On a Subclass of Multivalent Functions Defined by Differential Operator

by

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Abstract: In this paper, we introduce a class of multivalent functions defined by differential operator. We obtain some geometric properties, like, coefficient inequalities, distortion and growth theorem, closure theorem, extreme points and Hadamard product (or convolution).

Keyword: Multivalent functions, Differential operator, Extreme points and Hadamard product.

2019 Mathematics Subject classification: 30C45, 30C50.

1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions of the form:

$$f(z) = z^p - \sum_{p+1}^{\infty} a_n z^n \,, (a_n \geq 0, n \geq p+1, p \in \mathbb{N} = \{1,2,\dots\}, \qquad (1.1)$$

which are p- valent in the unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$.

For a function $f(z)$ in the class $\mathcal{A}(p)$ we define

$$D_p^0 f(z) = f(z),$$

$$D_p^1 f(z) = D_p f(z) = \frac{z}{p} f'(z)$$

$$D_p^2 f(z) = D\left(D_p f(z)\right),$$

$$D_p^{\mu} f(z) = D\left(D_p^{\mu-1} f(z)\right) = z^p - \sum_{n=p+1}^{\infty} \left(\frac{n}{p}\right)^{\mu} a_n z^n \,, \mu \in \mathbb{N}. \qquad (1.2)$$

For $p = 1$, the differential operator D^{μ} was introduced by salagean [4].

Let $(f * g)(z)$ denote the Hadamard product of the functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z^p - \sum_{n=p+1}^{\infty} b_n z^n \,, (b_n \geq 0), \qquad (1.3)$$

then

$$(f * g)(z) = z^p - \sum_{n=p+1}^{\infty} a_n b_n z^n . \qquad (1.4)$$

For a function $f(z)$ of the form (1.1), we define the class $W_p^{\mu}(\beta, \alpha, u, v)$ as follows:

Definition (1.1): Let $f(z) \in \mathcal{A}(p)$ be given by (1.1). Then the class $W_p^{\mu}(\beta,\alpha,u,v)$ is

$$W_p^{\mu}(\beta,\alpha,u,v) = \left\{ f \in \mathcal{A}(p): \left| \frac{z\left(D_p^{\mu} f(z)\right)'' + (2-p)\left(D_p^{\mu} f(z)\right)'}{(u+v+2\beta)\left(D_p^{\mu} f(z)\right)' + \beta z\left(D_p^{\mu} f(z)\right)''} \right| < \alpha, 0 \le \beta < 1, 0 < u \le 1, 0 \le v < 1, \mu, p \in \mathbb{N} \text{ and } 0 < \alpha < 1 \right\} \quad (1.5)$$

The same properties have been found for other classes in [2], [3] and [1].

2. Coefficient inequality

The following theorem has given a necessary and sufficient condition for a function f to be in the class $W_p^{\mu}(\beta,\alpha,u,v)$.

Theorem (2.1): Let a function $f(z) \in \mathcal{A}(p)$. Then $f(z) \in W_p^{\mu}(\beta,\alpha,u,v)$ if and only if

$$\sum_{n=p+1}^{\infty} n[n+1+\alpha(\beta(n+1)+u+v)-p]\left(\frac{n}{p}\right)^{\mu} a_n \le p[1+\alpha(\beta(p+1)+u+1)], \quad (2.1)$$

where $0 \le \beta < 1, 0 < u \le 1, \mu, p \in \mathbb{N}$ and $0 < \alpha < 1$.

Proof: Suppose that the inequality (2.1) holds true and $|z| = 1$. Then, we have

$$\left|z\left(D_p^{\mu} f(z)\right)'' + (2-p)\left(D_p^{\mu} f(z)\right)'\right| - \alpha\left|(u+v+2\beta)\left(D_p^{\mu} f(z)\right)' + \beta z\left(D_p^{\mu} f(z)\right)''\right|$$

$$= \left| \sum_{n=p+1}^{\infty} n(n-p+1)\left(\frac{n}{p}\right)^{\mu} a_n z^{n-1} - p z^{p-1} \right|$$

$$- \left| \alpha p[\beta(p+1)+u+v] z^{p-1} \right.$$

$$\left. - \sum_{n=p+1}^{\infty} n\alpha(\beta(n+1)+u+v)\left(\frac{n}{p}\right)^{\mu} a_n z^{n-1} \right|$$

$$\leq \sum_{n=p+1}^{\infty} n(n-p+1)\left(\frac{n}{p}\right)^{\mu} a_n |z|^{n-1} - p z^{p-1}$$

$$- \alpha p(\beta(p+1)+u+v)|z|^{p-1}$$

$$+ \sum_{n=p+1}^{\infty} n\alpha(\beta(n+1)+u+v)\left(\frac{n}{p}\right)^{\mu} a_n |z|^{n-1}$$

$$= \sum_{n=p+1}^{\infty} n[n+1+\alpha(\beta(n+1)+u+v)-p]\left(\frac{n}{p}\right)^{\mu} a_n$$

$$- p[1+\alpha(\beta(p+1)+u+v)] \leq 0,$$

by hypothesis.

Hence, by maximum modulus principle, $f \in W_p^{\mu}(\beta, \alpha, u, v)$. Then from (1.5), we have

$$\left| \frac{z\left(D_p^{\mu} f(z)\right)'' + (2-p)\left(D_p^{\mu} f(z)\right)'}{(u+v+2\beta)\left(D_p^{\mu} f(z)\right)' + \beta z\left(D_p^{\mu} f(z)\right)''} \right|$$

$$= \left| \frac{\sum_{n=p+1}^{\infty} n(n-p+1)\left(\frac{n}{p}\right)^{\mu} a_n z^{n-1} - p z^{p-1}}{p[\beta(p+1)+u+v] z^{p-1} - \sum_{n=p+1}^{\infty} n(\beta(n+1)+u+v)\left(\frac{n}{p}\right)^{\mu} a_n z^{n-1}} \right|.$$

Since $Re(z) \leq |z|$ for all $z (z \in U)$, we get

$$Re\left\{\frac{\sum_{n=p+1}^{\infty} n(n-p+1)\left(\frac{n}{p}\right)^{\mu} a_n z^{n-1} - p z^{p-1}}{p[\beta(p+1)+u+v]z^{p-1} - \sum_{n=p+1}^{\infty} n(\beta(n+1)+u+v)\left(\frac{n}{p}\right)^{\mu} a_n z^{n-1}}\right\} < \alpha.$$

Letting $z \to 1^{-}$, through real values, we obtain the inequality (2.1), so the proof is complete.

Corollary (2.1): Let $f \in W_p^{\mu}(\beta, \alpha, u, v)$. Then

$$a_n \leq \frac{p[1+\alpha(\beta(p+1)+u+v)]}{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]}, n = p+1, p+2, \ldots \quad (2.2)$$

3. Distortion and Growth Theorem

Theorem (3.1): Let the function $f(z)$ defined by (1.2) be in the class $W_p^{\mu}(\beta, \alpha, u, v)$. Then for $|z| = r < 1$ and $n \geq p+1$, we have

$$\left|D_p^{\mu} f(z)\right| \geq \left[1 + \frac{p[1+\alpha(\beta(p+1)+u+v)]}{(p+1)[\alpha(\beta(p+2)+u+v)+2]} r\right] r^p, \quad (3.1)$$

and

$$\left|D_p^{\mu} f(z)\right| \leq \left[1 - \frac{p[1+\alpha(\beta(p+1)+u+v)]}{(p+1)[\alpha(\beta(p+2)+u+v)+2]} r\right] r^p. \quad (3.2)$$

The result is sharp for the function $f(z)$ is given by

$$f(z) = z^p - \frac{p[1+\alpha(\beta(p+2)+u+v)]}{\left(\frac{p+1}{p}\right)^{\mu} (p+1)[\alpha(\beta(p+2)+u+v)+2]} z^{p+1}, (z \in U). \quad (3.3)$$

Proof: Let $f \in W_p^{\mu}(\beta, \alpha, u, v)$. Then by Theorem (2.1), we get

$$\left(\frac{p+1}{p}\right)^{\mu}(p+1)[\alpha(\beta(p+2)+u+v)+2]\sum_{n=p+1}^{\infty}a_n$$
$$\leq \sum_{n=p+1}^{\infty} n[n+1+\alpha(\beta(p+2)+u+v)-p]\left(\frac{n}{p}\right)^{\mu}$$
$$\leq p[1+\alpha(\beta(p+2)+u+v)]$$

or

$$\sum_{n=p+1}^{\infty}a_n \leq \frac{p[1+\alpha(\beta(p+2)+u+v)]}{\left(\frac{p+1}{p}\right)^{\mu}(p+1)[\alpha(\beta(p+2)+u+v)+2]}. \quad (3.4)$$

Hence,

$$\left|D_p^{\mu}f(z)\right| \leq |z|^p+\sum_{n=p+1}^{\infty}\left(\frac{n}{p}\right)^{\mu}a_n|z|^n \leq |z|^p+\left(\frac{p+1}{p}\right)^{\mu}|z|^{p+1}\sum_{n=p+1}^{\infty}a_n$$
$$= r^p+\left(\frac{p+1}{p}\right)^{\mu}r^{p+1}\sum_{n=p+1}^{\infty}a_n$$
$$\leq r^p+\frac{p[1+\alpha(\beta(p+1)+u+v)]}{(p+1)[\alpha(\beta(p+1)+u+v)+2]}r^{p+1}$$
$$= \left[1+\frac{p[1+\alpha(\beta(p+1)+u+v)]}{(p+1)[\alpha(\beta(p+1)+u+v)+2]}r\right]r^p. \quad (3.5)$$

Similarly,

$$\left|D_p^{\mu}f(z)\right| \geq |z|^p-\sum_{n=p+1}^{\infty}\left(\frac{n}{p}\right)^{\mu}a_n|z|^n \geq |z|^p-\left(\frac{p+1}{p}\right)^{\mu}|z|^{p+1}\sum_{n=p+1}^{\infty}a_n$$
$$= r^p-\left(\frac{p+1}{p}\right)^{\mu}r^{p+1}\sum_{n=p+1}^{\infty}a_n$$
$$\geq r^p-\frac{p[1+\alpha(\beta(p+1)+u+v)]}{(p+1)[\alpha(\beta(p+1)+u+v)+2]}r^{p+1}$$
$$= \left[1-\frac{p[1+\alpha(\beta(p+1)+u+v)]}{(p+1)[\alpha(\beta(p+1)+u+v)+2]}r\right]r^p. \quad (3.6)$$

From (3.5) and (3.6), we get (3.1) and (3.2) and the proof is complete.

Theorem (3.2): Let the function $f(z)$ defined by (1.2) be in the class $W_p^{\mu}(\beta,\alpha,u,v)$. Then for $|z|=r<1$ and $n\geq p+1$, we have

$$\left|\left(D_p^{\mu}f(z)\right)'\right|\geq\left[\frac{p}{r}+\frac{p[1+\alpha(\beta(p+1)+u+v)]}{\alpha(\beta(p+1)+u+v)+2}\right]r^p, \tag{3.7}$$

and

$$\left|\left(D_p^{\mu}f(z)\right)'\right|\leq\left[\frac{p}{r}-\frac{p[1+\alpha(\beta(p+1)+u+v)]}{\alpha(\beta(p+1)+u+v)+2}\right]r^p, \tag{3.8}$$

The result is sharp for the function f is given by (3.3).

Proof: The proof is similar to that of Theorem (3.1)

4. Closure Theorem

Let the function$f_i(z)(i=1,2,\dots,m)$ be defined by

$$f_i(z)=z^p-\sum_{n=p+1}^{\infty}a_{n,i}z^n,\left(a_{n,i}\geq 0\right). \tag{4.1}$$

We shall prove the following results for the closure functions in the class

$W_p^{\mu}(\beta,\alpha,u,v)$.

Theorem (4.1): Let the functions $f_i(z)(i=1,2,\dots,m)$ be defined by (4.1) be in the class $W_p^{\mu}(\beta,\alpha,u,v)$. Then the function $h(z)$ defined by

$$h(z)=\sum_{i=1}^{m}c_if_i(z),(c_i\geq 0),$$

is also in theclass $W_p^{\mu}(\beta,\alpha,u,v)$, where

$$\sum_{i=1}^{m}c_i=1.$$

Proof: According to the definition of $h(z)$, it can be written as

$$h(z)=\sum_{i=1}^{m} c_i\left[z^p-\sum_{n=p+1}^{\infty} a_{n,i}z^n\right]$$

$$=\sum_{i=1}^{m} c_i z^p-\sum_{i=1}^{m}\sum_{n=p+1}^{\infty} c_i a_{n,i}z^n$$

$$=z^p-\sum_{n=p+1}^{\infty}\sum_{i=1}^{m} c_i a_{n,i}z^n.$$

Furthermore, since the functions $f_i(z)$ $(i=1,2,\dots,m)$ are in the class $W_p^{\mu}(\beta,\alpha,u,v)$, then

$$\sum_{n=p+1}^{\infty} n[n+1+\alpha(\beta(n+1)+u+v)-p]\left(\frac{n}{p}\right)^{\mu} a_{n,i}$$
$$\leq p[1+\alpha(\beta(n+1)+u+v)].$$

Hence

$$\sum_{n=p+1}^{\infty} n[n+1+\alpha(\beta(n+1)+u+v)-p]\left(\frac{n}{p}\right)^{\mu}\left(\sum_{i=1}^{m} c_i a_{n,i}\right)$$

$$=\sum_{i=1}^{m} c_i\left\{\sum_{n=p+1}^{\infty} n[n+1+\alpha(\beta(n+1)+u+v)-p]\left(\frac{n}{p}\right)^{\mu} a_{n,i}\right\}$$

$$\leq p[1+\alpha(\beta(n+1)+u+v)],$$

which implies that $h(z)$ is in the class $W_p^{\mu}(\beta,\alpha,u,v)$.

Corollary (4.1): Let the function $f_i(z)$ $(i=1,2)$ defined by (4.1) be in the class$W_p^{\mu}(\beta,\alpha,u,v)$.

Then the function $h(z)$ defined by

$h(z)=(1-\gamma)f_1(z)+\gamma f_2(z),(0\leq\gamma<1)$, is also in the class $W_p^{\mu}(\beta,\alpha,u,v)$.

5. Extreme points:

Theorem (5.1): Let $f_p = z^p$ and

$$f_p = z^p - \frac{p[1+\alpha(\beta(p+1)+u+v)]}{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]} z^n, (n \geq p+1; p \in \mathbb{N}). \quad (5.1)$$

Then the function $f(z)$ is in the class $W_p^{\mu}(\beta,\alpha,u,v)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=p+1}^{\infty} \sigma_n f_n(z),$$

where

$$\sigma_n \geq 0 \text{ and} \sum_{n=p}^{\infty} \sigma_n = 1. \quad (5.2)$$

Proof: Suppose that $f(z)$ is expressed in the form

$$f(z) \sum_{n=p}^{\infty} \sigma_n f_n(z)$$

$$= \sigma_p z^p + \sum_{n=p+1}^{\infty} \sigma_n \left[z^p - \frac{p[1+\alpha(\beta(p+1)+u+v)]}{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]} z^n \right]$$

$$= z^p \left(\sigma_p + \sum_{n=p+1}^{\infty} \sigma_n \right) - \sum_{n=p+1}^{\infty} \frac{p[1+\alpha(\beta(p+1)+u+v)]}{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]} \sigma_n z^n$$

$$= z^p - \sum_{n=p+1}^{\infty} q_n z^n,$$

where

$$q_n = \frac{p[1+\alpha(\beta(p+1)+u+v)]}{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]} \sigma_n.$$

Therefore $f \in W_p^{\mu}(\beta,\alpha,u,v)$, since

$$\sum_{n=p+1}^{\infty} \frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]}{p[1+\alpha(\beta(p+1)+u+v)]} q_n$$

$$= \sum_{n=p+1}^{\infty} \sigma_n = 1 - \sigma_p \le 1.$$

Conversely, assume that $f \in W_p^{\mu}(\beta, \alpha, u, v)$, then by (2.1) we may set

$$\sigma_n = \frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]}{p[1+\alpha(\beta(p+1)+u+v)]} a_n \,, n \ge p+1$$

and

$$1 - \sum_{n=p+1}^{\infty} \sigma_n = \sigma_p.$$

Then

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n$$

$$= z^p - \sum_{n=p+1}^{\infty} \frac{p[1+\alpha(\beta(p+1)+u+v)]}{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]} z^n,$$

$$= z^p - \sum_{n=p+1}^{\infty} \sigma_n(z^p - f_{\sigma}(z)) = z^p(1 - \sum_{n=p+1}^{\infty} \sigma_n) + \sum_{n=p+1}^{\infty} \sigma_n f_n(z)$$

$$= z^p \sigma_p + \sum_{n=p+1}^{\infty} \sigma_n f_n(z)$$

$$= \sum_{n=p}^{\infty} \sigma_n f_n(z).$$

This completes the proof.

6. Hadamard product:

Let the function $f_i(z)(i = 1,2)$ defined by (4.1). The Hadamard product of the functions $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p + \sum_{n=p+1}^{\infty} a_{n,1} a_{n,2} z^n. \qquad (6.1)$$

Theorem (6.1): Let the functions $f_i(z)(i = 1,2)$ defined by (4.1) be in the class $W_p^{\mu}(\beta,\alpha,u,v)$ and $(n \geq p+1)$. Then $(f_1 * f_2)(z) \in W_p^{\mu}(\beta,\delta,u,v)$, where

$$\delta \leq \frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]^2 - (n+1-p)p[1+\alpha(\beta(p+1)+u+v)]^2}{(\beta(p+1)+u+v)\left(p[1+\alpha(\beta(p+1)+u+v)]^2 - \left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]^2\right)}.$$

The result is sharp for the functions $f_i(z)$ given by

$$f_i(z) = \frac{p[1+\alpha(\beta(p+1)+u+v)]}{\left(\frac{p+1}{p}\right)^{\mu} (p+1)[\alpha(\beta(p+2)+u+v)+2]} z^{p+1}, (i = 1,2). \quad (6.2)$$

Proof: Since the functions $f_i(z)(i = 1,2)$ belong to the class $W_p^{\mu}(\beta,\alpha,u,v)$, then from Theorem (2.1), we have

$$\sum_{n=p+1}^{\infty} \frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]}{p[1+\alpha(\beta(p+1)+u+v)]} a_{n,1} \leq 1. \qquad (6.3)$$

Employing the technique used earlier by Schild and Silverman [5], we need to find the largest δ such that

$$\sum_{n=p+1}^{\infty} \frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\delta(\beta(n+1)+u+v)-p]}{p[1+\delta(\beta(p+1)+u+v)]} a_{n,1} a_{n,2} \leq 1. \quad (6.4)$$

By Cauchy – Schwarz inequality, we get

$$\sum_{n=p+1}^{\infty} \frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]}{p[1+\alpha(\beta(p+1)+u+v)]} \sqrt{a_{n,1}a_{n,2}} \leq 1. \quad (6.5)$$

Thus, it is sufficient to show that

$$\frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\delta(\beta(n+1)+u+v)-p]}{p[1+\delta(\beta(p+1)+u+v)]} \sqrt{a_{n,1}a_{n,2}} \leq \frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]}{p[1+\alpha(\beta(p+1)+u+v)]} a_{n,1}a_{n,2}$$

that is, that

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{[1+\delta(\beta(p+1)+u+v)][n+1+\alpha(\beta(n+1)+u+v)-p]}{[1+\alpha(\beta(p+1)+u+v)][n+1+\delta(\beta(n+1)+u+v)-p]}. \quad (6.6)$$

But from (6.5), we have

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{p[1+\alpha(\beta(p+1)+u+v)]}{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]}. \quad (6.7)$$

Thus it is enough to show that

$$\frac{p[1+\alpha(\beta(p+1)+u+v)]}{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]} \leq \frac{[1+\delta(\beta(p+1)+u+v)][n+1+\alpha(\beta(n+1)+u+v)-p]}{[1+\alpha(\beta(p+1)+u+v)][n+1+\delta(\beta(n+1)+u+v)-p]},$$

which implies

$$\delta \leq \frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]^2 - (n+1-p)p[1+\alpha(\beta(p+1)+u+v)]^2}{(\beta(p+1)+u+v)\left(p[1+\alpha(\beta(p+1)+u+v)]^2 - \left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]^2\right)}.$$

This completes the proof.

Theorem (6.2): Let the functions $f_i(z)(i=1,2)$ defined by (4.1) be in the class$W_p^{\mu}(\beta,\alpha,u,v)$. Then the function

$$h(z) = z^p - \sum_{n=p+1}^{\infty} \left(a_{n,1}^2 + a_{n,2}^2\right) z^n, \quad (6.8)$$

is in the class $W_p^{\mu}(\beta,\delta,u,v)$, where

$$\delta \le \frac{(n+1-p)[1+\alpha(\beta(p+1)+u+v)]-[n+1+\alpha(\beta(n+1)+u+v)-p]}{(\beta(n+1)+u+v)[1+\alpha(\beta(p+1)+u+v)]-(\beta(p+1)+u+v)[n+1+\alpha(\beta(n+1)+u+v)-p]}.$$

The result is sharp for the functions $f_i(z)(i=1,2)$ given by (6.2).

Proof: From Theorem (2.1), we have

$$\sum_{n=p+1}^{\infty}\left(\frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]}{p[1+\alpha(\beta(p+1)+u+v)]}\right)^{2} a_{n,i}^{2}$$
$$\le \left(\sum_{n=p+1}^{\infty}\frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]}{p[1+\alpha(\beta(p+1)+u+v)]} a_{n,i}\right)^{2} \le 1,$$

it follows that

$$\sum_{n=p+1}^{\infty}\frac{1}{2}\left(\frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]}{p[1+\alpha(\beta(p+1)+u+v)]}\right)^{2}\left(a_{n,1}^{2}+a_{n,2}^{2}\right) \le 1. \quad (6.9)$$

But $h \in W_p^{\mu}(\beta,\delta,u,v)$ if and only if

$$\sum_{n=p+1}^{\infty}\frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\delta(\beta(n+1)+u+v)-p]}{p[1+\delta(\beta(p+1)+u+v)]}\left(a_{n,1}^{2}+a_{n,2}^{2}\right) \le 1, \qquad (6.10)$$

the inequality (6.10) will satisfies if

$$\frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\delta(\beta(n+1)+u+v)-p]}{p[1+\delta(\beta(p+1)+u+v)]} \le \frac{\left(\frac{n}{p}\right)^{\mu} n[n+1+\alpha(\beta(n+1)+u+v)-p]}{p[1+\alpha(\beta(p+1)+u+v)]}, n \ge p+1$$

so that

$$\delta \le \frac{(n+1-p)[1+\alpha(\beta(p+1)+u+v)]-[n+1+\alpha(\beta(n+1)+u+v)-p]}{(\beta(n+1)+u+v)[1+\alpha(\beta(p+1)+u+v)]-(\beta(p+1)+u+v)[n+1+\alpha(\beta(n+1)+u+v)-p]}.$$

This completes the proof.

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