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One dimensional nonlinear integral operator with Newton–Kantorovich method

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Abstract The Newton–Kantorovich method (NKM) is widely used to find approximate solutions for nonlinear problems that occur in many fields of applied mathematics. This method linearizes the problems and then attempts to solve the linear problems by generating a sequence of functions. In this study, we have applied NKM to Volterra-type nonlinear integral equations then the method of Nystrom type Gauss–Legendre quadrature formula (QF) was used to find the approximate solution of a linear Fredholm integral equation. New concept of determining the solution based on subcollocation points is proposed. The existence and uniqueness of the approximated method are proven. In addition, the convergence rate is established in Banach space. Finally illustrative examples are provided to validate the accuracy of the presented method.

1. Introduction

Nonlinear integral equations occur in many scientific fields, including fluid mechanics (Ladopoulos, 2003), physics (Agarwal and Khan, 2015), chemical kinetics (Tsokos and Padgett, 1974, pp.180), and economic systems (Boikov and Tynda, 2003). The difficulty lies in determining the exact solution for such equations. Therefore, an alternative option is to find an approximate solution to the problems. A well-known approximate method is the Newton–Kantorovich method (NKM), which reduces a nonlinear integral equation into a sequence of linear integral equations. The solution is then approximated by processing the convergent sequence.

Particularly, in Boikov and Tynda (2015), weakly singular Volterra integral equations of the different types are considered. The construction of accuracy-optimal numerical methods for one-dimensional and multidimensional equations is discussed. Since this question is closely related with the optimal approximation problem, the orders of the Babenko and Kolmogorov n-widths of compact sets from some classes of functions have been evaluated. Construction of complexity order optimal numerical methods for Volterra integral equations with different types of weakly singular kernels is shown

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in Tynda (2006) as well as it is shown that for Volterra equations (in contrast to Fredholm integral equations) using the "Block-by-Block" technique it is not necessary to employ the additional iterations to construct complexity optimal methods. The NKM is also used for nonlinear functional equations. For instance, the authors of Uko and Argyros (2008) proved a weak Kantorovich-type theorem that generates the same conclusions as obtained in Argyros (2004) by the combination of weak Lipschitz and center-Lipschitz conditions. A local convergence analysis is presented in Argyros and Hilout (2013) for a fast two-step Newton-like method to find the approximate solution of nonlinear equations in a Banach space. The authors in Argyros and Khattri (2015) developed sufficient convergence conditions of Newton's method based on the majorizing principle. The work (Argyros, 1998) presents results about polynomial equations as well as analyzes iterative methods for their numerical solution in various general space settings. A Kantorovich-type convergence criterion was established in Shena and Li (2009) for inexact Newton methods. This criterion assumed that the first derivative of an operator satisfies the Lipschitz condition. The inexact Newton method was proved in Ferreira and Svaiter (2012) given a fixed relative residual error tolerance that was Q-linearly convergent to a zero of the nonlinear operator. The authors in Saberi-Nadjafi and Heidari (2010) developed a new method that combines the NKM with quadrature methods to solve nonlinear integral equations in Urysohn form. Mixed Hammerstein-type nonlinear integral equations were also solved in Ezquerro et al. (2012) using the NKM based on the concept of sequence majorizing provided by Kantorovich. The authors of Ezquerro et al. (2013) studied the semilocal convergence of Newton's method in Banach spaces upon modifying the classic conditions of Kantorovich and applied to two Hammerstein integral equations of the second type. In Akyüz-Daşcioğlu and Yaslan (2006), Chebyshev collocation method has been presented to solve nonlinear integral equations. This method transforms the integral equation to a matrix equation which corresponds to a system of nonlinear algebraic equations with unknown Chebyshev coefficients. Finally, some examples are presented to illustrate the method and results discussed. The authors in Hameed et al. (2015) and Eshkuvatova et al. (2010) consider the system of nonlinear integral equations of different types and proved the existence and uniqueness of the solution together with the rate of convergence of the approximate solution as well as numerical examples provided to validate the proposed method.

In this note we consider the Volterra-type nonlinear integral equation of the form.

$$x(t) - \lambda \int_{y(t)}^{t} H(t,\tau)\mathbb{F}(x(\tau))d\tau = f(t), \qquad (1)$$

where $0 < t_0 \leqslant t \leqslant T, \lambda$ is a real or complex number, the known function $y(t) \in C^1_{[t_0,T]}$ provide that $y(t) < t$ and $f(t) \in C_{[t_0,T]}$. The given kernel $H(t,\tau) \in C_{[t_0,T]\times[t_0,T]}$. $\mathbb{F}(\xi)$ is a differentiable continuous function.

The current paper is structured as follows. In Section 2, we describe (NKM). In Section 3, we solve the system of algebraic linear Fredholm integral equation by Nystrom type Gauss–Legendre quadrature formula (QF). Section 4 discusses the convergence rate of the approximate method and the error estimation. In Section 5, we apply the proposed method to three examples to demonstrate the accuracy and efficiency of the method. Finally, Section 6 summarizes the main concepts of the approximation method.

2. Newton–Kantorovich approach for nonlinear integral equation

Rewrite Eq. (1) in the operator equation

$$P(x) = x(t) - f(t) - \lambda \int_{y(t)}^{t} H(t,\tau)\mathbb{F}(x(\tau))d\tau = 0. \qquad (2)$$

Consider the initial iteration of NKM which is of the form

$$P'(x_0(t))(x(t) - x_0(t)) + P(x_0(t)) = 0, \qquad (3)$$

where $x_0(t)$ is the initial guess that might be any continuous function. The Frechet derivative of $P(x(t))$ at the initial condition $x_0(t)$ is defined as

$$\begin{aligned} P'(x_0)x &= \lim_{s\to 0}\frac{1}{s}[P(x_0+sx) - P(x_0)] \\ &= \lim_{s\to 0}\frac{1}{s}\left[\frac{dP(x_0)}{dx}sx + \frac{1}{2}\frac{d^2P}{dx^2}(x_0+\theta sx)s^2x^2\right], \quad \theta \in (0,1) \\ &= \frac{dP(x_0)}{dx}x. \end{aligned} \qquad (4)$$

From Eqs. (3) and (4) we obtain

$$\left.\frac{dP}{dx}\right|_{x_0}(\Delta x(t)) = -P(x_0(t)), \qquad (5)$$

where $\Delta x(t) = x_1(t) - x_0(t)$, and $x_0(t)$ is the initial guess. To solve Eq. (5) for $\Delta x(t)$ we need to compute the derivative

$$\begin{aligned} \left.\frac{dP}{dx}\right|_{x_0} &= \lim_{s\to 0}\frac{1}{s}[P(x_0+sx) - P(x_0)] \\ &= \lim_{s\to 0}\frac{1}{s}\left[sx - \lambda\int_{y(t)}^{t} H(t,\tau)[\mathbb{F}(x_0(\tau)+sx(\tau)) - \mathbb{F}(x_0(\tau))]d\tau\right], \\ &= x(t) - \lambda\int_{y(t)}^{t} H(t,\tau)\mathbb{F}'(x_0(t))x(\tau)d\tau, \end{aligned} \qquad (6)$$

From Eqs. (5) and (6) we obtain

$$\Delta x(t) - \lambda \int_{y(t)}^{t} H_0(t,\tau)\Delta x(\tau)d\tau = G_0(t), \qquad (7)$$

where

$$H_0(t,\tau) = H(t,\tau)\mathbb{F}'(x_0(\tau)) \qquad (8)$$

$$G_0(t) = f(t) + \lambda \int_{y(t)}^{t} H(t,\tau)\mathbb{F}(x_0(\tau))d\tau - x_0(t). \qquad (9)$$

Eq. (7) is linear with respect to $\Delta x(t)$, and it is easy to find $x_1(t) = x_0(t) + \Delta x(t)$.

By continuing this process, a sequence of approximate solution $x_m(t), (m = 2, 3, \ldots)$ can be evaluated from the equation

$$P'(x_0)\Delta x_m + P(x_m) = 0 \qquad (10)$$

which is equivalent to the equation

$$\Delta x_m(t) - \lambda \int_{y(t)}^{t} H_0(t,\tau)\Delta x_m(\tau)d\tau = G_{m-1}(t), \qquad (11)$$

where

$$\Delta x_m(t) = x_m(t) - x_{m-1}(t), \quad m = 1, 2, \ldots, \qquad (12)$$

and

$$G_{m-1}(t) = f(t) + \lambda \int_{y(t)}^{t} H(t,\tau)\mathbb{F}(x_{m-1}(\tau))d\tau - x_{m-1}(t).$$

Solving Eq. (11) for $\Delta x_m(t)$ gives a sequence of approximate solution $x_m(t)$.

3. Gauss–Legendre quadrature formula for a numerical solution

Introducing a uniform grid $\omega_1 = \{t_i : t_i = t_0 + ih,\ h = \frac{T-t_0}{n}, i = 1,2,\ldots,n\}$, where n refers to the number of partitions in $[t_0, T]$, Eq. (11) becomes

$$\Delta x_m(t_i) - \lambda \int_{y(t_i)}^{t_i} H_0(t_i,\tau)\Delta x_m(\tau)d\tau = G_{m-1}(t_i), \tag{13}$$

where

$$G_{m-1}(t_i) = f(t_i) + \lambda \int_{y(t_i)}^{t_i} H(t_i,\tau)\mathbb{F}(x_{m-1}(\tau))d\tau - x_{m-1}(t_i).$$

The robust way to approximate the integration in the system (13) is Gauss–Legendre QF. It is known that the Legendre polynomials $P_n(t)$ are orthogonal on $[-1,1]$ with weight $w = 1$. Consider the Gauss–Legendre QF (Jeffrey, 2000, pp. 318)

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n}\omega_i f(s_i) + R_i(f), \tag{14}$$

where

$$\omega_i = \frac{2}{(1-s_i^2)[P_n'(s_i)]^2}, \quad \sum_{i=1}^{n}\omega_i = 2, \quad P_n(s_i) \equiv 0,$$
$i = 1,2,\ldots,n,$

s_i, $i = 1,2,\ldots,n$ are roots of the Legendre polynomial $P_n(t)$ on the interval $[-1,1]$. The error term of Gauss–Legendre QF is

$$R_n(f) = \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{2n}(\xi), \quad -1 < \xi < 1.$$

The Gauss–Legendre (QF) formula for arbitrary interval $[a,b]$ has the form

$$\int_a^b f(x)dx = \frac{b-a}{2}\sum_{i=1}^{n}\omega_i f(t_i) + R_i(f), \tag{15}$$

where the nodes $t_i = \left(\frac{b-a}{2}\right)s_i + \left(\frac{b+a}{2}\right)$.

Let us describe the new idea to solve the Eq. (13). Firstly, we introduce a subgrid $\omega_2 = \{\tau_i^j\}$ of ω_1 at each subinterval $[y(t_i), t_i] \subseteq [t_0, T]$, where

$$\tau_i^j = \frac{t_i - y(t_i)}{2}s_j + \frac{t_i + y(t_i)}{2}, \quad i = 1,2,\ldots,n,$$
$$j = 1,2,\ldots,\ell \tag{16}$$

where $\tau_i^j \neq t_i$ and ℓ refers to the number of sub partitions of $[y(t_i), t_i]$. Secondly, we apply Gauss–Legendre quadrature formula to the kernel integral of (13) at each subinterval $[y(t_i), t_i]$ which yields

$$\int_{y(t_i)}^{t_i} H(t_i,\tau)\mathbb{F}(x_{m-1}(\tau))d\tau \simeq \frac{t_i - y(t_i)}{2}\sum_{j=1}^{\ell} H_0(\tau_i^k, \tau_i^j)\Delta x_m(\tau_i^j)w_j, \tag{17}$$

where H_0 is defined by (8).

Thirdly, from (13) and (17) it follows that

$$\Delta x_m(\tau_i^k) - \lambda\frac{t_i - y(t_i)}{2}\sum_{j=1}^{\ell} H_0(\tau_i^k, \tau_i^j)\Delta x_m(\tau_i^j)w_j = G_{m-1}(\tau_i^k), \tag{18}$$

$i = 1,2,\ldots,n; \quad k = 1,2,\ldots,\ell,$

where

$$G_{m-1}(\tau_i^k) = f(\tau_i^j) + \lambda\frac{t_i - y(t_i)}{2}\sum_{j=1}^{\ell} H(\tau_i^k, \tau_i^j)\mathbb{F}\left(x_{m-1}\left(\tau_{i(m-1)}^j\right)\right)w_j - x_{m-1}(\tau_i^k).$$

Eq. (18) is a linear algebraic system of $n \times \ell$ equations and $n \times \ell$ unknowns. If its matrix is non singular then it has a unique solution in terms of $\Delta x_m(\tau_i^k)$, $i = 1,2,\ldots,n$, $k = 1,2,\ldots,\ell$, then $x_m(\tau_i^k)$ can be evaluated as

$$x_m(\tau_i^k) = \Delta x_m(\tau_i^k) + x_{m-1}(\tau_i^k), \quad m = 2,3,\ldots. \tag{19}$$

Since the values of the functions $x_m(\tau_i^k)$ is known at ℓ Legendre grid points in each subinterval $(y(t_i), t_i)$ for each m iteration, the values of unknown function $x_m(t_i)$ can be found by using Newton forward interpolation formula given below

$$\begin{aligned} x_m(t) &\equiv P_\ell(t) \\ &= x_m(\tau_i^\ell) + x_m(\tau_i^\ell, \tau_i^{\ell-1})(t - \tau_i^\ell) \\ &\quad + x_m(\tau_i^\ell, \tau_i^{\ell-1}, \tau_i^{\ell-2})(t - \tau_i^\ell)(t - \tau_i^{\ell-1}) \\ &\quad + x_m(\tau_i^\ell, \tau_i^{\ell-1}, \tau_i^{\ell-2}, \ldots, \tau_i^1)(t - \tau_i^\ell)(t - \tau_i^{\ell-1})\cdots(t - \tau_i^1), \end{aligned} \tag{20}$$

with the error (Atkinson, 1997, pp.110)

$$\|x_m(t) - P_\ell(t)\| \leqslant \frac{M}{(\ell+1)!},$$

where

$$M = max\{|x_m^{\ell+1}(\xi)||(t - \tau_i^\ell), \ldots, (t - \tau_i^1)|\}.$$

4. Convergence analysis

Based on the general theorems of (NKM) and their applications to functional equations, we state the following theorem with respect to the successive approximations which are characterized by Eq. (11).

First, since $f(t), x_0(t), H(t,\tau), \mathbb{F}(\xi), \mathbb{F}'(\xi)$ and $\mathbb{F}''(\xi)$ are continuous functions in their domains of definitions, then they are bounded (Zeidler, 1995, pp.33), i.e.

$$|f(t)| \leqslant M_1, \quad |x_0(t)| \leqslant M_2, \quad |H(t,\tau)| \leqslant M_3, \quad |\mathbb{F}(x_0(t))| \leqslant M_4,$$
$$|\mathbb{F}'(x_0(t))| \leqslant M_5, \quad |\mathbb{F}''(x_0(t))| \leqslant M_6, \quad M_7 = \min_{t\in[t-0,T]}|y(t)|.$$

Then, we use the majorant function (Kantorovich and Akilov, 1982, pp.533)

$$\psi(t) = Kt^2 - 2t + 2\eta, \tag{21}$$

where $K = M_3M_6(T - M_7)$ and η to be nonnegative real number.

Theorem 1. *Let the operator $P(x) = 0$ in (2) is defined in $\Omega = \{x \in C_{[t_0,T]} : |x - x_0| \leqslant R\}$ and has a continuous second derivative in $\Omega_0 = \{x \in C_{[t_0,T]} : |x - x_0| \leqslant r\}$, where $T = t_0 + r \leqslant t_0 + R$. Moreover, let the functions $f(t) \in C_{[t_0,T]}$, $x_0(t) \in C^1_{[t_0,T]}$, $\mathbb{F}(\xi) \in C_{(-\infty,\infty)}$, $\mathbb{F}'(\xi) \in C_{(-\infty,\infty)}$ and the kernel $H(t,\tau) \in C_{[t_0,T]\times[t_0,T]}$, then if*

1. *The linear Volterra integral equation in Eq. (10) has a resolvent kernel $\Gamma(t,\tau;\lambda)$ where $\|\Gamma\| \leqslant M_3 M_5 e^{\lambda M_3 M_5 (T-M_6)}$,*
2. *$|\Delta x| \leqslant \eta$,*
3. *$|P''(x)| \leqslant K$.*

Then Eq. (1) has a unique solution x^* in the closed ball Ω_0 and the sequence $x_m(t), m \geqslant 0$ of successive approximations

$$\Delta x_m(t) - \lambda \int_{y(t)}^{t} H_0(t,\tau)\Delta x_m(\tau)d\tau = G_{m-1}(t), \tag{22}$$

where $\Delta x_m(t) = x_m(t) - x_m(t)$ converges to the solution $x^*(t)$. The rate of convergence is

$$\|x^* - x_m\| \leqslant \frac{1}{K}\left(1 - \sqrt{1 - 2K\eta}\right)^{m+1}, \quad m = 1, 2, \ldots \tag{23}$$

Proof. Since Eq. (7) is a linear integral equation of the second kind, it has a unique solution in term of $\Delta x(t)$ provided that its kernel $H_0(t,\tau)$ is a continuous function. Hence the existence of Γ_0 is accomplished.

To prove Γ_0 is bounded we need to establish the resolvent kernel $\Gamma_0(t,\tau;\lambda)$ of Eq. (7). Assume the integral operator U from $C[t_0,T] \to [t_0,T]$ is given by

$$Z = U(\Delta x), \quad Z(t) = \int_{y(t)}^{t} H_0(t,\tau)\Delta x(\tau)d\tau, \tag{24}$$

where $H_0(t,\tau)$ is defined in Eq. (8). Due to Eq. (24), Eq. (7) can be written as

$$\Delta x - \lambda U(\Delta x) = G_0(t). \tag{25}$$

The solution Δx^* of Eq. (25) is written in terms of G_0 as

$$\Delta x^* = G_0 + B(G_0), \tag{26}$$

where B is an integral operator and can be expressed as a power series of U (Atkinson, 1997, Theorem 1, pp.378)

$$B(G_0) = I + \lambda U(G_0) + \lambda^2 U^2(G_0) + \cdots + \lambda^n U^n(G_0) + \cdots, \tag{27}$$

and it is well known that the powers of U are also integral operators. In fact

$$Z_n = U^n, \quad Z_n(t) = \int_{y(t)}^{t} H_0^{(n)}(t,\tau)\Delta x(\tau)d\tau, \quad (n = 1, 2, \ldots), \tag{28}$$

where $H_0^{(n)}$ is the iterated kernel. Substituting (28) into (26) we obtain the solution of Eq. (25) which is of the form

$$\Delta x^*(t) = G_0(t) + \lambda \int_{y(t)}^{t} \Gamma_0(t,\tau;\lambda)G_0(\tau)d\tau, \tag{29}$$

where

$$\Gamma_0(t,\tau;\lambda) = \sum_{j=0}^{\infty} \lambda^j H_0^{j+1}(t,\tau), \tag{30}$$

and $\Gamma_0(t,\tau,\lambda)$ is the resolvent kernel. Next, we elucidate that the series in Eq. (29) is convergent uniformly for all $t \in [t_0,T]$. Since

$$|H_0(t,\tau)| = |H(t,\tau)\mathbb{F}(x_0(\tau))| \leqslant |H(t,\tau)||\mathbb{F}'(x_0(\tau))| \leqslant M_3 M_5. \tag{31}$$

Let $M = M_3 M_5$, then by mathematical induction we obtain

$$\left|H_0^{(2)}(t,\tau)\right| \leqslant \int_{y_0(t)}^{t} |H_0(t,u)H_0(u,\tau)|du \leqslant \frac{M^2(t-M_7)}{(1)!},$$

$$\left|H_0^{(3)}(t,\tau)\right| \leqslant \int_{y_0(t)}^{t} \left|H_0(t,u)H_0^{(2)}(u,\tau)\right|du \leqslant \frac{M^3(t-M_7)^2}{(2)!},$$

$$\vdots$$

$$\left|H_0^{(n)}(t,\tau)\right| \leqslant \int_{y_0(t)}^{t} \left|H_0(t,u)H_0^{(n-1)}(u,\tau)\right|du \leqslant \frac{M^n(t-M_7)^{n-1}}{(n-1)!},$$

$(n = 1, 2 \ldots)$,

then

$$\begin{aligned}\|\Gamma_0\| = \|B(G_0)\| &\leqslant \sum_{j=0}^{N} |\lambda|^j |H_0^{j+1}(t,\tau), | \\ &\leqslant \sum_{j=0}^{\infty} |\lambda|^j M^{j+1} \frac{(T-M_7)^j}{j!}, \\ &= M \sum_{j=0}^{\infty} M^j \frac{(T-M_7)^j}{j!}, \\ &= M e^{|\lambda| M (T-M_7)}.\end{aligned} \tag{32}$$

Table 1 Numerical results for Eq. (34).

$n = 2,\ \ell = 4,\ h = 0.5$

m	ϵ_x
1	0.07407
2	0.02250
3	0.00734
4	0.00245
5	8.20482E−004
10	3.53286E−006
20	6.57169E−011

Table 2 Numerical results for Eq. (35).

$n = 2,\ \ell = 4,\ h = 0.5$

m	ϵ_x
1	0.08227
2	0.04207
3	0.00861
4	0.00355
5	10.27756−004
10	5.22045E−005
20	1.25811E−010

Table 3 Error analysis of Example 3.

t	Korobov's	Sidi's	Laurie's	Chebyshev's	NKM
0.1	0.12E−004	0.66E−008	0.12E−006	0.36E−007	0.28E−016
0.2	0.31E−004	0.34E−007	0.27E−006	0.63E−007	0.56E−016
0.3	0.60E−004	0.11E−006	0.47E−006	0.35E−007	0.11E −015
0.4	0.11E−003	0.30E−006	0.71E−006	0.88E−007	0.00
0.5	0.18E−003	0.71E−006	0.95E−006	0.23E−007	0.00
0.6	0.29E−003	0.15E−005	0.12E−005	0.70E−007	0.11E−015
0.7	0.49E−003	0.31E−005	0.13E−005	0.69E−007	0.44E−015
0.8	0.82E−003	0.62E−005	0.11E−005	0.14E−007	0.33E−015
0.9	0.14E−002	0.12E−004	0.27E−006	0.12E−007	0.11E−010
1	0.25E−002	0.23E−004	0.20E−005	0.86E−007	0.92E−006

Therefore, the infinite series in Eq. (30) for $\Gamma_0(t,\tau;\lambda)$ is absolutely and uniformly convergent for all values of λ in the case of continuous Volterra kernel. Furthermore, we state that $\|P''(x)\| \leqslant K$ for all $x \in \Omega_0$. The second derivative $P''(x_0)(x)$ of nonlinear operator $P(x)$ is represented as

$$P''(x_0)x = \lim_{s\to 0}\frac{1}{s}[P'(x_0+sx) - P'(x_0)],$$
$$= \lim_{s\to 0}\frac{1}{s}\left(\frac{d^2P}{dx^2}(x_0)s\overline{x} + \frac{1}{2}\frac{d^3P}{dx^3}(x_0+\theta s\overline{x})s^2\overline{x}^2\right), = \frac{d^2P}{dx^2}\bigg|_{x_0}\overline{x},$$

then the norm of $\left\|\frac{dp^2}{dx^2}\right\|$ has the estimate

$$\left\|\frac{dp^2}{dx^2}\right\| = \max_{|x|\leqslant 1,|\overline{x}|\leqslant 1}\left|\int_{y(t)}^{t} H(t,\tau)\mathbb{F}''(x_0(\tau))x(\tau)\overline{x}(\tau)d\tau\right|$$
$$\leqslant M_3M_6(T - M_7).$$

Therefore, the second derivative is bounded and by using (Kantorovich and Akilov, 1982, Theorem 6, pp.532) implies that $x^*(t)$ is the unique solution of operator Eq. (2) and

$$\|x^* - x_m\| \leqslant \frac{1}{K}\left(1 - \sqrt{1-2K\eta}\right)^{m+1}, \quad m = 1,2,\ldots \quad \square \tag{33}$$

5. Numerical result

Example 1. Consider the following integral equation

$$x(t) - \int_{y(t)}^{t} t\tau x^3(\tau)d\tau = t - \frac{13446}{1000000}t^6, \tag{34}$$

where $t \in [0,1]$ and $y(t) = \frac{4}{5}t$.

The exact solution is

$$x^*(t) = t.$$

Consider the initial guess as

$$x_0(t) = \frac{t}{2},$$

Example 2. Consider the following integral equation

$$x(t) - \int_{y(t)}^{t} t\tau x^2(\tau)d\tau = \frac{t^2}{2} + \frac{t^3}{3} - \frac{229t^6}{2560} - \frac{137t^7}{1344} - \frac{3817t^8}{129024}, \tag{35}$$

where $t \in [0,1]$, and $y(t) = \frac{t}{2}$.

The exact solution and initial guess are

$$x^*(t) = \frac{t^2}{2} + \frac{t^3}{3},$$

$$x_0(t) = \frac{t^2}{4} + \frac{t^3}{6}.$$

Example 3. Consider the following integral equation

$$y(t) = e^t - 0.5(e^t - 1) + \int_0^t y^2(\tau)d\tau, \tag{36}$$

where $t \in [0,1]$, and $y(t) = 0$.

The exact solution is

$$x^*(t) = e^t.$$

Consider the initial guess as

$$x_0(t) = 0.5 + 2t,$$

Taken $h = 0.1$, $\ell = 4$, $n = 10$, and $m = 20$. In Table 3, the absolute errors of NKM are compared with the errors given by Korobov's polynomial transformation, Sidi's trigonometric transformation, and Laurie's special polynomial type transformation (Galperin et al., 2000) and Chebyshev collocation method (Akyüz-Daşcioğlu and Yaslan, 2006).

It is noted from the Table 1 and Table 2 that only a few iterations are needed for $x_m(t)$ to be very close to the exact solution $x^*(t)$. Furthermore, Table 3 shows that the results obtained by NKM are more accurate than other methods for different nodes $t_i \in [0,1]$, except the last point $t = 1$. For this point the result of Chebyshev's is better than the NKM result.

Notations used here are: n is the number of partitions on $[t_0, T], \ell$ is the number of subpartitions on $(y(t_i), t_i)$, $i = 1,2,\ldots,n$, where m is the number of iterations, and

$$\epsilon_x = \max_{t\in(0,1]}|x_m(t) - x^*(t)|,$$

6. Conclusion

In this note, the NKM is presented to solve the nonlinear integral equations of Volterra type. We have proposed a new idea by introducing subgrid collocation points $\tau_i^k, i = 1, 2, \ldots, n$, $k = 1, 2, \ldots, \ell$ which lie in the intervals $(y_0(t_i), t_i)$ and $(y_{m-1}(t_i), t_i)$. Gauss–Legendre QF is used for each subgrid interval. The theorem of existence and uniqueness of the approximate solution are established based on the general theorems of Kantorovich. Numerical examples revealed that the accuracy of the NKM can be achieved by a few number of iterations.

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