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An Integer Linear Programming Model for Solving Radio Mean Labeling Problem

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ABSTRACT A Radio mean labeling of a connected graph G is an injective function h from the vertex set, $V(G)$, to the set of natural numbers N such that for any two distinct vertices x and y of G, $\left\lceil \frac{h(x)+h(y)}{2} \right\rceil \geq diam+1-d(x,y)$. The radio mean number of h, $rmn(h)$, is the maximum number assigned to any vertex of G. The radio mean number of G, $rmn(G)$, is the minimum value of $rmn(h)$, taken over all radio mean labeling h of G. This work has three contributions. The first one is proving two theorems which find the radio mean number for cycles and paths. The second contribution is proposing an approximate algorithm which finds an upper bound for radio mean number of a given graph. The third contribution is that we introduce a novel integer linear programing formulation for the radio mean problem. Finally, the experimental results analysis and statistical test proved that the Integer Linear Programming Model overcame the proposed approximate algorithm according to CPU time only. On the other hand, both the Integer Linear Programming Model and the proposed approximate algorithm had the same upper bound of the radio mean number of G.

INDEX TERMS Channel assignment problem, radio mean number, upper bound, path and cycle.

I. INTRODUCTION

Let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges for the graph G respectively. Hale [1] proposed the channel assignment problem. The radio labeling of graphs (multilevel distance labeling) is proposed by Chartrand *et al.* [2] in 2001 due to the regulations for channel assignments of FM radio stations. Zhang [3] determined the upper bounds of the radio numbers of cycles. Liu and Zhu [4] introduced the exact formula for the radio numbers for paths and cycles. Badr and Moussa [5] introduced the algorithm that determines the upper bound of radio k-chromatic number for a graph. This algorithm overcame the algorithm that was due to Saha and Panigrahi [6]. Saha and Panigrahi [7] proposed two radio k-coloring methods for a given graph which will find radio k-colorings.

Ponraja *et al.* [8] and Ponraja and Narayanan [9] proposed the radio mean labeling of graphs as follows: let h be an injective function from the vertex set, $V(G)$, to the set of natural numbers N where where for any two distinct vertices x and y of the graph G, $\left\lceil \frac{h(x)+h(y)}{2} \right\rceil \geq diam + 1 - d(x, y)$, where diam is the diameter of G and $d(x, y)$ denotes the distance between the two vertices x and y. The number of radio mean of $h, rmn(h)$, is the maximum number assigned to any vertex of G. The number of radio mean of G, $rmn(G)$, is the minimum value of $rmn(h)$, taken over all radio mean labeling h of G. Ponraja *et al.* [8] found the number of radio mean for networks with diameter 3, lotus with a circle, Sunflower networks and Helms. Ponraja and Narayanan [9] determined the number of radio mean for some networks that are related to cycles and complete graph. In [10] they found the number of radio mean for triangular ladder network, $P_n \odot \bar{K}_2$ (It consists of a path P_n in which every vertex x_i joined to two vertices y_i and z_i of $\bar{K}_2$), $K_n \odot \bar{K}_2$ (It consists of a complete graph K_n in which every vertex x_i joined to two vertices y_i and z_i of $\bar{K}_2$) and $W_n \odot \bar{K}_2$ (It consists of a wheel W_n in which every vertex x_i joined to two vertices y_i and z_i of $\bar{K}_2$). Since the radio mean labeling problem is derived from the radio

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k-coloring application, so, the radio mean labeling application is NP-hard problem for a graph. In [11] the authors introduced an application for radio frequency identification (RFID). On the other hand, the metaheuristic approaches for the linear wireless sensor networks were proposed in [12]

This work has three contributions. The first contribution is proving two theorems which find the radio mean number for cycles and paths. The second contribution is proposing an approximate algorithm which finds an upper bound for a radio mean number of a given graph. The third contribution is that we introduce a novel integer linear programing formulation for the radio mean problem. Finally, the experimental results analysis and statistical test proved that the Integer Linear Programming Model overcame the proposed approximate algorithm according to CPU time only. On the other hand, both the Integer Linear Programming Model and the proposed approximate algorithm had the same upper bound of the radio mean number of G.

The rest of this work is organized as the following: the radio mean number of cycle and path are introduced in Section 2. An approximate algorithm which finds the upper bound of radio mean number of a graph is proposed with an example in Section 3. A novel integer linear programing formulation for finding a radio mean number of a graph is introduced with an example in Section 4. In Section 5 the numerical results analysis and statistical test between the Integer Linear Programming Model and the proposed approximate algorithm are provided. Finally, conclusions are drawn in Section 6.

II. MAIN RESULTS

In this section, we introduce some basic definitions before proving the theorems that determine the radio mean number of cycle and path.

Definition 1: The distance from a vertex u to a vertex v is the number of edges in a shortest $u - v$ path in G and it is denoted by $d(u, v)$.

Definition 2: Let G be a connected graph, the eccentricity $e(v)$ of a vertex v is the distance between v and a vertex farthest from v in G.

Definition 3: The diameter $diam(G)$ of G is the greatest eccentricity among the vertices of G.

Theorem 1: The radio mean number of the cycles C_n is given by:

$$rmn(C_n) = \begin{cases} n & if\ 3 \le n \le 7 \\ \left\lfloor \frac{3n}{2} - 4 \right\rfloor & if\ n \ge 8 \end{cases}$$

Proof: Let $x_1, x_2, \ldots, x_n$ be cycle of length n so $diam(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$. Define a function $\rightarrow h{:}V(C_n)N$ by the following cases:

Case a: For $3 \le n \le 7$, we have three subcases as the following:

Case a.1: At $3 \le n \le 5$, in this subcase the vertices are labeled by the following function:

$$h(x_i) = i;\ 1 \le i \le n.$$

Case a.2: At $n = 6$, in this subcase the vertices are labeled by the following function:

$$\begin{cases} h(x_{i+1}) = n-1-2i;\ 0 \le i < \frac{n}{2} - 1 \\ h(x_{n-j}) = 2j+2;\ 0 \le j < \frac{n}{2} - 1 \\ h\left(x_{\frac{n}{2}}\right) = n,\ h\left(x_{\frac{n}{2}+1}\right) = 1 \end{cases}$$

Case a.3: At $n = 7$, we may label the vertices of C_7 as the following

$$\begin{cases} h(x_{i+1}) = n-1-2i;\ 0 \le\ i < \frac{n+1}{2} \\ h(x_{n-j}) = 2j+2;\ 0 \le\ j < \frac{n-1}{2} \end{cases}$$

Therefor for any pair (x_i, x_j), $i \ne j$, $1 \le i \le n$, $1 \le j \le n$ we have:

$$d(x_i, x_j) + \left\lceil \frac{h(x_i) + h(x_j)}{2} \right\rceil \ge 1 + \left\lfloor \frac{n}{2} \right\rfloor = 1 + dim(C_n)$$

Case b: For $n \equiv 0(mod4)$ i.e $n = 4k + 8$, $k \ge 0$, we may label the vertices of C_n as follows:

$$h\left(x_{\frac{n}{2}}\right) = 1,\ h(x_1) = n + 2k - 1$$
$$h(x_{i+2}) = \frac{n}{2} - 1 + 2i,\ 0 \le i < \frac{n}{2} - 2$$
$$h(x_{n-j}) = \frac{n}{2} - 2 + 2j,\ 0 \le j < \frac{n}{2}$$

Therefore for any pair (x_i, x_j), $i \ne j$, $1 \le i \le n$, $1 \le j \le n$, we have:

$$d(x_i, x_j) + \left\lceil \frac{h(x_i) + h(x_j)}{2} \right\rceil \ge 1 + \left\lfloor \frac{n}{2} \right\rfloor = 1 + dim(C_n)$$

Case c: $n \equiv 1(mod4)$ i.e $n = 4k + 9$, $k \ge 0$, in this case the vertices are labeled by the following function:

$$h\left(x_{\frac{n+1}{2}}\right) = 1,\ h(x_1) = n + 2k,\ h(x_n) = \left\lfloor \frac{n}{2} \right\rfloor - 2$$
$$h(x_{n-1}) = \left\lfloor \frac{n}{2} \right\rfloor + 2,\ h(x_{n-2}) = \left\lfloor \frac{n}{2} \right\rfloor$$
$$h(x_{i+2}) = \left\lfloor \frac{n}{2} \right\rfloor - 1 + 2i,\ 0 \le i < \left\lfloor \frac{n}{2} \right\rfloor - 1$$
$$h(x_{n-3-j}) = \left\lfloor \frac{n}{2} \right\rfloor + 4 + 2j,\ 0 \le j < \left\lfloor \frac{n}{2} \right\rfloor - 3$$

Therefore for any pair (x_i, x_j), $i \ne j$, $1 \le i \le n$, $1 \le j \le n$, we have:

$$d(x_i, x_j) + \left\lceil \frac{h(x_i) + h(x_j)}{2} \right\rceil \ge 1 + \left\lfloor \frac{n}{2} \right\rfloor = 1 + dim(C_n)$$

Case d: $n \equiv 2(mod4)$ i.e $n = 4k + 10$, $k \ge 0$,, in this case the vertices are labeled by the following function:

$$h\left(x_{\frac{n}{2}+1}\right) = 1,\ h(x_1) = n + 2k + 1$$
$$h(x_{i+2}) = \frac{n}{2} - 1 + 2i,\ 0 \le i \le \frac{n}{2} - 2$$
$$h(x_{n-j}) = \frac{n}{2} - 2 + 2j,\ 0 \le j < \frac{n}{2} - 1$$

Therefore for any pair (x_i, x_j), $i \neq j, 1 \leq i \leq n, 1 \leq j \leq n$, we have:

$$d(x_i, x_j) + \left\lceil \frac{h(x_i) + h(x_j)}{2} \right\rceil \geq 1 + \left\lfloor \frac{n}{2} \right\rfloor = 1+dim(C_n)$$

Case e: $n \equiv 3(mod4)$ i.e $n= 4k+7, k \geq 1$, in this case the vertices are labeled by the following function:

$$h\left(x_{\frac{n+1}{2}}\right) = 1, h(x_1) = n + 2k-1, \quad h(x_n) = \left\lfloor \frac{n}{2} \right\rfloor - 2$$
$$h(x_{n-1}) = \left\lfloor \frac{n}{2} \right\rfloor + 2, \quad h(x_{n-2}) = \left\lfloor \frac{n}{2} \right\rfloor$$
$$h(x_{i+2}) = \left\lfloor \frac{n}{2} \right\rfloor - 1 + 2i, \quad 0 \leq i < \left\lfloor \frac{n}{2} \right\rfloor - 1$$
$$h(x_{n-3-j}) = \left\lfloor \frac{n}{2} \right\rfloor + 4 + 2j, \quad 0 \leq j < \left\lfloor \frac{n}{2} \right\rfloor - 3$$

Therefore for any pair (x_i, x_j), $i \neq j, \leq 1i \leq n, 1 \leq j \leq n$, we have:

$$d(x_i, x_j) + \left\lceil \frac{h(x_i) + h(x_j)}{2} \right\rceil \geq 1 + \left\lfloor \frac{n}{2} \right\rfloor = 1+dim(C_n)$$

Thus, the radio mean condition is satisfied for all pairs of vertices. Now, we have the upper bound of the radio mean labeling of C_n as the following inequality:

$$rmn(C_n) \leq rmn(h) = \begin{cases} n & if\ 3 \leq n \leq 7 \\ \left\lfloor \frac{3n}{2} - 4 \right\rfloor & if\ n \geq 8 \end{cases} \quad (1)$$

Since h is an injective mapping (i.e. we can't label two or more vertices in $V(C_n)$ with the same natural number in N) then the lower bound of the radio mean labeling of C_n is determined by the following inequality:

$$rmn(C_n) \geq \begin{cases} n & if\ 3 \leq n \leq 7 \\ \left\lfloor \frac{3n}{2} - 4 \right\rfloor & if\ n \geq 8 \end{cases} \quad (2)$$

From Inequalities 1 and 2, we have:

$$rmn(C_n) = \begin{cases} n & if\ 3 \leq n \leq 7 \\ \left\lfloor \frac{3n}{2} - 4 \right\rfloor & if\ n \geq 8 \end{cases}$$

Therefore, the labeling $h:V(C_n) \rightarrow N$ defined by the above cases satisfies the radio mean condition. ■

Theorem 2: The number of radio mean for the path graph P_n is given by:

$$rmn(P_n) = \begin{cases} 1 & if\ n = 1 \\ 2 & if\ n = 2 \\ 2n-3 & if\ n > 2 \end{cases}$$

Proof: Let $x_1, x_2, \ldots, x_n$ be path of length $n - 1$, i.e. $diam(P_n) = n - 1$. Define a function $h:V(P_n) \rightarrow N$, as follows:

$$h(x_1) = 1,$$
$$h(x_{n-i}) = n - 1 + i, 0 \leq i < n - 1$$

Therefore for any pair (x_i, x_j), $i \neq j, 0 \leq i, j \leq n$, we have

$$d(x_i, x_j) + \left\lceil \frac{h(x_i) + h(x_j)}{2} \right\rceil \geq 1+n-1 = 1 + dim(P_n)$$

for any pair (x_i, x_j), $i \neq j, 0 \leq i \leq n, 0 \leq j \leq n$. Thus, the radio mean condition is satisfied with all pairs of vertices. Now, we have the upper bound of the radio mean labeling of P_n as the following inequality:

$$rmn(P_n) \leq rmn(h) = 2n - 3 \quad (3)$$

Since h is an injective mapping (i.e. we can't label two or more vertices in $V(C_n)$ with the same natural number in N) then the lower bound of the radio mean labeling of P_n is determined by the following inequality:

$$rmn(P_n) \geq 2n - 3 \quad (4)$$

for all radio mean labeling h. From Inequalities 3 and 4, we have:

$$rmn(P_n) = 2n-3, n \geq 3.$$

Therefore, the labeling $h:V(P_n) \rightarrow N$ defined by the above satisfies the radio mean condition. ■

Algorithm 1 The Upper Bound of Bound of the Radio Mean Number of a Graph G

Input: The adjacency matrix of the graph G and the diameter of G (dim).
Output: The upper bound of radio mean number o Gf.
Begin
1: Choose a vertex u and $lab(u) = dim$.
2: $S= \{u$.
3: For all $v \in V(G) - S$, compute,

$$temp(v) = \max_{t \in s} \left\{ lab(t) + \text{ceil}\left(\frac{\max\{(dim+1-d(u,v), 1\}}{dim} \right) \right\}$$

4: Let min $= \min_{v \in V(G)-S} \{temp(v)\}$
5: Choose a vertex $v \in V(G) - S$, where

$$temp(v) = min.$$

6: Assign, $lab(v) = min$.
7: $S = S \cup \{v$
8: Repeat Step 3 to Step 6 until all vertices are labeled.
9: Repeat Step 1 to Step 7 for every verte $x \in V(G)$x.
End

III. A NEW GRAPH RADIO MEAN ALGORITHM

In this section, we propose an algorithm that determines an upper bound of the radio mean for an arbitrary graph G. The main idea of the proposed algorithm is that the algorithm changes the initial vertex to improve the upper bound. The diameter of G is assigned to some vertex. The next vertex is labeled by the minimum possible integer. After all vertices are labeled, the algorithm changes the initial vertex over all vertices.

Complexity of Algorithms 1: It is clear that step 1 and step 2 both have one operation. On the other hand, step 3 has a nested loop which has $O(n^2)$ time complexity. Three steps (step 4, step 5 and step 6) have $O(n)$ time complexity. Step 7 has one operation but the last two steps (step 8, step9) have $O(n^3)$ and $O(n^4)$ respectively. The proposed algorithm has the following time complexity:

$$2O(1) + 3O(n) + O(1) + O(n^3) + O(n^4) = O(n^4)$$

Example 1: This example shows how to compute the upper bound of the number of radio mean for the path P_5. We suppose that x_i are the labels of the vertices v_i such that $1 \le i \le 5$. So, Algorithm 1determines the upper bound of the number of radio mean as the following:

It is known hat $diam(P_5) = 4$. We choose a vertex x_1 and $\text{lab}(x_1) = 4$. Let $S= \{x_1$ and for all $v \in V(G) - S$, compute,

$$temp\,(x_2) = \max_{x_1}\left\{4 + \text{ceil}\left(\frac{\text{MAX}\,\{(4+1-1, 1\}}{10}\right)\right\} = 5$$
$$temp\,(x_3) = \max_{x_1}\left\{4 + \text{ceil}\left(\frac{\text{MAX}\,\{(4+1-2, 1\}}{10}\right)\right\} = 5$$
$$temp\,(x_4) = \max_{x_1}\left\{4 + \text{ceil}\left(\frac{\text{MAX}\,\{(4+1-3, 1\}}{10}\right)\right\} = 5$$
$$temp\,(x_5) = \max_{x_1}\left\{4 + \text{ceil}\left(\frac{\text{MAX}\,\{(4+1-4, 1\}}{10}\right)\right\} = 5.$$

Let $\min = \min_{v\in V(G)-S}\{temp(v)\} = 5$ we choose a vertex $x_2 \in V(G) - S$. Such that $temp\,(x_2) = 5$. Give $lab\,(x_2) = 5$ and $S = \{x_1, x_1$

$$temp\,(x_3) = \max_{x_1,x_2}\left\{\begin{matrix} 4 + \text{ceil}\left(\frac{\max\{(4+1-2,1\}}{10}\right) \\ 5 + \text{ceil}\left(\frac{\max\{(4+1-1,1\}}{10}\right)\end{matrix}\right\} = 6$$
$$temp\,(x_4) = \max_{x_1,x_2}\left\{\begin{matrix} 4 + \text{ceil}\left(\frac{\max\{(4+1-3,1\}}{10}\right) \\ 5 + \text{ceil}\left(\frac{\max\{(4+1-2,1\}}{10}\right)\end{matrix}\right\} = 6$$
$$temp\,(x_5) = \max_{x_1,x_2}\left\{\begin{matrix} 4 + \text{ceil}\left(\frac{\max\{(4+1-4,1\}}{10}\right) \\ 5 + \text{ceil}\left(\frac{\max\{(4+1-3,1\}}{10}\right)\end{matrix}\right\} = 6$$

Let $\min = \min_{v\in V(G)-S}\{temp(v)\} = 6$, we choose a vertex $x_3 \in V(G) - S$, where $temp\,(x_3) = 6$. Give $lab\,(x_3) = 6$ and $S= \{x_1, x_2, x_3$

$$temp\,(x_4) = \max_{x_1,x_2,x_3}\left\{\begin{matrix} 4 + \text{ceil}\left(\frac{\max\{(4+1-3,1\}}{10}\right) \\ 5 + \text{ceil}\left(\frac{\max\{(4+1-2,1\}}{10}\right) \\ 6 + \text{ceil}\left(\frac{\max\{(4+1-1,1\}}{10}\right)\end{matrix}\right\} = 7$$
$$temp\,(x_5) = \max_{x_1,x_2,x_3}\left\{\begin{matrix} 4 + \text{ceil}\left(\frac{\max\{(4+1-4,1\}}{10}\right) \\ 5 + \text{ceil}\left(\frac{\max\{(4+1-3,1\}}{10}\right) \\ 6 + \text{ceil}\left(\frac{\max\{(4+1-2,1\}}{10}\right)\end{matrix}\right\} = 7$$

Let $\min = \min_{v\in V(G)-S}\{temp(v)\} = 7$, we choose a vertex $x_4 \in V(G) - S$, where $temp\,(x_4) = 7$. Give $col\,(x_4) = 7$ and $S= \{x_1, x_2, x_3, x_4$

$$temp\,(x_5) = \max_{x_1}\left\{\begin{matrix} 4 + \text{ceil}\left(\frac{\max\{(4+1-4,1\}}{10}\right) \\ 5 + \text{ceil}\left(\frac{\max\{(4+1-3,1\}}{10}\right) \\ 6 + \text{ceil}\left(\frac{\max\{(4+1-2,1\}}{10}\right) \\ 7 + \text{ceil}\left(\frac{\max\{(4+1-1,1\}}{10}\right)\end{matrix}\right\} = 8$$

TABLE 1. Description of the computing environment.

CPU	Intel (R) Core (TM) i3-3217U CPU@ 1.80 GHz
RAM Size	4 GB RAM
MATLAB version	R2018a (9.4.0.813654)

TABLE 2. Comparison between Standard Radio mean number, Algorithm and Integer Linear Programming for the upper bound of radio mean number for the path graph.

Path Graph					
n	Standard RM	Proposed Algorithm		Integer Linear Programming	
		$rmn(P_n)$	CPU Time	$rmn(P_n)$	CPU Time
1	1	-	-	-	-
2	2	2	0.006218	2	0.18282
3	3	4	0.007495	4	0.205361
4	5	6	0.026153	6	0.215519
5	7	8	0.036307	8	0.218075
6	9	10	0.036767	10	0.2199
7	11	12	0.044163	12	0.223014
8	13	14	0.058617	14	0.224061
9	15	16	0.069201	16	0.227619
10	17	18	0.094584	18	0.228267
11	19	20	0.149424	20	0.23177
12	21	22	0.152461	22	0.236957
13	23	24	0.195782	24	0.237275
14	25	26	0.294814	26	0.238958
15	27	28	0.439655	28	0.240135
16	29	30	0.466819	30	0.245314
17	31	32	0.497643	32	0.25213
18	33	34	0.597840	34	0.252392
19	35	36	0.715104	36	0.254802
20	37	38	1.077264	38	0.256105
21	39	40	1.079634	40	0.25981
22	41	42	1.284285	42	0.26076
23	43	44	1.490910	44	0.269241
24	45	46	1.724434	46	0.27758
25	47	48	1.987043	48	0.278393
26	49	50	2.346911	50	0.281428
27	51	52	3.063563	52	0.281446
28	53	54	3.123361	54	0.284219
29	55	56	3.598407	56	0.305907
30	57	58	4.082001	58	0.317522
50	97	98	30.19102	98	0.356951

Let $\min = \min_{v\in V-S}\{temp\,(v)\} = 8$, we choose a vertex $x_5 \in V(G) - S$, such that $temp\,(x_5) = 8$. Give $col\,(x_5) = 8$ and $S = \{x_1, x_2, x_3, x_4, x_5\}$. It is clear that all vertices are labeled and $rmn\,(P_5) = 8$.

IV. CASTING AS AN INTEGER LINEAR PROGRAMMING MODEL

In this section, we introduce a new mathematical model [13]–[21] for the radio mean labeling application.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ are the vertices of the connected graph G with order n, and let $D= [d_{ij}]$ be the distance matrix of G, that is, $d_{ij} = d(v_i, v_j)$ for $1 \le i, j \le n$. Let x_i be the labels of the vertices v_i where, $1 \le i \le n$. We define the function F by $F = x_1 + x_2 + \ldots + x_n$.

Minimizing F subject to the $\binom{n}{2}$ constraints

$$2n\left\lceil\frac{x_i + x_j}{2}\right\rceil \ge diam + 1 - d(v_i, v_j) \text{ for}$$
$$1 \le i \le n-1;\ 2 \le j \le n \text{ and } i < j \text{ where}$$
$$x_1, x_2, \ldots, x_n \in \{0, 1\}$$

TABLE 3. **Comparison between Standard Radio mean number, Algorithm and Integer Linear Programming for the upper bound of radio mean number for the cycle graph.**

Cycle Graph					
n	Standard RM	Proposed Algorithm		Integer Linear Programming	
		rmn(C_n)	CPU Time		rmn(C_n)
1	-	-	-	-	-
2	-	-	-	-	-
3	3	3	0.00388	3	0.192434
4	4	5	0.007196	5	0.197589
5	5	6	0.014144	6	0.204881
6	6	8	0.01999	8	0.208477
7	7	9	0.034647	9	0.210394
8	8	11	0.048613	11	0.210555
9	9	12	0.073299	12	0.212059
10	10	14	0.087588	14	0.217347
11	12	15	0.131079	15	0.219025
12	14	17	0.133592	17	0.219235
13	15	18	0.219834	18	0.220924
14	17	20	0.225044	20	0.222914
15	18	21	0.31399	21	0.225055
16	20	23	0.375647	23	0.225115
17	21	24	0.474422	24	0.229262
18	23	26	0.574174	26	0.231136
19	24	27	0.683679	27	0.237609
20	26	29	0.827998	29	0.243028
21	27	30	1.177096	30	0.245212
22	29	32	1.184903	32	0.248772
23	30	33	1.454892	33	0.250364
24	32	35	1.683995	35	0.253585
25	33	36	2.27022	36	0.254351
26	35	38	2.549199	38	0.256721
27	36	39	2.979835	39	0.26002
28	38	41	3.442036	41	0.2711
29	40	43	3.460139	43	0.302073
30	41	44	3.936796	44	0.302639
50	71	74	31.025694	74	0.459412

Example 2: This example shows how to formulate the radio mean problem as the mathematical model for cycle C_3. Let x_i labels the vertices v_i where $1 \leq i \leq 3$. Now, this mathematical model (integer programming model) is the following:

$$\begin{aligned} &\min f = x_1 + x_2 + x_3 \\ &\textit{subject to}: \; 6|x_1 - x_2| \geq diam + 1 - d(v_1, v_2); \\ &\qquad 6|x_1 - x_3| \geq diam + 1 - d(v_1, v_3); \\ &\qquad 6|x_2 - x_3| \geq diam + 1 - d(v_2, v_3) \\ &\qquad where x_1, x_2, x_3 \geq 0 \end{aligned}$$

Since $diam = \lfloor n/2 \rfloor$ (the diameter of C_n) then diam $= 1$ for C_3 and the distance matrix of the cyclic graph C_3 is

$$D = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

So, the final form of the above mathematical model is:

$$\begin{aligned} &\min f = x_1 + x_2 + x_3 \\ &\textit{subject to}: \; 6|x_1 - x_2| \geq 1; 6|x_1 - x_3| \geq 1; \\ &\qquad 6|x_2 - x_3| \geq 1; \quad x_1, x_2, x_3 \geq 0 \end{aligned}$$

MATLAB solver has 3 for the solution of the above example.

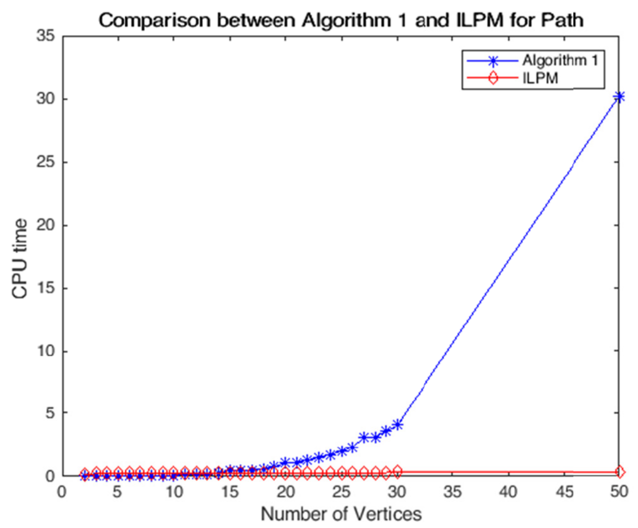

FIGURE 1. **A comparison between Algorithm 1 and Integer Linear Programming Model according to CPU time for paths.**

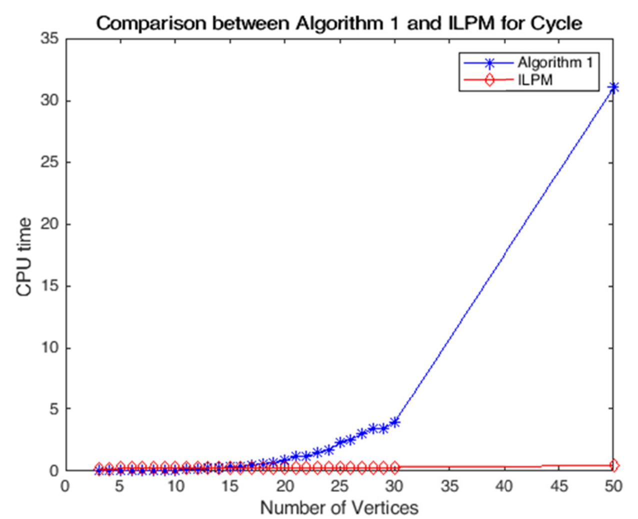

FIGURE 2. **A comparison between Algorithm 1 and Integer Linear Programming Model according to CPU time for cycles.**

V. COMPUTATIONAL STUDY

In this section, we describe our numerical experiments and present computational results, which prove that the Integer Linear Programming Model overcomes the proposed approximate algorithm according to CPU time only. We test the proposed approaches on path graphs and cycle graphs. Table 1 describes the computing environment. MATLAB solver was used to solve the mathematical model.

In Tables 2 and 3 the abbreviations Standard RM, rmn(P_n), and CPU Time are used to denote the exact radio mean number for path graphs and cycle graphs respectively. Table 2 shows that the proposed Integer Linear Programming Model and the proposed Algorithm 1 determine the same upper bound for the number of radio mean which closes to the exact radio mean number of the path P_n. On the other hand, Integer Linear Programming Model overcomes Algorithm 1 according to CPU time as shown in Figure 1.

Table 3 shows that the proposed Integer Linear Programming Model and the proposed Algorithm 1 determine the same upper bound for the number of radio mean which is close to the exact radio mean number of the cycle C_n. On the other hand, Table 2 and Table 3 show that the Integer Linear Programming Model overcomes the proposed algorithm Algorithm1 according to CPU time only as shown in Figure 2.

VI. CONCLUSION

In this work, we propose three contributions. The first contribution is that we proved two theorems which determine the radio mean number for cycle graphs and path graphs. The second contribution is proposing a new approximate algorithm which finds the upper bound for the number of radio mean for a given graph. The third contribution is that we propose a new mathematical model for finding the upper bound for the number of radio mean of a graph. Finally, the experimental results analysis and statistical test prove that the Integer Linear Programming Model overcame the proposed approximate algorithm according to CPU time only. In future work, we will adopt new approaches for determining the radio mean number of large graphs. These approaches are parallel processing and metaheuristic algorithms.

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