, \alpha, u, v)$ . Then from (1.5), we have

$$\begin{aligned}
&\left| \frac{z \left(D_p^{\mu} f(z)\right)'' + (2-p) \left(D_p^{\mu} f(z)\right)'}{(u+v+2\beta) \left(D_p^{\mu} f(z)\right)' + \beta z \left(D_p^{\mu} f(z)\right)''} \right| \\
&= \left| \frac{\sum_{n=p+1}^{\infty} n(n-p+1) \left(\frac{n}{p}\right)^{\mu} a_n z^{n-1} - p z^{p-1}}{p [\beta(p+1) + u + v] z^{p-1} - \sum_{n=p+1}^{\infty} n (\beta(n+1) + u + v) \left(\frac{n}{p}\right)^{\mu} a_n z^{n-1}} \right|.
\end{aligned}$$

Since  $Re(z) \leq |z|$  for all  $z (z \in U)$ , we get



$$\operatorname{Re} \left\{ \frac{\sum_{n=p+1}^{\infty} n(n-p+1) \left(\frac{n}{p}\right)^{\mu} a_n z^{n-1} - p z^{p-1}}{p[\beta(p+1)+u+v] z^{p-1} - \sum_{n=p+1}^{\infty} n(\beta(n+1)+u+v) \left(\frac{n}{p}\right)^{\mu} a_n z^{n-1}} \right\} < \alpha.$$

Letting  $z \rightarrow 1^-$ , through real values, we obtain the inequality (2.1), so the proof is complete.

**Corollary (2.1):** Let  $f \in W_p^{\mu}(\beta, \alpha, u, v)$ . Then

$$a_n \leq \frac{p[1 + \alpha(\beta(p+1) + u + v)]}{\left(\frac{n}{p}\right)^{\mu} n[n + 1 + \alpha(\beta(n+1) + u + v) - p]}, n = p + 1, p + 2, \dots \quad (2.2)$$

### 3. Distortion and Growth Theorem

**Theorem (3.1):** Let the function  $f(z)$  defined by (1.2) be in the class  $W_p^{\mu}(\beta, \alpha, u, v)$ . Then for  $|z| = r < 1$  and  $n \geq p + 1$ , we have

$$|D_p^{\mu} f(z)| \geq \left[ 1 + \frac{p[1 + \alpha(\beta(p+1) + u + v)]}{(p+1)[\alpha(\beta(p+2) + u + v) + 2]} r \right] r^p, \quad (3.1)$$

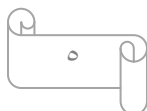
and

$$|D_p^{\mu} f(z)| \leq \left[ 1 - \frac{p[1 + \alpha(\beta(p+1) + u + v)]}{(p+1)[\alpha(\beta(p+2) + u + v) + 2]} r \right] r^p. \quad (3.2)$$

The result is sharp for the function  $f(z)$  is given by

$$f(z) = z^p - \frac{p[1 + \alpha(\beta(p+2) + u + v)]}{\left(\frac{p+1}{p}\right)^{\mu} (p+1)[\alpha(\beta(p+2) + u + v) + 2]} z^{p+1}, (z \in U). \quad (3.3)$$

**Proof:** Let  $f \in W_p^{\mu}(\beta, \alpha, u, v)$ . Then by Theorem (2.1), we get



$$\begin{aligned}
& \left(\frac{p+1}{p}\right)^\mu (p+1)[\alpha(\beta(p+2)+u+v)+2] \sum_{n=p+1}^{\infty} a_n \\
& \leq \sum_{n=p+1}^{\infty} n[n+1+\alpha(\beta(p+2)+u+v)-p] \left(\frac{n}{p}\right)^\mu \\
& \leq p[1+\alpha(\beta(p+2)+u+v)]
\end{aligned}$$

or

$$\sum_{n=p+1}^{\infty} a_n \leq \frac{p[1+\alpha(\beta(p+2)+u+v)]}{\left(\frac{p+1}{p}\right)^\mu (p+1)[\alpha(\beta(p+2)+u+v)+2]}. \quad (3.4)$$

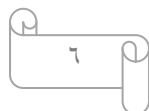
Hence,

$$\begin{aligned}
|D_p^\mu f(z)| & \leq |z|^p + \sum_{n=p+1}^{\infty} \left(\frac{n}{p}\right)^\mu a_n |z|^n \leq |z|^p + \left(\frac{p+1}{p}\right)^\mu |z|^{p+1} \sum_{n=p+1}^{\infty} a_n \\
& = r^p + \left(\frac{p+1}{p}\right)^\mu r^{p+1} \sum_{n=p+1}^{\infty} a_n \\
& \leq r^p + \frac{p[1+\alpha(\beta(p+1)+u+v)]}{(p+1)[\alpha(\beta(p+1)+u+v)+2]} r^{p+1} \\
& = \left[1 + \frac{p[1+\alpha(\beta(p+1)+u+v)]}{(p+1)[\alpha(\beta(p+1)+u+v)+2]} r\right] r^p. \quad (3.5)
\end{aligned}$$

Similarly,

$$\begin{aligned}
|D_p^\mu f(z)| & \geq |z|^p - \sum_{n=p+1}^{\infty} \left(\frac{n}{p}\right)^\mu a_n |z|^n \geq |z|^p - \left(\frac{p+1}{p}\right)^\mu |z|^{p+1} \sum_{n=p+1}^{\infty} a_n \\
& = r^p - \left(\frac{p+1}{p}\right)^\mu r^{p+1} \sum_{n=p+1}^{\infty} a_n \\
& \geq r^p - \frac{p[1+\alpha(\beta(p+1)+u+v)]}{(p+1)[\alpha(\beta(p+1)+u+v)+2]} r^{p+1} \\
& = \left[1 - \frac{p[1+\alpha(\beta(p+1)+u+v)]}{(p+1)[\alpha(\beta(p+1)+u+v)+2]} r\right] r^p. \quad (3.6)
\end{aligned}$$

From (3.5) and (3.6), we get (3.1) and (3.2) and the proof is complete.



**Theorem (3.2):** Let the function  $f(z)$  defined by (1.2) be in the class  $W_p^\mu(\beta, \alpha, u, v)$ . Then for  $|z| = r < 1$  and  $n \geq p + 1$ , we have

$$\left| \left( D_p^\mu f(z) \right)' \right| \geq \left[ \frac{p}{r} + \frac{p[1 + \alpha(\beta(p+1) + u + v)]}{\alpha(\beta(p+1) + u + v) + 2} \right] r^p, \quad (3.7)$$

and

$$\left| \left( D_p^\mu f(z) \right)' \right| \leq \left[ \frac{p}{r} - \frac{p[1 + \alpha(\beta(p+1) + u + v)]}{\alpha(\beta(p+1) + u + v) + 2} \right] r^p, \quad (3.8)$$

The result is sharp for the function  $f$  is given by (3.3).

Proof: The proof is similar to that of Theorem (3.1)

#### 4. Closure Theorem

Let the function  $f_i(z)$  ( $i = 1, 2, \dots, m$ ) be defined by

$$f_i(z) = z^p - \sum_{n=p+1}^{\infty} a_{n,i} z^n, \quad (a_{n,i} \geq 0). \quad (4.1)$$

We shall prove the following results for the closure functions in the class

$W_p^\mu(\beta, \alpha, u, v)$ .

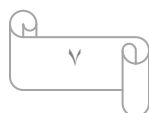
Theorem (4.1): Let the functions  $f_i(z)$  ( $i = 1, 2, \dots, m$ ) be defined by (4.1) be in the class  $W_p^\mu(\beta, \alpha, u, v)$ . Then the function  $h(z)$  defined by

$$h(z) = \sum_{i=1}^m c_i f_i(z), \quad (c_i \geq 0),$$

is also in the class  $W_p^\mu(\beta, \alpha, u, v)$ , where

$$\sum_{i=1}^m c_i = 1.$$

Proof: According to the definition of  $h(z)$ , it can be written as



$$\begin{aligned}
h(z) &= \sum_{i=1}^m c_i \left[ z^p - \sum_{n=p+1}^{\infty} a_{n,i} z^n \right] \\
&= \sum_{i=1}^m c_i z^p - \sum_{i=1}^m \sum_{n=p+1}^{\infty} c_i a_{n,i} z^n \\
&= z^p - \sum_{n=p+1}^{\infty} \sum_{i=1}^m c_i a_{n,i} z^n.
\end{aligned}$$

Furthermore, since the functions  $f_i(z)$  ( $i = 1, 2, \dots, m$ ) are in the class  $W_p^\mu(\beta, \alpha, u, v)$ , then

$$\begin{aligned}
\sum_{n=p+1}^{\infty} n[n+1+\alpha(\beta(n+1)+u+v)-p] \left(\frac{n}{p}\right)^\mu a_{n,i} \\
\leq p[1+\alpha(\beta(n+1)+u+v)].
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{n=p+1}^{\infty} n[n+1+\alpha(\beta(n+1)+u+v)-p] \left(\frac{n}{p}\right)^\mu \left(\sum_{i=1}^m c_i a_{n,i}\right) \\
&= \sum_{i=1}^m c_i \left\{ \sum_{n=p+1}^{\infty} n[n+1+\alpha(\beta(n+1)+u+v)-p] \left(\frac{n}{p}\right)^\mu a_{n,i} \right\} \\
&\leq p[1+\alpha(\beta(n+1)+u+v)],
\end{aligned}$$

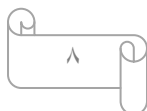
which implies that  $h(z)$  is in the class  $W_p^\mu(\beta, \alpha, u, v)$ .

**Corollary (4.1):** Let the function  $f_i(z)$  ( $i = 1, 2$ ) defined by (4.1) be in the class  $W_p^\mu(\beta, \alpha, u, v)$ .

Then the function  $h(z)$  defined by

$h(z) = (1 - \gamma)f_1(z) + \gamma f_2(z)$ , ( $0 \leq \gamma < 1$ ) , is also in the class  $W_p^\mu(\beta, \alpha, u, v)$ .

## 5. Extreme points:





**Theorem (5.1):** Let  $f_p = z^p$  and

$$f_p = z^p - \frac{p[1 + \alpha(\beta(p+1) + u + v)]}{\binom{n}{p}^\mu n[n+1 + \alpha(\beta(n+1) + u + v) - p]} z^n, (n \geq p+1; p \in \mathbb{N}). \quad (5.1)$$

Then the function  $f(z)$  is in the class  $W_p^\mu(\beta, \alpha, u, v)$  if and only if it can be expressed in the form

$$f(z) = \sum_{n=p+1}^{\infty} \sigma_n f_n(z),$$

where

$$\sigma_n \geq 0 \text{ and } \sum_{n=p}^{\infty} \sigma_n = 1. \quad (5.2)$$

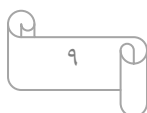
**Proof:** Suppose that  $f(z)$  is expressed in the form

$$\begin{aligned} f(z) &= \sum_{n=p}^{\infty} \sigma_n f_n(z) \\ &= \sigma_p z^p + \sum_{n=p+1}^{\infty} \sigma_n \left[ z^p - \frac{p[1 + \alpha(\beta(p+1) + u + v)]}{\binom{n}{p}^\mu n[n+1 + \alpha(\beta(n+1) + u + v) - p]} z^n \right] \\ &= z^p \left( \sigma_p + \sum_{n=p+1}^{\infty} \sigma_n \right) - \sum_{n=p+1}^{\infty} \frac{p[1 + \alpha(\beta(p+1) + u + v)]}{\binom{n}{p}^\mu n[n+1 + \alpha(\beta(n+1) + u + v) - p]} \sigma_n z^n \\ &= z^p - \sum_{n=p+1}^{\infty} q_n z^n, \end{aligned}$$

where

$$q_n = \frac{p[1 + \alpha(\beta(p+1) + u + v)]}{\binom{n}{p}^\mu n[n+1 + \alpha(\beta(n+1) + u + v) - p]} \sigma_n.$$

Therefore  $f \in W_p^\mu(\beta, \alpha, u, v)$ , since



$$\begin{aligned} & \sum_{n=p+1}^{\infty} \frac{\binom{n}{p}^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]}{p[1+\alpha(\beta(p+1)+u+v)]} q_n \\ &= \sum_{n=p+1}^{\infty} \sigma_n = 1 - \sigma_p \leq 1. \end{aligned}$$

Conversely, assume that  $f \in W_p^{\mu}(\beta, \alpha, u, v)$ , then by (2.1) we may set

$$\sigma_n = \frac{\binom{n}{p}^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]}{p[1+\alpha(\beta(p+1)+u+v)]} a_n, n \geq p+1$$

and

$$1 - \sum_{n=p+1}^{\infty} \sigma_n = \sigma_p.$$

Then

$$\begin{aligned} f(z) &= z^p - \sum_{n=p+1}^{\infty} a_n z^n \\ &= z^p - \sum_{n=p+1}^{\infty} \frac{p[1+\alpha(\beta(p+1)+u+v)]}{\binom{n}{p}^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]} z^n, \\ &= z^p - \sum_{n=p+1}^{\infty} \sigma_n (z^p - f_{\sigma}(z)) = z^p (1 - \sum_{n=p+1}^{\infty} \sigma_n) + \sum_{n=p+1}^{\infty} \sigma_n f_n(z) \\ &= z^p \sigma_p + \sum_{n=p+1}^{\infty} \sigma_n f_n(z) \\ &= \sum_{n=p}^{\infty} \sigma_n f_n(z). \end{aligned}$$

This completes the proof.



## 6. Hadamard product:

Let the function  $f_i(z)$  ( $i = 1, 2$ ) defined by (4.1). The Hadamard product of the functions  $f_1(z)$  and  $f_2(z)$  is defined by

$$(f_1 * f_2)(z) = z^p + \sum_{n=p+1}^{\infty} a_{n,1} a_{n,2} z^n. \quad (6.1)$$

**Theorem (6.1):** Let the functions  $f_i(z)$  ( $i = 1, 2$ ) defined by (4.1) be in the class  $W_p^\mu(\beta, \alpha, u, v)$  and ( $n \geq p + 1$ ). Then  $(f_1 * f_2)(z) \in W_p^\mu(\beta, \delta, u, v)$ , where

$$\delta \leq \frac{\binom{n}{p}^\mu n[n+1 + \alpha(\beta(n+1) + u + v) - p]^2 - (n+1-p)p[1 + \alpha(\beta(p+1) + u + v)]^2}{(\beta(p+1) + u + v) \left( p[1 + \alpha(\beta(p+1) + u + v)]^2 - \binom{n}{p}^\mu n[n+1 + \alpha(\beta(n+1) + u + v) - p]^2 \right)}.$$

The result is sharp for the functions  $f_i(z)$  given by

$$f_i(z) = \frac{p[1 + \alpha(\beta(p+1) + u + v)]}{\left(\frac{p+1}{p}\right)^\mu (p+1)[\alpha(\beta(p+2) + u + v) + 2]} z^{p+1}, \quad (i = 1, 2). \quad (6.2)$$

Proof: Since the functions  $f_i(z)$  ( $i = 1, 2$ ) belong to the class  $W_p^\mu(\beta, \alpha, u, v)$ , then from Theorem (2.1), we have

$$\sum_{n=p+1}^{\infty} \frac{\binom{n}{p}^\mu n[n+1 + \alpha(\beta(n+1) + u + v) - p]}{p[1 + \alpha(\beta(p+1) + u + v)]} a_{n,1} \leq 1. \quad (6.3)$$

Employing the technique used earlier by Schild and Silverman [5], we need to find the largest  $\delta$  such that

$$\sum_{n=p+1}^{\infty} \frac{\binom{n}{p}^\mu n[n+1 + \delta(\beta(n+1) + u + v) - p]}{p[1 + \delta(\beta(p+1) + u + v)]} a_{n,1} a_{n,2} \leq 1. \quad (6.4)$$

By Cauchy – Schwarz inequality, we get



$$\sum_{n=p+1}^{\infty} \frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]}{p[1+\alpha(\beta(p+1)+u+v)]} \sqrt{a_{n,1}a_{n,2}} \leq 1. \quad (6.5)$$

Thus, it is sufficient to show that

$$\frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\delta(\beta(n+1)+u+v)-p]}{p[1+\delta(\beta(p+1)+u+v)]} \sqrt{a_{n,1}a_{n,2}} \leq \frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]}{p[1+\alpha(\beta(p+1)+u+v)]} a_{n,1}a_{n,2}$$

that is, that

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{[1+\delta(\beta(p+1)+u+v)][n+1+\alpha(\beta(n+1)+u+v)-p]}{[1+\alpha(\beta(p+1)+u+v)][n+1+\delta(\beta(n+1)+u+v)-p]}. \quad (6.6)$$

But from (6.5), we have

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{p[1+\alpha(\beta(p+1)+u+v)]}{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]}. \quad (6.7)$$

Thus it is enough to show that

$$\begin{aligned} & \frac{p[1+\alpha(\beta(p+1)+u+v)]}{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]} \\ & \leq \frac{[1+\delta(\beta(p+1)+u+v)][n+1+\alpha(\beta(n+1)+u+v)-p]}{[1+\alpha(\beta(p+1)+u+v)][n+1+\delta(\beta(n+1)+u+v)-p]}, \end{aligned}$$

which implies

$$\delta \leq \frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]^2 - (n+1-p)p[1+\alpha(\beta(p+1)+u+v)]^2}{(\beta(p+1)+u+v) \left( p[1+\alpha(\beta(p+1)+u+v)]^2 - \left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]^2 \right)}.$$

This completes the proof.

**Theorem (6.2):** Let the functions  $f_i(z)$  ( $i = 1, 2$ ) defined by (4.1) be in the class  $W_p^{\mu}(\beta, \alpha, u, v)$ . Then the function

$$h(z) = z^p - \sum_{n=p+1}^{\infty} (a_{n,1}^2 + a_{n,2}^2) z^n, \quad (6.8)$$

is in the class  $W_p^{\mu}(\beta, \delta, u, v)$ , where

$$\delta \leq \frac{(n+1-p)[1+\alpha(\beta(p+1)+u+v)] - [n+1+\alpha(\beta(n+1)+u+v)-p]}{(\beta(n+1)+u+v)[1+\alpha(\beta(p+1)+u+v)] - (\beta(p+1)+u+v)[n+1+\alpha(\beta(n+1)+u+v)-p]}$$

The result is sharp for the functions  $f_i(z)$  ( $i = 1, 2$ ) given by (6.2).

Proof: From Theorem (2.1), we have

$$\begin{aligned} \sum_{n=p+1}^{\infty} \left( \frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]}{p[1+\alpha(\beta(p+1)+u+v)]} \right)^2 a_{n,i}^2 \\ \leq \left( \sum_{n=p+1}^{\infty} \frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]}{p[1+\alpha(\beta(p+1)+u+v)]} a_{n,i} \right)^2 \leq 1, \end{aligned}$$

it follows that

$$\sum_{n=p+1}^{\infty} \frac{1}{2} \left( \frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]}{p[1+\alpha(\beta(p+1)+u+v)]} \right)^2 (a_{n,1}^2 + a_{n,2}^2) \leq 1. \quad (6.9)$$

But  $h \in W_p^{\mu}(\beta, \delta, u, v)$  if and only if

$$\sum_{n=p+1}^{\infty} \frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\delta(\beta(n+1)+u+v)-p]}{p[1+\delta(\beta(p+1)+u+v)]} (a_{n,1}^2 + a_{n,2}^2) \leq 1, \quad (6.10)$$

the inequality (6.10) will satisfies if

$$\frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\delta(\beta(n+1)+u+v)-p]}{p[1+\delta(\beta(p+1)+u+v)]} \leq \frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]}{p[1+\alpha(\beta(p+1)+u+v)]}, n \geq p+1$$

so that

$$\delta \leq \frac{(n+1-p)[1+\alpha(\beta(p+1)+u+v)] - [n+1+\alpha(\beta(n+1)+u+v)-p]}{(\beta(n+1)+u+v)[1+\alpha(\beta(p+1)+u+v)] - (\beta(p+1)+u+v)[n+1+\alpha(\beta(n+1)+u+v)-p]}$$

This completes the proof.

## References:

- [1] M. K. Aouf, on certain subclass of multivalent functions with negative coefficients defined by using a differential operator, Bulletin of Institute of Mathematics, 5(2) (2010), 181-200.

- [2] M. K. Aout A. O. Msotafa, Certain class of p-valent functions defined by convolution, *General Math.*, 20(1) (2012), 85-98.
- [3] W. G. Atshan, S. R. Kulkarni, same application of generalized Ruscheweyh derivatives involving a general fractional derivative operator to a class of analytic functions with negative coefficient, *surveys in Mathematics and its Applications*, 5(2010), 35-47.
- [4] G. S .Sălăgean, Subclass of univalent functions, *Lecture Notes in Math.*, Springer-verlag, 1013, (1983), 362-372.
- [5] A.Schild and H.Silverman, Convolutions of univalent functions with negative coefficients, *Ann.Univ.Mariae curie-Sklodowska, sect.A*, 29(1975), 99-107.