E-duality results for E-differentiable vector optimization problems under (generalized) E-convexity

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Abstract: In this paper, a class of E-differentiable multiobjective programming problems with both inequality and equality constraints is considered. The so-called vector mixed E-dual problem is defined for the considered E-differentiable multiobjective programming problem with both inequality and equality constraints. Then, several mixed E-duality theorems are established under (generalized) E-convexity hypotheses. Further, so-called vector Mond-Weir E-dual and vector Wolfe E-dual problems are also defined for the considered E-differentiable multiobjective programming problem as special cases of its vector mixed E-dual problem.

1. Introduction

Several classes of functions have been defined for the purpose of weakening the limitations of convexity in mathematical programming. One of the notions of generalized convexity introduced into optimization theory is the concept of E-convexity. The definitions of E-convex set and E-convex function were introduced by Youness [15]. This kind of generalized convexity is based on the effect of an operator $E : R^n \to R^n$ on the sets and the domains of functions. However, some results and proofs presented by Youness [15] were incorrect as it was pointed out by Yang [14]. Megahed et al. [11] presented the concept of an E-differentiable convex function which transforms a (not necessarily) differentiable convex function to a differentiable function based on the effect of an operator $E : R^n \to R^n$. Recently, Antczak and Abdulaleem [1] proved the so-called E-optimality conditions and Wolfe E-duality for E-differentiable vector optimization problems with both inequality and equality constraints.

In the recent years, duality in vector optimization has been attracting the interest of many researches. Such optimization problems with several objectives conflicting with one another reflect the complexity of the real world and are encountered in various fields. Many authors have defined the duality results for various classes of nonconvex vector optimization problems (see, for example, [1–9], and others).

In this paper, a class of nonconvex E-differentiable vector optimization problems with both inequality and equality constraints is considered in which the involved functions are (generalized) E-convex. For such a (not necessarily) differentiable multiobjective programming problem, its vector mixed E-dual problem is defined. Further, so-called vector Mond-Weir E-dual and vector Wolfe E-dual problems are also defined for the considered E-differentiable multiobjective programming problem as special cases of its vector mixed E-dual problem. Then, several E-duality theorems are established between the considered E-differentiable multicriteria optimization problem and its vector E-duals under appropriate (generalized) E-convexity hypotheses.

2. Preliminaries

Let R^n be the n-dimensional Euclidean space and R^n_+ be its nonnegative orthant. The following convention for equalities and inequalities will be used in the paper. For any vectors $x=(x_1,x_2,\cdots,x_n)^T$ and $y=(y_1,y_2,\cdots,y_n)^T$ in R^n, we define:

(i) $x=y$ if and only if $x_i=y_i$ for all $i=1,2,\cdots,n$;

(ii) $x>y$ if and only if $x_i>y_i$ for all $i=1,2,\cdots,n$;

(iii) $x\geqq y$ if and only if $x_i\geqq y_i$ for all $i=1,2,\cdots,n$;

(iv) $x\geq y$ if and only if $x_i\geqq y_i$ for all $i=1,2,\cdots,n$ but $x\neq y$;

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(v) $x \ngtr y$ is the negation of $x > y$.

Definition 1 ([15]). *A set $M \subseteq R^n$ is said to be E-convex set (with respect to an operator $E : R^n \to R^n$) if and only if the following relation*

$$E(u) + \lambda (E(x) - E(u)) \in M \tag{1}$$

holds for all $x, u \in M$ and any $\lambda \in [0,1]$.

Definition 2 ([15]). *A real-valued function $f : M \to R$ is said to be E-convex (with respect to an operator $E : R^n \to R^n$) on M if and only if the following inequality*

$$f(\lambda E(x) + (1-\lambda)E(u)) \leqq \lambda f(E(x)) + (1-\lambda) f(E(u)) \tag{2}$$

holds for all $x, u \in M$ and any $\lambda \in [0,1]$.

It is clear that every convex function is E-convex (if E is the identity map).

Definition 3 ([15]). *A real-valued function $f : M \to R$ is said to be strictly E-convex (with respect to an operator $E : R^n \to R^n$) on M if and only if the following inequality*

$$f(\lambda E(x) + (1-\lambda)E(u)) < \lambda f(E(x)) + (1-\lambda) f(E(u)) \tag{3}$$

holds for all $x, u \in M$, $E(x) \neq E(u)$, and any $\lambda \in (0,1)$.

Definition 4 ([11]). *Let $E : R^n \to R^n$ and $f : M \to R$ be a (not necessarily) differentiable function at a given point $u \in M$. It is said that f is an E-differentiable function at u if and only if $f \circ E$ is a differentiable function at u (in the usual sense) and, moreover,*

$$(f \circ E)(x) = (f \circ E)(u) + \nabla (f \circ E)(u)(x-u) + \theta(u, x-u)\|x-u\|, \tag{4}$$

where $\theta(u, x-u) \to 0$ as $x \to u$.

Proposition 5 ([1]). *Let $E : R^n \to R^n$, M be an E-convex subset of R^n, $f : R^n \to R$ be an E-convex (strictly E-convex) function on M and $u \in M$. Further, assume that f is E-differentiable at u. Then, the following inequality*

$$f(E(x)) - f(E(u)) \geqq \nabla f(E(u))(E(x) - E(u)), \quad (>) \tag{5}$$

holds for all $x \in M$.

Definition 6 ([1]). *Let $E : R^n \to R^n$, M be a nonempty E-convex subset of R^n and $f : R^n \to R$ be an E-differentiable function at $u \in M$. f is said to be a pseudo-E-convex function at u on M if the following relation*

$$(f \circ E)(x) < (f \circ E)(u) \Rightarrow \nabla (f \circ E)(u)(E(x) - E(u)) < 0 \tag{6}$$

holds for all $x \in M$. If (6) is satisfied for each $u \in M$, then f is said to be a pseudo-E-convex function on M.

Definition 7 ([1]). *Let $E : R^n \to R^n$, M be a nonempty E-convex subset of R^n and $f : R^n \to R$ be an E-differentiable function at $u \in M$. f is said to be quasi-E-convex function at u on M if the following relation*

$$(f \circ E)(x) \leqq (f \circ E)(u) \Rightarrow \nabla (f \circ E)(u)(E(x) - E(u)) \leqq 0 \tag{7}$$

holds for all $x \in M$. If (7) is satisfied for each $u \in M$, then f is said to be a quasi E-convex function on M.

In this paper, we consider the following (not necessarily differentiable) multiobjective programming problem (MOP) with both inequality and equality constraints:

$$\begin{aligned} &\text{minimize } f(x) = (f_1(x), \dots, f_p(x)) \\ &\text{subject to } g_j(x) \leqq 0, \ \ j \in J = \{1, \dots, m\}, \\ &\qquad h_t(x) = 0, \ \ t \in T = \{1, \dots, q\}, \\ &\qquad x \in R^n, \end{aligned} \qquad \text{(MOP)}$$

where $f_i : R^n \to R$, $i \in I = \{1, \dots, p\}$, $g_j : R^n \to R$, $j \in J$, $h_t : R^n \to R$, $t \in T$, are real-valued functions defined on R^n. We shall write $g := (g_1, \dots, g_m) : R^n \to R^m$ and $h := (h_1, \dots, h_q) : R^n \to R^q$ for convenience.

For the purpose of simplifying our presentation, we will next introduce some notations which will be used frequently throughout this paper. Let

$$\Omega := \left\{ x \in X : g_j(x) \leqq 0, \ \ j \in J, \, h_t(x) = 0, \ \ t \in T \right\}$$

be the set of all feasible solutions of (MOP). Further, we denote by $J(x)$ the set of inequality constraint indices that are active at a feasible solution x, that is, $J(x) = \left\{ j \in J : g_j(x) = 0 \right\}$.

For such multicriterion optimization problems, the following concepts of (weak) Pareto optimal solutions are defined as follows:

Definition 8. *A feasible point $\overline{x}$ is said to be a weak Pareto (weakly efficient) solution of (MOP) if and only if there exists no other feasible point x such that*

$$f(x) < f(\overline{x}).$$

Definition 9. *A feasible point $\overline{x}$ is said to be a Pareto (efficient) solution of (MOP) if and only if there exists no other feasible point x such that*

$$f(x) \leq f(\overline{x}).$$

Let $E : R^n \to R^n$ be a given one-to-one and onto operator. Throughout the paper, we shall assume that the functions constituting the considered multiobjective programming problem (MOP) are E-differentiable at any feasible solution.
Now, for the considered multiobjective programming problem (MOP), we define its associated differentiable vector optimization problem as follows:

$$\begin{aligned} &\text{minimize } f(E(x)) = (f_1(E(x)),...,f_p(E(x))) \\ &\text{subject to } g_j(E(x)) \leqq 0, \;\; j \in J = \{1,...,m\}, \\ &\qquad h_t(E(x)) = 0, \;\; t \in T = \{1,...,q\}, \\ &\qquad x \in R^n. \end{aligned} \tag{VP$_E$}$$

We call the problem (VP$_E$) an E-vector optimization problem associated to (MOP). Let

$$\Omega_E := \left\{ x \in R^n : g_j(E(x)) \leqq 0, \;\; j \in J, h_t(E(x)) = 0, \;\; t \in T \right\}$$

be the set of all feasible solutions of (VP$_E$). Since the functions constituting the problem (MOP) are assumed to be E-differentiable at any feasible solution of (MOP), by Definition 4, the functions constituting the E-vector optimization problem (VP$_E$) are differentiable at any its feasible solution (in the usual sense). Further, we denote by $J_E(x)$ the set of inequality constraint indices that are active at a feasible solution $x \in \Omega_E$, that is, $J_E(x) = \left\{ j \in J : (g_j \circ E)(x) = 0 \right\}$.
Now, we give the definitions of a weak Pareto (a weakly efficient) solution and a Pareto (an efficient) solution of the vector optimization problem (VP$_E$), which are, at the same time, a weak E-Pareto solution (a weakly E-efficient solution) and an E-Pareto solution (an E-efficient solution) of the considered multiobjective programming problem (MOP).

Definition 10. *A feasible point $E(\overline{x})$ is said to be a weak E-Pareto solution (weakly E-efficient solution) of (MOP) if and only if there exists no other feasible point $E(x)$ such that*

$$f(E(x)) < f(E(\overline{x})).$$

Definition 11. *A feasible point $E(\overline{x})$ is said to be an E-Pareto solution (E-efficient solution) of (MOP) if and only if there exists no other feasible point $E(x)$ such that*

$$f(E(x)) \leq f(E(\overline{x})).$$

Lemma 12 ([1])**.** *Let $E : R^n \to R^n$ be a one-to-one and onto. Then $E(\Omega_E) = \Omega$.*

Lemma 13 ([1])**.** *Let $\overline{x} \in \Omega$ be a weak Pareto solution (a Pareto solution) of the considered multiobjective programming problem (MOP). Then, there exists $\overline{z} \in \Omega_E$ such that $\overline{x} = E(\overline{z})$ and $\overline{z}$ is a weak Pareto (a Pareto) solution of the E-vector optimization problem (VP$_E$).*

Lemma 14 ([1])**.** *Let $\overline{z} \in \Omega_E$ be a weak Pareto (a Pareto) solution of the E-vector optimization problem (VP$_E$). Then $E(\overline{z})$ is a weak Pareto solution (a Pareto solution) of the considered multiobjective programming problem (MOP).*

Remark 15. *As it follows from Lemma 14, if $\overline{z} \in \Omega_E$ is a weak Pareto (a Pareto) solution of the E-vector optimization problem (VP$_E$), then $E(\overline{z})$ is a weak Pareto solution (a Pareto solution) of the considered multiobjective programming problem (MOP). We call $E(\overline{z})$ a weak E-Pareto (an E-Pareto) solution of the problem (MOP).*

As it follows from the above lemmas, there is some equivalence between the vector optimization problem (MOP) and (VP$_E$). Therefore, if we prove optimality results for the differentiable E-vector optimization problem (VP$_E$), they will be applicable also for the original nondifferentiable multiobjective programming problem (MOP) in which the involved functions are E-differentiable.

Definition 16. *For the E-vector optimization problem (VP$_E$), the E-linearized cone at $\overline{x} \in \Omega_E$, denoted by $L_E(\overline{x})$, is defined as follows*

$$L_E(\overline{x}) = \left\{ d \in R^n : \nabla g_j(E(\overline{x}))\, d \leqq 0, \; j \in J_E(\overline{x}), \; \nabla h_t(E(\overline{x}))\, d = 0, \; t \in T \right\}.$$

Definition 17. *The tangent cone (also called contingent cone or Bouligand cone) of Ω_E at $\overline{x} \in cl\,\Omega_E$ is defined by*

$$T_{\Omega_E}(\overline{x}) = \left\{ d \in R^n : \exists_{\{d_n\} \subset R^n} d_n \to d, \; \exists_{\{t_n\} \subset R} t_n \downarrow 0 \; s.t. \; \overline{x} + t_n d_n \in \Omega_E \right\}$$

or, equivalently,

$$T_{\Omega_E}(\overline{x}) = \left\{ d \in R^n : \exists_{\{\beta_n\} \subset R} \beta_n \to \infty, \; \exists_{\{x_n\} \subset \Omega_E} x_n \to \overline{x} \; s.t. \; \beta_n (x_n - \overline{x}) \to d \right\}$$

A vector $d \in R^n$ belonging to $T_{\Omega_E}(\overline{x})$ is called a tangent direction to Ω_E from $\overline{x} \in cl\,\Omega_E$.

Now, we present the E-Abadie constraint qualification (ACQ_E) which were derived for E-differentiable multiobjective programming problems with both inequality and equality constraints by Antczak and Abdulaleem [1].

Definition 18 ([1])**.** *It is said that the so-called E-Abadie constraint qualification (ACQ_E) holds at $\bar{x} \in \Omega_E$ for the differentiable E-vector optimization problem (VP_E) with both inequality and equality constraints if*

$$T_{\Omega_E}(\bar{x}) = L_E(\bar{x}). \tag{8}$$

Now, we present the Karush-Kuhn-Tucker necessary optimality conditions for $\bar{x} \in \Omega_E$ to be a weak Pareto solution of the E-vector optimization problem (VP_E). These conditions are, at the same time, the E-Karush-Kuhn-Tucker necessary optimality conditions for $E(\bar{x}) \in \Omega$ to be a weak E-Pareto solution of the considered E-differentiable multiobjective programming problem (MOP).

Theorem 19 ([1])**.** *(E-Karush-Kuhn-Tucker necessary optimality conditions). Let $\bar{x} \in \Omega_E$ be a weak Pareto solution of the E-vector optimization problem (VP_E) (and, thus, $E(\bar{x})$ be a weak E-Pareto solution of the considered multiobjective programming problem (MOP)). Further, let the objective functions f_i, $i \in I$, the constraint functions g_j, $j \in J$, and h_t, $t \in T$, be E-differentiable at $\bar{x}$ and the E-Abadie constraint qualification (ACQ_E) be satisfied at $\bar{x}$. Then there exist Lagrange multipliers $\overline{\lambda} \in R^p$, $\overline{\mu} \in R^m$ and $\overline{\xi} \in R^q$ such that*

$$\sum_{i=1}^{p} \overline{\lambda}_i \nabla (f_i \circ E)(\bar{x}) + \sum_{j=1}^{m} \overline{\mu}_j \nabla (g_j \circ E)(\bar{x}) + \sum_{t=1}^{q} \overline{\xi}_t \nabla (h_t \circ E)(\bar{x}) = 0, \tag{9}$$

$$\overline{\mu}_j (g_j \circ E)(\bar{x}) = 0, \ \ j \in J, \tag{10}$$

$$\overline{\lambda} \geq 0, \ \ \overline{\mu} \geqq 0. \tag{11}$$

3. Mond-Weir E-duality

In this section, for the differentiable vector E-optimization problem (VP_E), we define its vector Mond-Weir dual problem. In other words, for the considered E-differentiable E-convex multiobjective programming problem (MOP), we define its vector E-dual problem (MWD_E) in the sense of Mond-Weir [10]. Then, we prove several E-duality results between vector optimization problems (MOP) and (MWD_E) under appropriate (generalized) E-convexity hypotheses.
Let $E : R^n \to R^n$ be a given one-to-one and onto operator. We define the following vector dual problem in the sense of Mond-Weir related for the differentiable multicriteria E-optimization problem (VP_E):

$$\begin{gathered}
(f \circ E)(y) = (f_1(E(y)), \ldots, f_p(E(y))) \to V - \max \\
\text{s.t. } \textstyle\sum_{i=1}^{p} \lambda_i \nabla (f_i \circ E)(y) + \sum_{j=1}^{m} \mu_j \nabla (g_j \circ E)(y) + \sum_{t=1}^{q} \xi_t \nabla (h_t \circ E)(y) = 0, \\
\textstyle\sum_{j=1}^{m} \mu_j (g_j \circ E)(y) + \sum_{t=1}^{q} \xi_t (h_t \circ E)(y) \geqq 0, \quad (\mathrm{MWD}_E) \\
\lambda \in R^p, \lambda \geq 0, \ \mu \in R^m, \mu \geqq 0, \ \xi \in R^q,
\end{gathered}$$

where all functions are defined in the similar way as for the considered E-vector optimization problem (VP_E). Further, let

$$\begin{gathered}
\Gamma_E = \Big\{ (y, \lambda, \mu, \xi) \in R^n \times R^p \times R^m \times R^q : \\
\textstyle\sum_{i=1}^{p} \lambda_i \nabla (f_i \circ E)(y) + \sum_{j=1}^{m} \mu_j \nabla (g_j \circ E)(y) + \sum_{t=1}^{q} \xi_t \nabla (h_t \circ E)(y) = 0, \\
\textstyle\sum_{j=1}^{m} \mu_j (g_j \circ E)(y) + \sum_{t=1}^{q} \xi_t (h_t \circ E)(y) \geqq 0, \ \lambda \geq 0, \ \mu \geqq 0 \Big\}
\end{gathered}$$

be the set of all feasible solutions of the problem (MWD_E). Let us denote, $Y_E = \{y \in R^n : (y, \lambda, \mu, \xi) \in \Gamma_E\}$. The formulated vector dual problem (MWD_E) is the vector Mond-Weir dual problem for the vector E-optimization problem (VP_E). At the same time, we call (MWD_E) the vector Mond-Weir E-dual problem or the vector E-dual problem in the sense of Mond-Weir for the considered E-differentiable multiobjective programming problem (MOP).
Now, under E-convexity hypotheses, we prove duality results in the sense of Mond-Weir between the E-vector problems (VP_E) and (MWD_E) and, thus, E-duality results in the sense of Mond-Weir between the problems (MOP) and (MWD_E).

Theorem 20. *(Mond-Weir weak duality between (VP_E) and (MWD_E)). Let x and (y, λ, μ, ξ) be any feasible solutions of the problems (VP_E) and (MWD_E), respectively. Further, assume that at least one of the following hypotheses is fulfilled:*

A) each objective function f_i, $i \in I$, is E-convex at y on $\Omega_E \cup Y_E$, each constraint function g_j, $j \in J$, is E-convex at y on $\Omega_E \cup Y_E$, the functions h_t, $t \in T^+(E(y))$ and the functions $-h_t$, $t \in T^-(E(y))$, are E-convex at y on $\Omega_E \cup Y_E$.

B) $(f \circ E)(y)$ is pseudo-E-convex at y on $\Omega_E \cup Y_E$, $\mu_j (g_j \circ E)(y)$ is quasi-E-convex at y on $\Omega_E \cup Y_E$, $\xi_t (h_t \circ E)(y)$ is quasi E-convex at y on $\Omega_E \cup Y_E$.

Then

$$(f \circ E)(x) \nless (f \circ E)(y). \tag{12}$$

Proof. Let x and (y, λ, μ, ξ) be any feasible solutions of the problems (VP_E) and (VMD_E), respectively.
The proof of this theorem under hypothesis A). If $x = y$, then the weak duality trivially holds. Now, we prove the weak duality theorem when $x \neq y$. We proceed by contradiction. Suppose, contrary to the result, that the inequality

$$(f \circ E)(x) < (f \circ E)(y) \tag{13}$$

holds. By the feasibility of (y, λ, μ, ξ) in the problem (MWD_E), the above inequality yields

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(E(x)) < \sum_{i=1}^{p} \lambda_i \nabla f_i(E((y)). \tag{14}$$

By assumption, x and (y, λ, μ, ξ) are feasible solutions for the problems (VP_E) and (MWD_E), respectively. Since the functions f_i, $i \in I$, g_j, $j \in J$, h_t, $t \in T^+(E(y))$, $-h_t$, $t \in T^-(E(y))$, are E-convex at y on $\Omega_E \cup Y_E$, by Proposition 5, the following inequalities

$$f_i(E(x)) - f_i(E(y)) \geqq \nabla f_i(E(y))(E(x) - E(y)), i \in I, \tag{15}$$

$$g_j(E(x)) - g_j(E(y)) \geqq \nabla g_j(E(y))(E(x) - E(y)), j \in J_E(y), \tag{16}$$

$$h_t(E(x)) - h_t(E(y)) \geqq \nabla h_t(E(y))(E(x) - E(y)), t \in T^+(E(y)), \tag{17}$$

$$-h_t(E(x)) + h_t(E(y)) \geqq -\nabla h_t(E(y))(E(x) - E(y)), \ t \in T^-(E(y)) \tag{18}$$

hold, respectively. Multiplying inequalities (15)-(18) by the corresponding Lagrange multiplier, respectively, we obtain that the inequality

$$\lambda_i f_i(E(x)) - \lambda_i f_i(E(y)) \geqq \lambda_i \nabla f_i(E(y))(E(x) - E(y)), i \in I, \tag{19}$$

$$\mu_j g_j(E(x)) - \mu_j g_j(E(y)) \geqq \mu_j \nabla g_j(E(y))(E(x) - E(y)), j \in J_E(y), \tag{20}$$

$$\xi_t h_t(E(x)) - \xi_t h_t(E(y)) \geqq \xi_t \nabla h_t(E(y))(E(x) - E(y)), t \in T^+(E(y)), \tag{21}$$

$$\xi_t h_t(E(x)) - \xi_t h_t(E(y)) \geqq \xi_t \nabla h_t(E(y))(E(x) - E(y)), \ t \in T^-(E(y)) \tag{22}$$

hold, respectively. Then adding both sides of (19)-(22), we obtain that the inequality

$$\sum_{i=1}^{p} \lambda_i f_i(E(x)) - \sum_{i=1}^{p} \lambda_i f_i(E(y)) + \sum_{j=1}^{m} \mu_j g_j(E(x)) -$$
$$\sum_{j=1}^{m} \mu_j g_j(E(y)) + \sum_{t=1}^{q} \xi_t h_t(E(x)) - \sum_{t=1}^{q} \xi_t h_t(E(y)) \geqq$$
$$\left[\sum_{i=1}^{p} \lambda_i \nabla f_i(E(y)) + \sum_{j=1}^{m} \mu_j \nabla g_j(E(y)) + \sum_{t=1}^{q} \xi_t \nabla h_t(E(y))\right](E(x) - E(y))$$

holds. Thus, by $x \in \Omega_E$, it follows that the following inequality

$$\begin{gathered}\sum_{i=1}^{p} \lambda_i f_i(E(x)) - \sum_{i=1}^{p} \lambda_i f_i(E(y)) - \sum_{j=1}^{m} \mu_j g_j(E(y)) - \sum_{t=1}^{q} \xi_t h_t(E(y)) \geqq \\ \left[\sum_{i=1}^{p} \lambda_i \nabla f_i(E(y)) + \sum_{j=1}^{m} \mu_j \nabla g_j(E(y)) + \sum_{t=1}^{q} \xi_t \nabla h_t(E(y))\right](E(x) - E(y))\end{gathered} \tag{23}$$

holds. Thus, by (23) and the first constraint of (MWD_E), we have

$$\sum_{i=1}^{p} \lambda_i f_i(E(x)) - \sum_{i=1}^{p} \lambda_i f_i(E(y)) - \sum_{j=1}^{m} \mu_j g_j(E(y)) - \sum_{t=1}^{q} \xi_t h_t(E(y)) \geqq 0. \tag{24}$$

Since (y, λ, μ, ξ) is a feasible solutions of (MWD_E), the inequality

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(E(x)) \geqq \sum_{i=1}^{p} \lambda_i \nabla f_i(E((y)) \tag{25}$$

holds. Hence, by (25) and $\lambda_i \geqq 0$, $i = 1, 2, ..., p$, $\sum_{i=1}^{p} \lambda_i = 1$, it follows that the inequality (14) cannot hold which means that the proof of the Mond-Weir weak duality theorem between the E-vector optimization problems (VP_E) and (MWD_E) is completed under hypothesis A).
The proof of this theorem under hypothesis B). We proceed by contradiction. Suppose, contrary to the result, that (13) holds. Since the function $(f \circ E)(\cdot)$ is pseudo-E-convex at y on $\Omega_E \cup Y_E$, by Definition 6, the inequality

$$\sum_{i=1}^{p} \lambda_i \nabla (f_i \circ E)(y)(E(x) - E(y)) < 0 \tag{26}$$

holds. Since $\mu_j(g_j \circ E)(y)$ and $\xi_t \nabla (h_t \circ E)(y)$ are quasi-E-convex at y on $\Omega_E \cup Y_E$, Definition 7 implies that the inequalities

$$\sum_{j=1}^{m} \mu_j \nabla (g_j \circ E)(y)(E(x) - E(y)) \leqq 0, \tag{27}$$

$$\sum_{t=1}^{q} \xi_t \nabla (h_t \circ E)(y)(E(x) - E(y)) \leqq 0 \tag{28}$$

hold, respectively. Combining (26), (27) and (28), it follows that the inequality

$$\left[\sum_{i=1}^{p} \lambda_i \nabla f_i(E(y)) + \sum_{j=1}^{m} \mu_j \nabla g_j(E(y)) + \sum_{t=1}^{q} \xi_t \nabla h_t(E(y))\right](E(x) - E(y)) < 0 \tag{29}$$

holds, contradicting the first constraint of the vector Mond-Weir E-dual problem (MWD$_E$). This means that the proof of the Mond-Weir weak duality theorem between the E-vector optimization problems (VP$_E$) and (MWD$_E$) is completed under hypothesis B). □

Theorem 21. *(Mond-Weir weak E-duality between (MOP) and (MWD$_E$)). Let $E(x)$ and (y, λ, μ, ξ) be a feasible solutions of the problems (MOP) and (MWD$_E$), respectively. Further, assume that all hypotheses of Theorem 20 are fulfilled. Then, Mond-Weir weak E-duality between (MOP) and (MWD$_E$) holds, that is,*

$$(f \circ E)(x) \nless (f \circ E)(y).$$

Proof. Let $E(x)$ and (y, λ, μ, ξ) be any feasible solutions of the problems (MOP) and (MWD$_E$), respectively. Then, by Lemma 12. it follows that x is any feasible solution of (VP$_E$). Since all hypotheses of Theorem 20 are fulfilled, the Mond-Weir weak E-duality theorem between the problems (MOP) and (MWD$_E$) follows directly from Theorem 20. □

If some stronger E-convexity hypotheses are imposed on the functions constituting the considered E-differentiable multiobjective programming problem, then the stronger result is true.

Theorem 22. *(Mond-Weir weak duality between (VP$_E$) and (MWD$_E$)). Let x and (y, λ, μ, ξ) be any feasible solutions of the problems (VP$_E$) and (MWD$_E$), respectively. Further, assume that at least one of the following hypotheses is fulfilled:*

- *A) each objective function f_i, $i \in I$, is strictly E-convex at y on $\Omega_E \cup Y_E$, each constraint function g_j, $j \in J$, is E-convex at y on $\Omega_E \cup Y_E$, the functions h_t, $t \in T^+(E(y))$ and the functions $-h_t$, $t \in T^-(E(y))$, are E-convex at y on $\Omega_E \cup Y_E$.*
- *B) $(f \circ E)(y)$ is strictly pseudo-E-convex at y on $\Omega_E \cup Y_E$, $\mu_j(g_j \circ E)(y)$ is quasi-E-convex at y on $\Omega_E \cup Y_E$, $\xi_t(h_t \circ E)(y)$ is quasi-E-convex at y on $\Omega_E \cup Y_E$.*

Then

$$(f \circ E)(x) \nleq (f \circ E)(y). \tag{30}$$

Theorem 23. *(Mond-Weir weak E-duality between (MOP) and (MWD$_E$)). Let $E(x)$ and (y, λ, μ, ξ) be any feasible solutions of the problems (MOP) and (MWD$_E$), respectively. Further, assume that all hypotheses of Theorem 22 are fulfilled. Then, weak E-duality between (MOP) and (VMD$_E$) holds, that is,*

$$(f \circ E)(x) \nleq (f \circ E)(y).$$

Theorem 24. *(Mond-Weir strong duality between (VP$_E$) and (MWD$_E$) and also Mond-Weir strong E-duality between (MOP) and (MWD$_E$)). Let $\bar{x} \in \Omega_E$ be a weak Pareto solution (a Pareto solution) of the E-vector optimization problem (VP$_E$) (and, thus, $E(\bar{x})$ be a weak E-Pareto solution (an E-Pareto solution) of the E-vector optimization problem (MOP)). Further, assume that the E-Abadie constraint qualification (ACQ$_E$) is satisfied at $\bar{x}$. Then, there exist $\overline{\lambda} \in R^p$, $\overline{\mu} \in R^m$, $\overline{\mu} \geqq 0$, $\overline{\xi} \in R^q$ such that $\left(\bar{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is feasible for the problem (MWD$_E$) and the objective functions of (VP$_E$) and (MWD$_E$) are equal at these points. If also all hypotheses of the Mond-Weir weak duality (Theorem 20 (Theorem 22)) are satisfied, then $\left(\bar{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a weak efficient (an efficient) solution of a maximum type in the problem (MWD$_E$).*

In other words, if $E(\bar{x}) \in \Omega$ is a (weak) E-Pareto solution of the multiobjective programming problem (MOP), then $\left(\bar{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a (weak) efficient solution of a maximum type in the dual problem (MWD$_E$) in the sense of Mond-Weir. This means that the Mond-Weir strong E-duality holds between the problems (MOP) and (MWD$_E$).

Proof. Since $\bar{x} \in \Omega_E$ is a (weak) Pareto solution of the problem (VP$_E$) and the E-Abadie constraint qualification (ACQ$_E$) is satisfied at $\bar{x}$, by Theorem 19, there exist $\overline{\lambda} \in R^p$, $\overline{\mu} \in R^m$, $\overline{\mu} \geqq 0$, $\overline{\xi} \in R^q$ such that $\left(\bar{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a feasible solution of the problem (MWD$_E$). This means that the objective functions of (VP$_E$) and (MWD$_E$) are equal. If we assume that all

hypotheses of the Mond-Weir weak duality (Theorem 20 (Theorem 22)) are fulfilled, $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ is a (weak) efficient solution of a maximum type in the dual problem (MWD$_E$) in the sense of Mond-Weir.
Moreover, we have, by Lemma 12, that $E(\overline{x}) \in \Omega$. Since $\overline{x} \in \Omega_E$ is a weak Pareto solution of the problem (VP$_E$), by Lemma 14, it follows that $E(\overline{x})$ is a weak E-Pareto solution in the problem (MOP). Then, by the Mond-Weir strong duality between (VP$_E$) and (MWD$_E$), we conclude that also the Mond-Weir strong E-duality holds between the problems (MOP) and (MWD$_E$). This means that if $E(\overline{x}) \in \Omega$ is a weak E-Pareto solution of the problem (MOP), there exist $\overline{\lambda} \in R^p$, $\overline{\mu} \in R^m$, $\overline{\mu} \geqq 0$, $\overline{\xi} \in R^q$ such that $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ is a weakly efficient solution of a maximum type in the Mond-Weir dual problem (MWD$_E$). □

Theorem 25. *(Mond-Weir converse duality between (VP$_E$) and (MWD$_E$)). Let $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ be a (weakly) efficient solution of a maximum type in Mond-Weir dual problem (MWD$_E$) such that $\overline{x} \in \Omega_E$. Moreover, assume that the objective functions f_i, $i \in I$, are (E-convex) strictly E-convex at $\overline{x}$ on $\Omega_E \cup Y_E$, the constraint functions g_j, $j \in J$, are E-convex at $\overline{x}$ on $\Omega_E \cup Y_E$, the functions h_t, $t \in T^+(E(\overline{x}))$ and the functions $-h_t$, $t \in T^-(E(\overline{x}))$, are E-convex at $\overline{x}$ on $\Omega_E \cup Y_E$. Then $\overline{x}$ is a (weak) Pareto solution of the problem (VP$_E$).*

Proof. Let $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ be a (weakly) efficient solution of a maximum type in Mond-Weir dual problem (MWD$_E$) such that $\overline{x} \in \Omega_E$. By means of contradiction, we suppose that there exists $\widetilde{x} \in \Omega_E$ such that the inequalty

$$(f \circ E)(\widetilde{x}) < (f \circ E)(\overline{x}) \tag{31}$$

holds. By the feasibility of $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ in the problem (MWD$_E$), the above inequality yields

$$\sum_{i=1}^{p}\overline{\lambda}_i\nabla f_i(E(\widetilde{x})) < \sum_{i=1}^{p}\overline{\lambda}_i\nabla f_i(E((\overline{x})). \tag{32}$$

Since the functions f_i, $i \in I$, g_j, $j \in J$, h_t, $t \in T^+(E(\overline{x}))$, $-h_t$, $t \in T^-(E(\overline{x}))$, are E-convex at $\overline{x}$ on $\Omega_E \cup Y_E$, by Proposition 5, the following inequalities

$$f_i(E(\widetilde{x})) - f_i(E(\overline{x})) \geqq \nabla f_i(E(\overline{x}))(E(\widetilde{x}) - E(\overline{x})),\ i \in I, \tag{33}$$

$$g_j(E(\widetilde{x})) - g_j(E(\overline{x})) \geqq \nabla g_j(E(\overline{x}))(E(\widetilde{x}) - E(\overline{x})),\ j \in J_E(\overline{x}), \tag{34}$$

$$h_t(E(\widetilde{x})) - h_t(E(\overline{x})) \geqq \nabla h_t(E(\overline{x}))(E(\widetilde{x}) - E(\overline{x})),\ t \in T^+(E(\overline{x})), \tag{35}$$

$$-h_t(E(\widetilde{x})) + h_t(E(\overline{x})) \geqq -\nabla h_t(E(\overline{x}))(E(\widetilde{x}) - E(\overline{x})),\ \ t \in T^-(E(\overline{x})) \tag{36}$$

hold, respectively. Multiplying inequalities (33)-(36) by corresponding Lagrange multipliers, respectively, we obtain that the inequality

$$\overline{\lambda}_i f_i(E(\widetilde{x})) - \overline{\lambda}_i f_i(E(\overline{x})) \geqq \overline{\lambda}_i\nabla f_i(E(\overline{x}))(E(\widetilde{x}) - E(\overline{x})),\ i \in I, \tag{37}$$

$$\overline{\mu}_j g_j(E(\widetilde{x})) - \overline{\mu}_j g_j(E(\overline{x})) \geqq \overline{\mu}_j\nabla g_j(E(\overline{x}))(E(\widetilde{x}) - E(\overline{x})),\ j \in J_E(\overline{x}), \tag{38}$$

$$\overline{\xi}_t h_t(E(\widetilde{x})) - \overline{\xi}_t h_t(E(\overline{x})) \geqq \overline{\xi}_t\nabla h_t(E(\overline{x}))(E(\widetilde{x}) - E(\overline{x})),\ t \in T^+(E(\overline{x})), \tag{39}$$

$$\overline{\xi}_t h_t(E(\widetilde{x})) - \overline{\xi}_t h_t(E(\overline{x})) \geqq \overline{\xi}_t\nabla h_t(E(\overline{x}))(E(\widetilde{x}) - E(\overline{x})),\ \ t \in T^-(E(\overline{x})) \tag{40}$$

hold, respectively. Then, adding both sides of (37)-(40), we get that the inequality

$$\textstyle\sum_{i=1}^{p}\overline{\lambda}_i f_i(E(\widetilde{x})) - \sum_{i=1}^{p}\overline{\lambda}_i f_i(E(\overline{x})) + \sum_{j=1}^{m}\overline{\mu}_j g_j(E(\widetilde{x})) -$$

$$\textstyle\sum_{j=1}^{m}\overline{\mu}_j g_j(E(\overline{x})) + \sum_{t=1}^{q}\overline{\xi}_t h_t(E(\widetilde{x})) - \sum_{t=1}^{q}\overline{\xi}_t h_t(E(\overline{x})) \geqq$$
$$\left[\textstyle\sum_{i=1}^{p}\overline{\lambda}_i\nabla f_i(E(\overline{x})) + \sum_{j=1}^{m}\overline{\mu}_j\nabla g_j(E(\overline{x})) + \sum_{t=1}^{q}\overline{\xi}_t\nabla h_t(E(\overline{x}))\right](E(\widetilde{x}) - E(\overline{x}))$$

holds. Thus, by $\widetilde{x} \in \Omega_E$, $\overline{x} \in \Omega_E$ and (32), the following inequality

$$\left[\textstyle\sum_{i=1}^{p}\overline{\lambda}_i\nabla f_i(E(\overline{x})) + \sum_{j=1}^{m}\overline{\mu}_j\nabla g_j(E(\overline{x})) + \sum_{t=1}^{q}\overline{\xi}_t\nabla h_t(E(\overline{x}))\right](E(\widetilde{x}) - E(\overline{x})) < 0 \tag{41}$$

holds, contradicting the feasibility of $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ in (MWD$_E$). This means that the proof of the converse duality theorem between the E-vector optimization problems (VP$_E$) and (MWD$_E$) is completed. □

Theorem 26. *(Mond-Weir converse E-duality between (MOP) and (MWD$_E$)). Let $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ be a (weakly) efficient solution of a maximum type in Mond-Weir dual problem (MWD$_E$). Further, assume that all hypotheses of Theorem 25 are fulfilled. Then $E(\overline{x}) \in \Omega$ is a (weak) E-Pareto solution of the problem (MOP).*

Proof. The proof of this theorem follows directly from Lemma 14 and Theorem 25. □

Theorem 27. *(Mond-Weir restricted converse duality between (VP$_E$) and (MWD$_E$)). Let $\bar{x}$ be feasible of the considered E-differentiable multiobjective programming problem (VP$_E$) and $\left(\bar{y},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ be feasible of its vector Mond-Weir dual problem (MWD$_E$). Moreover, assume that the functions f_i, $i \in I$, are strictly E-convex at $\bar{y}$ on $\Omega_E \cup Y_E$, the constraint functions g_j, $j \in J$, are E-convex at $\bar{y}$ on $\Omega_E \cup Y_E$, the functions h_t, $t \in T^+(E(\bar{y}))$ and functions $-h_t$, $t \in T^-(E(\bar{y}))$, are E-convex at $\bar{y}$ on $\Omega_E \cup Y_E$ such that $f(E(\bar{x})) = f(E(\bar{y}))$. Then $\bar{x}$ is a (weak) Pareto solution of the problem (VP$_E$) and $\left(\bar{y},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ is a (weakly) efficient point of a maximum type for the problem (MWD$_E$).*

Proof. By means of contradiction, suppose that $\bar{x}$ is not a weak Pareto solution of the problem (VP$_E$). This means, by Definition 10, that there exists $\tilde{x} \in \Omega_E$ such that

$$f(E(\tilde{x})) < f(E(\bar{x})). \tag{42}$$

By assumption, $f(E(\bar{x})) = f(E(\bar{y}))$. Hence, (42) yields

$$f(E(\tilde{x})) < f(E(\bar{y})). \tag{43}$$

By assumption, $\left(\bar{y},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ is a feasible solution for (MWD$_E$). Then, it follows that $\overline{\lambda} \geq 0$. Hence, the above inequality yields

$$\sum_{i=1}^{p} \overline{\lambda}_i f_i(E(\tilde{x})) < \sum_{i=1}^{p} \overline{\lambda}_i f_i(E(\bar{y})). \tag{44}$$

By assumption, the functions f_i, $i \in I$, g_j, $j \in J$, h_t, $t \in T^+(E(\bar{y}))$, $-h_t$, $t \in T^-(E(\bar{y}))$, are E-convex at $\bar{y}$ on $\Omega_E \cup Y_E$. Then, by Proposition 5, the following inequalities

$$f_i(E(z)) - f_i(E(\bar{y})) \geqq \nabla f_i(E(\bar{y}))(E(z) - E(\bar{y})),\ \ i \in I, \tag{45}$$

$$g_j(E(z)) - g_j(E(\bar{y})) \geqq \nabla g_j(E(\bar{y}))(E(z) - E(\bar{y})),\ \ j \in J(E(\bar{y})), \tag{46}$$

$$h_t(E(z)) - h_t(E(\bar{y})) \geqq \nabla h_t(E(\bar{y}))(E(z) - E(\bar{y})),\ \ t \in T^+(E(\bar{y})), \tag{47}$$

$$-h_t(E(z)) + h_t(E(\bar{y})) \geqq -\nabla h_t(E(\bar{y}))(E(z) - E(\bar{y})),\ \ \ t \in T^-(E(\bar{y})) \tag{48}$$

hold for $z \in \Omega_E \cup Y_E$. Thus, they are also fulfilled for $z = \tilde{x} \in \Omega_E$. Hence, (45)-(48) yield, respectively,

$$f_i(E(\tilde{x})) - f_i(E(\bar{y})) \geqq \nabla f_i(E(\bar{y}))(E(\tilde{x}) - E(\bar{y})),\ \ i \in I, \tag{49}$$

$$g_j(E(\tilde{x})) - g_j(E(\bar{y})) \geqq \nabla g_j(E(\bar{y}))(E(\tilde{x}) - E(\bar{y})),\ \ j \in J(E(\bar{y})), \tag{50}$$

$$h_t(E(\tilde{x})) - h_t(E(\bar{y})) \geqq \nabla h_t(E(\bar{y}))(E(\tilde{x}) - E(\bar{y})),\ \ t \in T^+(E(\bar{y})), \tag{51}$$

$$-h_t(E(\tilde{x})) + h_t(E(\bar{y})) \geqq -\nabla h_t(E(\bar{y}))(E(\tilde{x}) - E(\bar{y})),\ \ t \in T^-(E(\bar{y})). \tag{52}$$

By the feasibility of $\left(\bar{y},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ in (MWD$_E$), it follows that

$$\overline{\lambda}_i f_i(E(\tilde{x})) - \overline{\lambda}_i f_i(E(\bar{y})) \geqq \overline{\lambda}_i \nabla f_i(E(\bar{y}))(E(\tilde{x}) - E(\bar{y})),\ \ i \in I, \tag{53}$$

$$\overline{\mu}_j g_j(E(\tilde{x})) - \overline{\mu}_j g_j(E(\bar{y})) \geqq \overline{\mu}_j \nabla g_j(E(\bar{y}))(E(\tilde{x}) - E(\bar{y})),\ \ j \in J(E(\bar{y})), \tag{54}$$

$$\overline{\xi}_t h_t(E(\tilde{x})) - \overline{\xi}_t h_t(E(\bar{y})) \geqq \overline{\xi}_t \nabla h_t(E(\bar{y}))(E(\tilde{x}) - E(\bar{y})),\ \ t \in T^+(E(\bar{y})), \tag{55}$$

$$\overline{\xi}_t h_t(E(\tilde{x})) - \overline{\xi}_t h_t(E(\bar{y})) \geqq \overline{\xi}_t \nabla h_t(E(\bar{y}))(E(\tilde{x}) - E(\bar{y})),\ \ t \in T^-(E(\bar{y})). \tag{56}$$

Adding both sides of (53)-(56), we obtain

$$\begin{array}{c} \sum_{i=1}^{p} \overline{\lambda}_i f_i(E(\tilde{x})) - \sum_{i=1}^{p} \overline{\lambda}_i f_i(E(\bar{y})) + \sum_{j=1}^{m} \overline{\mu}_j g_j(E(\tilde{x})) \\ - \sum_{j=1}^{m} \overline{\mu}_j g_j(E(\bar{y})) + \sum_{t=1}^{q} \overline{\xi}_t h_t(E(\tilde{x})) - \sum_{t=1}^{q} \overline{\xi}_t h_t(E(\bar{y})) \geqq \\ \left[\sum_{i=1}^{p} \overline{\lambda}_i \nabla f_i(E(\bar{y})) + \sum_{j=1}^{m} \mu_j \nabla g_j(E(\bar{y})) + \sum_{t=1}^{q} \xi_t \nabla h_t(E(\bar{y}))\right](E(\tilde{x}) - E(\bar{y})) \end{array} \tag{57}$$

By (57), the first constraint of (MWD$_E$) and $\tilde{x} \in \Omega_E$ imply that the following inequality

$$\sum_{i=1}^{p} f_i(E(\tilde{x})) \geqq \sum_{i=1}^{p} f_i(E(\bar{y})) \tag{58}$$

holds, contradicting (44). Then, $\bar{x} = \bar{y}$ and this means, by Mond-Weir weak duality (Theorem 20), that $\bar{x}$ is a weak Pareto solution of the problem (VP$_E$) and $\left(\bar{y},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ is a weakly efficient solution of a maximum type for the problem (MWD$_E$). Thus, the proof of this theorem is completed. □

Theorem 28. *(Mond-Weir restricted converse E-duality between (MOP) and (MWD$_E$)). Let $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ be a feasible solution of the problem (MWD$_E$). Further, assume that there exist $E(\overline{x}) \in \Omega$ such that $\overline{x} = \overline{y}$. If all hypotheses of Theorem 36 are fulfilled, then $E(\overline{x})$ is an (weak) E-Pareto solution of the problem (MOP) and $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is an (weak) efficient solution of maximum type for the problem (MWD$_E$).*

Proof. The proof of this theorem follows directly from Lemma 14 and Theorem 36. □

4. Wolfe E-duality

In this section, a vector E-dual problem in the sense of Wolfe is considered for the class of E-convex vector optimization problems with inequality and equality constraints.
Let $E : R^n \to R^n$ be a given one-to-one and onto operator. Consider the following vector dual problem in the sense of Wolfe related to the vector optimization problem (VP$_E$):

$$\text{maximize } f(E(y)) + \left[\sum_{j=1}^{m} \mu_j g_j(E(y)) + \sum_{t=1}^{q} \xi_t h_t(E(y))\right] e$$

$$\text{s.t. } \sum_{i=1}^{p} \lambda_i \nabla (f_i \circ E)(y) + \sum_{j=1}^{m} \mu_j \nabla (g_j \circ E)(y) + \sum_{t=1}^{q} \xi_t \nabla (h_t \circ E)(y) = 0, \qquad (\text{WD}_E)$$

$$y \in R^n,\ \lambda \in R^p, \lambda \geq 0,\ \lambda e = 1,\ e = (1,1,...,1)^T \in R^p,$$
$$\mu \in R^m, \mu \geqq 0, \xi \in R^q,$$

where all functions are defined in the similar way as for the considered vector optimization problem (MOP) and $e = (1,...,1) \in R^p$. Further, let

$$\Gamma_E = \left\{ (y, \lambda, \mu, \xi) \in R^n \times R^p \times R^m \times R^q : \sum_{i=1}^{p} \lambda_i \nabla (f_i \circ E)(y) + \sum_{j=1}^{m} \mu_j \nabla (g_j \circ E)(y) \right.$$
$$\left. + \sum_{t=1}^{q} \xi_t \nabla (h_t \circ E)(y) = 0,\ \lambda \geq 0,\ \lambda e = 1,\ \mu \geqq 0 \right\}.$$

be the set of all feasible solutions of the problem (WD$_E$). Further, $Y_E = \{y \in R^n : (y, \lambda, \mu, \xi) \in \Gamma_E\}$. We call the vector dual problem (WD$_E$) Wolfe vector E-dual problem or vector E-dual problem in the sense of Wolfe.
Now, under (generalized) E-convexity hypotheses, we prove duality results between the E-vector problems (VP$_E$) and (WD$_E$) and, thus, E-duality results between the problems (MOP) and (WD$_E$).

Theorem 29. *(Wolfe weak duality between (VP$_E$) and (WD$_E$)). Let x and (y, λ, μ, ξ) be any feasible solutions of the problems (VP$_E$) and (WD$_E$), respectively. Further, assume that at least one of the following hypotheses is fulfilled:*

A) each objective function f_i, $i \in I$, is E-convex at y on $\Omega_E \cup Y_E$, each constraint function g_j, $j \in J$, is E-convex at y on $\Omega_E \cup Y_E$, the functions h_t, $t \in T^+(E(y))$ and the functions $-h_t$, $t \in T^-(E(y))$, are E-convex at y on $\Omega_E \cup Y_E$.

B) $f(E(y)) + \left[\sum_{j=1}^{m} \mu_j g_j(E(y)) + \sum_{t=1}^{q} \xi_t h_t(E(y))\right] e$ is pseudo-E-convex at y on $\Omega_E \cup Y_E$, $\mu_j (g_j \circ E)(y)$ is quasi-E-convex at y on $\Omega_E \cup Y_E$, $\xi_t (h_t \circ E)(y)$ is quasi-E-convex at y on $\Omega_E \cup Y_E$.

Then

$$f(E(x)) \nless f(E(y)) + \left[\sum_{j=1}^{m} \mu_j g_j(E(y)) + \sum_{t=1}^{q} \xi_t h_t(E(y))\right] e. \qquad (59)$$

Proof. Let x and (y, λ, μ, ξ) be any feasible solutions of the problems (VP$_E$) and (WD$_E$), respectively. The proof of this theorem under hypothesis A) (see [1]).
The proof of this theorem under hypothesis B). We proceed by contradiction. Suppose, contrary to the result, that

$$f(E(x)) < f(E(y)) + \left[\sum_{j=1}^{m} \mu_j g_j(E(y)) + \sum_{t=1}^{q} \xi_t h_t(E(y))\right], i \in I \qquad (60)$$

holds. Since the function $(f \circ E)(\cdot) + \left[\mu_j (g_j \circ E)(\cdot) + \xi_t (h_t \circ E)(\cdot)\right] e$ is pseudo-E-convex at y on $\Omega_E \cup Y_E$, by Definition 6, the inequality

$$\sum_{i=1}^{p} \lambda_i \nabla (f_i \circ E)(y) + \sum_{j=1}^{m} \mu_j \nabla (g_j \circ E)(y) + \sum_{t=1}^{q} \xi_t \nabla (h_t \circ E)(y) < 0 \qquad (61)$$

holds. From $x \in \Omega_E$ and $(y, \lambda, \mu, \xi) \in \Gamma_E$, the relations

$$\sum_{j=1}^{m} \mu_j (g_j \circ E)(x) \leqq \sum_{j=1}^{m} \mu_j (g_j \circ E)(y), \tag{62}$$

$$\sum_{t=1}^{q} \xi_t (h_t \circ E)(x) = \sum_{t=1}^{q} \xi_t (h_t \circ E)(y) \tag{63}$$

hold, respectively. Since $\mu_j (g_j \circ E)(y)$ and $\xi_t (h_t \circ E)(y)$ are quasi-E-convex at y on $\Omega_E \cup Y_E$, Definition 7 implies that the inequalities

$$\sum_{j=1}^{m} \mu_j \nabla (g_j \circ E)(y)(E(x) - E(y)) \leqq 0, \tag{64}$$

$$\sum_{t=1}^{q} \xi_t \nabla (h_t \circ E)(y)(E(x) - E(y)) \leqq 0 \tag{65}$$

hold, respectively. Combining (61), (64) and (65), it follows that the inequality

$$\left[\sum_{i=1}^{p} \lambda_i \nabla (f_i \circ E)(y) + \sum_{j=1}^{m} \mu_i \nabla (g_j \circ E)(y) + \sum_{t=1}^{q} \xi_i \nabla (h_t \circ E)(y)\right](E(x) - E(y)) < 0 \tag{66}$$

holds, contradicting the first constraint of the vector Wolfe E-dual problem (WD$_E$). This means that the proof of the Wolfe weak duality theorem between the E-vector optimization problems (VP$_E$) and (WD$_E$) is completed under hypothesis B). □

Theorem 30. *(Wolfe weak E-duality between (MOP) and (WD$_E$)). Let $E(x)$ and (y, λ, μ, ξ) be any feasible solutions of the problems (MOP) and (WD$_E$), respectively. Further, assume that all hypotheses of Theorem 29 are fulfilled. Then, Wolfe weak E-duality between (MOP) and (WD$_E$) holds, that is,*

$$f(E(x)) \not< f(E(y)) + \left[\sum_{j=1}^{m} \mu_j g_j(E(y)) + \sum_{t=1}^{q} \xi_t h_t(E(y))\right] e.$$

Proof. Let $E(x)$ and (y, λ, μ, ξ) be any feasible solutions of the problems (MOP) and (WD$_E$), respectively. Then, by Lemma 12. it follows that x is any feasible solution of (VP$_E$). Since all hypotheses of Theorem 29 are fulfilled, the Wolfe weak E-duality theorem between the problems (MOP) and (WD$_E$) follows directly from Theorem 29. □
If some stronger E-convexity hypotheses are imposed on the functions constituting the considered E-differentiable multiobjective programming problem, then the stronger result is true.

Theorem 31. *(Wolfe weak duality between (VP$_E$) and (WD$_E$)). Let x and (y, λ, μ, ξ) be any feasible solutions of the problems (VP$_E$) and (WD$_E$), respectively. Further, assume that at least one of the following hypotheses is fulfilled:*

A) each objective function f_i, $i \in I$, is strictly E-convex at y on $\Omega_E \cup Y_E$, each constraint function g_j, $j \in J$, is E-convex at y on $\Omega_E \cup Y_E$, the functions h_t, $t \in T^+(E(y))$ and the functions $-h_t$, $t \in T^-(E(y))$, are E-convex at y on $\Omega_E \cup Y_E$.

B) $f(E(y)) + \left[\sum_{j=1}^{m} \mu_j g_j(E(y)) + \sum_{t=1}^{q} \xi_t h_t(E(y))\right] e$ is strictly pseudo-E-convex at y on $\Omega_E \cup Y_E$, $\mu_j (g_j \circ E)(y)$ is quasi-E-convex at y on $\Omega_E \cup Y_E$, $\xi_t (h_t \circ E)(y)$ is quasi-E-convex at y on $\Omega_E \cup Y_E$.

Then

$$(f \circ E)(x) \nleq f(E(y)) + \left[\sum_{j=1}^{m} \mu_j g_j(E(y)) + \sum_{t=1}^{q} \xi_t h_t(E(y))\right] e. \tag{67}$$

Theorem 32. *(Wolfe weak E-duality between (MOP) and (WD$_E$)). Let $E(x)$ and (y, λ, μ, ξ) be any feasible solutions of the problems (MOP) and (WD$_E$), respectively. Further, assume that all hypotheses of Theorem 31 are fulfilled. Then, Wolfe weak E-duality between (MOP) and (WD$_E$) holds, that is,*

$$(f \circ E)(x) \nleq f(E(y)) + \left[\sum_{j=1}^{m} \mu_j g_j(E(y)) + \sum_{t=1}^{q} \xi_t h_t(E(y))\right] e.$$

Theorem 33. *(Wolfe strong duality between (VP$_E$) and (WD$_E$) and also Wolfe strong E-duality between (MOP) and (WD$_E$)). Let $\overline{x} \in \Omega_E$ be a (weak) Pareto solution of the E-vector optimization problem (MOP) and the E-Abadie constraint qualification (ACQ$_E$) be satisfied at $\overline{x}$.Then there exist $\overline{\lambda} \in R^p$, $\overline{\mu} \in R^m$, $\overline{\xi} \in R^q$ such that $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi})$ is feasible for the problem (WD$_E$) and the objective functions of (VP$_E$) and (WD$_E$) are equal at these points. If also all hypotheses of the*

weak duality theorem (Theorem 29 or Theorem 31) are satisfied, then $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi})$ *is a (weak) efficient solution of maximum type for the problem* (WD_E).
In other words, if $E(\overline{x}) \in \Omega$ *is a (weak) E-Pareto solution of the multiobjective programming problem (MOP), then* $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ *is a (weak) efficient solution of a maximum type in the dual problem* (WD_E) *in the sense of Wolfe. This means that the strong Wolfe E-duality holds between the problems (MOP) and* (WD_E).

Proof. Since $\overline{x} \in \Omega_E$ is a (weak) Pareto solution of the problem (VP_E) and the E-Abadie constraint qualification (ACQ_E) is satisfied at $\overline{x}$, by Theorem 19, there exist $\overline{\lambda} \in R^p$, $\overline{\lambda} \neq 0$, $\overline{\mu} \in R^m$, $\overline{\mu} \geqq 0$, $\overline{\xi} \in R^q$, $\overline{\xi} \geqq 0$ such that $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a feasible solution of the problem (WD_E). This means that the objective functions of (VP_E) and (WD_E) are equal. If we assume that all hypotheses of Wolfe weak duality (Theorem 29 or Theorem 31) are fulfilled, $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is an (weakly) efficient solution of a maximum type in the dual problem (WD_E) in the sense of Wolfe.
Moreover, we have by Lemma 12, that $E(\overline{x}) \in \Omega$. Since $\overline{x} \in \Omega_E$ is a (weak) Pareto solution of the problem (VP_E), by Lemma 14, it follows that $E(\overline{x})$ is a weak E-Pareto solution in the problem (MOP). Then, by the strong duality between (VP_E) and (WD_E), we conclude that also the Wolfe strong E-duality holds between the problems (MOP) and (WD_E). This means that if $E(\overline{x}) \in \Omega$ is a (weak) E-Pareto solution of the problem (MOP), there exist $\overline{\lambda} \in R^p$, $\overline{\mu} \in R^m$, $\overline{\mu} \geqq 0$, $\overline{\xi} \in R^q$ such that $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is an (weakly) efficient solution of a maximum type in the Wolfe vector dual problem (WD_E). □

Theorem 34. *(Wolfe converse duality between* (VP_E) *and* (WD_E)*). Let* $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ *be a (weakly) efficient solution of a maximum type in the vector E-Wolfe dual problem* (WD_E) *such that* $\overline{x} \in \Omega_E$*. Moreover, assume that* $f(E(\overline{x})) + \left[\sum_{j=1}^m \overline{\mu}_j g_j (E(\overline{x})) + \sum_{t=1}^q \overline{\xi}_t h_t(E(\overline{x}))\right] e$ *is pseudo-E-convex at* $\overline{x}$ *on* $\Omega_E \cup Y_E$*,* $\overline{\mu}_j (g_j \circ E)(\overline{x})$ *is quasi-E-convex at* $\overline{x}$ *on* $\Omega_E \cup Y_E$*,* $\overline{\xi}_t (h_t \circ E)(\overline{x})$ *is quasi-E-convex at* $\overline{x}$ *on* $\Omega_E \cup Y_E$*. Then* $\overline{x}$ *is a (weak) Pareto solution of the problem* (VP_E)*.*

Proof. Let $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ be a weakly efficient solution of a maximum type in Wolfe E-dual problem (WD_E) such that $\overline{x} \in \Omega_E$. By means of contradiction, we suppose that there exists $\widetilde{x} \in \Omega_E$ such that

$$(f \circ E)(\widetilde{x}) < (f \circ E)(\overline{x}) \tag{68}$$

holds. By the feasibility of $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ in the problem (WD_E). Hence, by the E-Karush-Kuhn-Tucker necessary optimality conditions, we get

$$(f \circ E)(\widetilde{x}) + \left[\sum_{j=1}^m \mu_j (g_j \circ E)(\widetilde{x}) + \sum_{t=1}^q \xi_t (h_t \circ E)(\widetilde{x})\right] e <$$
$$(f \circ E)(\overline{x}) + \left[\sum_{j=1}^m \mu_j (g_j \circ E)(\overline{x}) + \sum_{t=1}^q \xi_t (h_t \circ E)(\overline{x})\right] e. \tag{69}$$

Since $\overline{\lambda}_i \in R^p, i \in I$, (69) yields

$$\sum_{i=1}^p \overline{\lambda}_i (f_i \circ E)(\widetilde{x}) + \left[\sum_{j=1}^m \mu_j (g_j \circ E)(\widetilde{x}) + \sum_{t=1}^q \xi_t (h_t \circ E)(\widetilde{x})\right] \sum_{i=1}^p \overline{\lambda}_i <$$
$$\sum_{i=1}^p \overline{\lambda}_i (f_i \circ E)(\overline{x}) + \left[\sum_{j=1}^m \mu_j (g_j \circ E)(\overline{x}) + \sum_{t=1}^q \xi_t (h_t \circ E)(\overline{x})\right] \sum_{i=1}^p \overline{\lambda}_i. \tag{70}$$

From the feasibility of $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ in the problem (WD_E), we have $\sum_{i=1}^p \overline{\lambda}_i = 1$. Then, the inequality above implies

$$\sum_{i=1}^p \overline{\lambda}_i (f_i \circ E)(\widetilde{x}) + \left[\sum_{j=1}^m \mu_j (g_j \circ E)(\widetilde{x}) + \sum_{t=1}^q \xi_t (h_t \circ E)(\widetilde{x})\right] <$$
$$\sum_{i=1}^p \overline{\lambda}_i (f_i \circ E)(\overline{x}) + \left[\sum_{j=1}^m \mu_j (g_j \circ E)(\overline{x}) + \sum_{t=1}^q \xi_t (h_t \circ E)(\overline{x})\right]. \tag{71}$$

Since the function $(f \circ E)(\cdot) + \left[\overline{\mu}_j (g_j \circ E)(\cdot) + \overline{\xi}_t (h_t \circ E)(\cdot)\right] e$ is pseudo-E-convex at $\overline{x}$ on $\Omega_E \cup Y_E$, by Definition 6, the inequality

$$\sum_{i=1}^p \overline{\lambda}_i \nabla (f_i \circ E)(\overline{x}) + \sum_{j=1}^m \overline{\mu}_j \nabla (g_j \circ E)(\overline{x}) + \sum_{t=1}^q \overline{\xi}_t \nabla (h_t \circ E)(\overline{x}) < 0 \tag{72}$$

holds. From $\widetilde{x} \in \Omega_E$ and $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Gamma_E$, the relations

$$\sum_{j=1}^{m} \overline{\mu}_j (g_j \circ E)(\widetilde{x}) \leqq \sum_{j=1}^{m} \overline{\mu}_j (g_j \circ E)(\overline{x}), \tag{73}$$

$$\sum_{t=1}^{q} \overline{\xi}_t (h_t \circ E)(\widetilde{x}) = \sum_{t=1}^{q} \overline{\xi}_t (h_t \circ E)(\overline{x}) \tag{74}$$

hold, respectively. Since $\overline{\mu}_j (g_j \circ E)(\overline{x})$ and $\overline{\xi}_t (h_t \circ E)(\overline{x})$ are quasi-E-convex at $\overline{x}$ on $\Omega_E \cup Y_E$, Definition 7 implies that the inequalities

$$\sum_{j=1}^{m} \overline{\mu}_j \nabla (g_j \circ E)(\overline{x}) (E(\widetilde{x}) - E(\overline{x})) \leqq 0, \tag{75}$$

$$\sum_{t=1}^{q} \overline{\xi}_t \nabla (h_t \circ E)(\overline{x}) (E(\widetilde{x}) - E(\overline{x})) \leqq 0 \tag{76}$$

hold, respectively. Combining (72), (75) and (76), it follows that the inequality

$$\left[\sum_{i=1}^{p} \overline{\lambda}_i \nabla f_i(E(\overline{x})) + \sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(E(\overline{x})) + \sum_{t=1}^{q} \overline{\xi}_t \nabla h_t(E(\overline{x}))\right] (E(\widetilde{x}) - E(\overline{x})) < 0, \tag{77}$$

which contradicting the feasibility of $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ in (WD$_E$). This means that the proof of the converse duality theorem between the E-vector optimization problems (VP$_E$) and (WD$_E$) is completed. □

Theorem 35. *(Wolfe converse E-duality between (MOP) and (WD$_E$)). Let $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ be a (weakly) efficient solution of a maximum type in Wolfe vector dual problem (WD$_E$). Further, assume that all hypotheses of Theorem 34 are fulfilled. Then $E(\overline{x}) \in \Omega$ is a (weak) E-Pareto solution of the problem (MOP).*

Proof. The proof of this theorem follows directly from Lemma 14 and Theorem 34. □

Theorem 36. *(Wolfe restricted converse duality between (VP$_E$) and (WD$_E$)). Let $\overline{x}$ and $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ be feasible solutions for the problems (VP$_E$) and (WD$_E$), respectively, such that*

$$(f \circ E)(\overline{x}) < (f \circ E)(\overline{y}) + \left[\sum_{j=1}^{m} \overline{\mu}_j (g_j \circ E)(\overline{y}) + \sum_{t=1}^{q} \overline{\xi}_t (h_t \circ E)(\overline{y})\right] e. \quad (\leq) \tag{78}$$

Moreover, assume that the objective functions f_i, $i \in I$, are quasi-E-convex at $\overline{y}$ on $\Omega_E \cup Y_E$, the constraint functions g_j, $j \in J$, are quasi-E-convex at $\overline{y}$ on $\Omega_E \cup Y_E$, the functions h_t, $t \in T^+(E(\overline{y}))$ and functions $-h_t$, $t \in T^-(E(\overline{y}))$, are quasi-E-convex at $\overline{y}$ on $\Omega_E \cup Y_E$. Then $\overline{x} = \overline{y}$, that is, $\overline{x}$ is a (weak) Pareto solution of the problem (VP$_E$) and $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a (weakly) efficient point of maximum type for the problem (WD$_E$).

Proof. Note that, by (78), it follows that

$$(f_i \circ E)(\overline{x}) < (f_i \circ E)(\overline{y}) + \sum_{j=1}^{m} \overline{\mu}_j (g_j \circ E)(\overline{y}) + \sum_{t=1}^{q} \overline{\xi}_t (h_t \circ E)(\overline{y}), \, i \in I. \tag{79}$$

Multiplying each inequality (79) by $\overline{\lambda}_i$, $i \in I$, and then adding both sides of the resulting inequalities, we get

$$\sum_{i=1}^{p} \overline{\lambda}_i (f \circ E)(\overline{x}) < \sum_{i=1}^{p} \overline{\lambda}_i (f \circ E)(\overline{y}) + \left[\sum_{j=1}^{m} \overline{\mu}_j (g_j \circ E)(\overline{y}) + \sum_{t=1}^{q} \overline{\xi}_t (h_t \circ E)(\overline{y})\right] \sum_{i=1}^{p} \lambda_i. \tag{80}$$

Since $\sum_{i=1}^{p} \lambda_i = 1$, (80) implies

$$\sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(\overline{x}) < \sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(\overline{y}) + \sum_{j=1}^{m} \overline{\mu}_j (g_j \circ E)(\overline{y}) + \sum_{t=1}^{q} \overline{\xi}_t (h_t \circ E)(\overline{y}). \tag{81}$$

Now, we proceed by contradiction. Suppose, contrary to the result, that $\overline{x} \neq \overline{y}$. From $\overline{x} \in \Omega_E$ and $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right) \in \Gamma_E$, the relations

$$\sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(\overline{y}) \leqq \sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(\overline{x}), \tag{82}$$

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$$\sum_{j=1}^{m}\overline{\mu}_j\left(g_j\circ E\right)\left(\overline{y}\right)\leqq\sum_{j=1}^{m}\overline{\mu}_j\left(g_j\circ E\right)\left(\overline{x}\right),\tag{83}$$

$$\sum_{t=1}^{q}\overline{\xi}_t\left(h_t\circ E\right)\left(\overline{y}\right)=\sum_{t=1}^{q}\overline{\xi}_t\left(h_t\circ E\right)\left(\overline{x}\right)\tag{84}$$

hold, respectively. By assumption, the functions f_i, $i\in I$, g_j, $j\in J(E(\overline{y}))$, h_t, $t\in T^{+}\left(E\left(\overline{y}\right)\right)$, and $-h_t$, $t\in T^{-}\left(E\left(\overline{y}\right)\right)$ are quasi-E-convex at $\overline{y}$ on $\Omega_E\cup Y$. Then, by Definition 7 implies that the inequalities

$$\sum_{i=1}^{p}\overline{\lambda}_i\nabla\left(f_i\circ E\right)\left(\overline{y}\right)\left(E\left(\overline{x}\right)-E\left(\overline{y}\right)\right)\leqq 0,\tag{85}$$

$$\sum_{j=1}^{m}\overline{\mu}_j\nabla\left(g_j\circ E\right)\left(\overline{y}\right)\left(E\left(\overline{x}\right)-E\left(\overline{y}\right)\right)\leqq 0,\tag{86}$$

$$\sum_{t=1}^{q}\overline{\xi}_t\nabla\left(h_t\circ E\right)\left(\overline{y}\right)\left(E\left(\overline{x}\right)-E\left(\overline{y}\right)\right)\leqq 0\tag{87}$$

hold, respectively. Combining (82)-(84), it follows that the inequality

$$\sum_{i=1}^{p}\overline{\lambda}_i\left(f_i\circ E\right)\left(\overline{x}\right)+\sum_{j=1}^{m}\overline{\mu}_j\left(g_j\circ E\right)\left(\overline{x}\right)+\sum_{t=1}^{q}\overline{\xi}_t\left(h_t\circ E\right)\left(\overline{x}\right)$$

$$\geqq\sum_{i=1}^{p}\overline{\lambda}_i\left(f_i\circ E\right)\left(\overline{y}\right)+\sum_{j=1}^{m}\overline{\mu}_j\left(g_j\circ E\right)\left(\overline{y}\right)+\sum_{t=1}^{q}\overline{\xi}_t\left(h_t\circ E\right)\left(\overline{y}\right).$$

Hence, by $\overline{x}\in\Omega_E$, we get that the following inequality

$$\sum_{i=1}^{p}\overline{\lambda}_i\left(f_i\circ E\right)\left(\overline{x}\right)\geqq\sum_{i=1}^{p}\overline{\lambda}_i\left(f_i\circ E\right)\left(\overline{y}\right)+\sum_{j=1}^{m}\overline{\mu}_j\left(g_j\circ E\right)\left(\overline{y}\right)+\sum_{t=1}^{q}\overline{\xi}_t\left(h_t\circ E\right)\left(\overline{y}\right).\tag{88}$$

holds, contradicting (81). Then, $\overline{x}=\overline{y}$ and this means, by weak duality (Theorem 29 or Theorem 31) that $\overline{x}$ is a (weak) Pareto solution of the problem (VP$_E$) and $\left(\overline{y},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ is a (weakly) efficient solution of maximum type for the problem (WD$_E$). Thus, the proof of this theorem is completed. □

Theorem 37. *(Wolfe restricted converse E-duality between (MOP) and (WD$_E$)). Let $\left(\overline{y},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ be a feasible solution of the problem (WD$_E$). Further, assume that there exist $E\left(\overline{x}\right)\in\Omega$ such that $\overline{x}=\overline{y}$. If all hypotheses of Theorem 36 are fulfilled, then $E\left(\overline{x}\right)$ is an E-Pareto solution of the problem (VP) and $\left(\overline{y},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ is a weakly efficient solution of maximum type for the problem (WD$_E$).*

Proof. The proof of this theorem follows directly from Lemma 14 and Theorem 36. □

5. Mixed E-duality

In this section, a vector mixed E-dual problem is defined for the considered E-differentiable multiobjective programming problem (MOP) with inequality and equality constraints.
Before we define the foregoing vector dual problem, we introduce some notations which will be helpful in presenting its formulation.
Let the index set J be partitioned into two disjoint subset J_1 and J_2 such that $J=J_1\cup J_2$ and the index set T be partitioned into two disjoint subset T_1 and T_2 such that $T=T_1\cup T_2$. Let J_1 be an index set such that $J_1\subseteq J$ and $J_2=J\backslash J_1$ and, moreover, $|J_1|$ and $|J_2|$ denote the cardinality of the index sets J_1 and J_2, respectively. Further, let T_1 be an index set such that $T_1\subseteq T$ and $T_2=T\backslash T_1$ and, moreover, $|T_1|$ and $|T_2|$ denote the cardinality of the index sets T_1 and T_2, respectively. Let us denote the set

$$\Omega^2=\left\{x\in R^n:g_j(x)\leqq 0,\ \ j\in J_2,h_t(x)=0,\ \ t\in T_2\right\}.$$

Now, for the define E-differentiable multiobjective programming problem (MOP), we introduce the definition of the mixed scalar Lagrange function $L:\Omega\times R^p_+\times R^{|J_1|}_+\times R^{|T_1|}\to R$ as follows

$$L(x,\lambda,\mu_{J_1},\xi_{T_1}):=\sum_{i=1}^{p}\lambda_if_i(x)+\sum_{j\in J_1}\mu_jg_j(x)+\sum_{t\in T_1}\xi_th_t(x).\tag{89}$$

Further, let $E : R^n \to R^n$ be a given one-to-one and onto operator. Further, let us define the following set

$$\Omega_E^2 := \left\{ x \in R^n : (g_j \circ E)(x) \leqq 0,\ \ j \in J_2,\ (h_t \circ E)(x) = 0,\ \ t \in T_2 \right\}.$$

Now, we define the following vector mixed E-dual problem for the considered E-differentiable multiobjective programming problem (MOP):

$$\begin{aligned} &\text{maximize } (f \circ E)(y) + \left[\mu_{J_1}(g_{J_1} \circ E)(y) + \xi_{T_1}(h_{T_1} \circ E)(y) \right] e \\ &\text{s.t. } \lambda \nabla (f \circ E)(y) + \mu \nabla (g \circ E)(y) + \xi \nabla (h \circ E)(y) = 0, \\ &\quad \mu_{J_2}(g_{J_2} \circ E)(y) \geqq 0, \\ &\quad \xi_{T_2} \nabla (h_{T_2} \circ E)(y) = 0, \\ &\quad \lambda \in R^p, \lambda \geq 0, \lambda e = 1,\ \mu \in R^m, \mu \geqq 0, \xi \in R^q, \end{aligned} \qquad (\mathrm{VMD}_E)$$

where all functions are defined in the similar way as for the considered vector optimization problem (MOP). Further, let Γ_E denote the set of all feasible solutions of (VMD_E), that is,

$$\Gamma_E = \Bigg\{ (y, \lambda, \mu, \xi) \in R^n \times R^p \times R^m \times R^q :$$

$$\lambda \nabla (f \circ E)(y) + \mu \nabla (g \circ E)(y) + \xi \nabla (h \circ E)(y) = 0,$$

$$\sum_{j \in J_2} \mu_j (g_j \circ E)(y) \geqq 0, \sum_{t \in T_2} \xi_t \nabla (h_t \circ E)(y) = 0, \lambda \geq 0,\ \lambda e = 1, \mu \geqq 0 \Bigg\}.$$

Further, $Y_E = \{ y \in R^n : (y, \lambda, \mu, \xi) \in \Gamma_E \}$. We call ($\mathrm{VMD}_E$) the vector mixed E-dual problem for the E-differentiable multiobjective optimization problem (MOP).

Note that if set $J_1 = \varnothing$ and $T_1 = \varnothing$ in (VMD_E), then we get a vector Mond-Weir E-dual problem for (MOP) and, moreover, if we set $J_2 = \varnothing$ and $T_2 = \varnothing$ in (VMD_E), then we obtain a vector Wolfe E-dual problem for (MOP).

Now, we shall prove several mixed duality results between E-vector optimization problems (VP_E) and (VMD_E) under (generalized) E-convexity assumptions. Then, we use these duality results in proving several mixed E-duality results between vector optimization problems (MOP) and (VMD_E).

Theorem 38. *(Mixed weak duality between (VP_E) and (VMD_E)). Let x and (y, λ, μ, ξ) be any feasible solutions of the problems (VP_E) and (VMD_E), respectively. Further, assume that at least one of the following hypotheses is fulfilled:*

A) each objective function f_i, $i \in I$, is E-convex at y on $\Omega_E \cup Y_E$, each constraint function g_j, $j \in J$, is an E-convex function at y on $\Omega_E \cup Y_E$, the functions h_t, $t \in T^+(E(y))$ and the functions $-h_t$, $t \in T^-(E(y))$, are E-convex at y on $\Omega_E \cup Y_E$.

B) $(f \circ E)(y) + \left[\mu_{J_1}(g_{J_1} \circ E)(y) + \xi_{T_1}(h_{T_1} \circ E)(y) \right] e$ is pseudo-E-convex at y on $\Omega_E \cup Y_E$, $\mu_{J_2}(g_{J_2} \circ E)(y)$ is quasi-E-convex at y on $\Omega_E \cup Y_E$, $\xi_{T_2} \nabla (h_{T_2} \circ E)(y)$ is quasi-E-convex at y on $\Omega_E \cup Y_E$.

Then

$$(f \circ E)(x) \not< (f \circ E)(y) + \left[\mu_{J_1}(g_{J_1} \circ E)(y) + \xi_{T_1}(h_{T_1} \circ E)(y) \right] e. \tag{90}$$

Proof. Let x and (y, λ, μ, ξ) be any feasible solutions of the problems (VP_E) and (VMD_E), respectively.
The proof of this theorem under hypothesis A). By means of contradiction, suppose that

$$(f \circ E)(x) < (f \circ E)(y) + \left[\mu_{J_1}(g_{J_1} \circ E)(y) + \xi_{T_1}(h_{T_1} \circ E)(y) \right] e.$$

Thus,

$$(f_i \circ E)(x) < (f_i \circ E)(y) + \left[\sum_{j \in J_1} \mu_j (g_j \circ E)(y) + \sum_{t \in T_1} \xi_t (h_t \circ E)(y) \right], i \in I. \tag{91}$$

Multiplying each inequality (91) by λ_i and then adding both sides of the resulting inequalities, we get

$$\sum_{i=1}^{p} \lambda_i (f_i \circ E)(x) < \sum_{i=1}^{p} \lambda_i (f_i \circ E)(y) + \left[\sum_{j \in J_1} \mu_j (g_j \circ E)(y) + \sum_{t \in T_1} \xi_t (h_t \circ E)(y) \right] \sum_{i=1}^{p} \lambda_i.$$

Since $\sum_{i=1}^{p} \lambda_i = 1$, the following inequality

$$\sum_{i=1}^{p} \lambda_i (f_i \circ E)(x) < \sum_{i=1}^{p} \lambda_i (f_i \circ E)(y) + \sum_{j \in J_1} \mu_j (g_j \circ E)(y) + \sum_{t \in T_1} \xi_t (h_t \circ E)(y)$$

holds. By $x \in \Omega_E$ and $(y, \lambda, \mu, \xi) \in \Gamma_E$, we have

$$\sum_{i=1}^{p} \lambda_i (f_i \circ E)(x) + \sum_{j \in J_1} \mu_j (g_j \circ E)(x) + \sum_{t \in T_1} \xi_t (h_t \circ E)(x) < \tag{92}$$

$$\sum_{i=1}^{p} \lambda_i (f_i \circ E)(y) + \sum_{j \in J_1} \mu_j (g_j \circ E)(y) + \sum_{t \in T_1} \xi_t (h_t \circ E)(y),$$

$$\sum_{j \in J_2} \mu_j (g_j \circ E)(x) \leqq \sum_{j \in J_2} \mu_j (g_j \circ E)(y), \tag{93}$$

$$\sum_{t \in T_2} \xi_t (h_t \circ E)(x) = \sum_{t \in T_2} \xi_t (h_t \circ E)(y). \tag{94}$$

Combining (92), (93) and (94), we get

$$\sum_{i=1}^{p} \lambda_i (f_i \circ E)(x) + \sum_{j=1}^{m} \mu_j (g_j \circ E)(x) + \sum_{t=1}^{q} \xi_t (h_t \circ E)(x) < \tag{95}$$

$$\sum_{i=1}^{p} \lambda_i (f_i \circ E)(y) + \sum_{j=1}^{m} \mu_j (g_j \circ E)(y) + \sum_{t=1}^{q} \xi_t (h_t \circ E)(y).$$

Since the functions f_i, $i \in I$, g_j, $j \in J$, h_t, $t \in T^+$, $-h_t$, $t \in T^-$, are E-convex on $\Omega_E \cup Y_E$, by Proposition 5, the following inequalities

$$(f_i \circ E)(x) - (f_i \circ E)(y) \geqq \nabla (f_i \circ E)(y)(E(x) - E(y)), \; i \in I, \tag{96}$$

$$(g_j \circ E)(x) - (g_j \circ E)(y) \geqq \nabla (g_j \circ E)(y)(E(x) - E(y)), \; j \in J_E(y), \tag{97}$$

$$(h_t \circ E)(x) - (h_t \circ E)(y) \geqq \nabla (h_t \circ E)(y)(E(x) - E(y)), \; t \in T^+(E(y)), \tag{98}$$

$$-(h_t \circ E)(x) + (h_t \circ E)(y) \geqq -\nabla (h_t \circ E)(y)(E(x) - E(y)), \;\; t \in T^-(E(y)) \tag{99}$$

hold, respectively. Multiplying inequalities (93)-(96) by corresponding Lagrange multipliers, respectively, and then adding both sides of resulting inequalities, we obtain

$$\sum_{i=1}^{p} \lambda_i (f_i \circ E)(x) - \sum_{i=1}^{p} \lambda_i (f_i \circ E)(y) + \sum_{j=1}^{m} \mu_i (g_j \circ E)(x) - \sum_{j=1}^{m} \mu_i (g_j \circ E)(y)$$
$$+ \sum_{t=1}^{q} \xi_i (h_t \circ E)(x) - \sum_{t=1}^{q} \xi_i (h_t \circ E)(y) \geqq \Big[\sum_{i=1}^{p} \lambda_i \nabla (f_i \circ E)(y) +$$
$$\sum_{j=1}^{m} \mu_i \nabla (g_j \circ E)(y) + \sum_{t=1}^{q} \xi_i \nabla (h_t \circ E)(y) \Big] (E(x) - E(y)).$$

Hence, by (95), the inequality above implies that the following inequality

$$\left[\sum_{i=1}^{p} \lambda_i \nabla (f_i \circ E)(y) + \sum_{j=1}^{m} \mu_i \nabla (g_j \circ E)(y) + \sum_{t=1}^{q} \xi_i \nabla (h_t \circ E)(y) \right] (E(x) - E(y)) < 0 \tag{100}$$

holds, contradicts the first constraint of the vector mixed E-dual problem (VMD$_E$). This means that the proof of the mixed weak duality theorem between the E-vector optimization problems (VP$_E$) and (VMD$_E$) is completed under hypothesis A). The proof of this theorem under hypothesis B). We proceed by contradiction. Suppose, contrary to the result, that (91) holds. Since the function $(f \circ E)(\cdot) + \Big[\mu_{J_1} (g_{J_1} \circ E)(\cdot) + \xi_{T_1} (h_{T_1} \circ E)(\cdot) \Big] e$ is pseudo-E-convex at y on $\Omega_E \cup Y_E$, by Definition 6, the inequality

$$\left[\sum_{i=1}^{p} \lambda_i \nabla (f_i \circ E)(y) + \sum_{j \in J_1} \mu_j \nabla (g_j \circ E)(y) + \sum_{t \in T_1} \xi_t \nabla (h_t \circ E)(y) \right] (E(x) - E(y)) < 0 \tag{101}$$

holds. By $x \in \Omega_E$ and $(y, \lambda, \mu, \xi) \in \Gamma_E$, it follows that the relations (93) and (94) are fulfilled. Since $\mu_{J_2} (g_{J_2} \circ E)(y)$ and $\xi_{T_2} (h_{T_2} \circ E)(y)$ are quasi-E-convex at y on $\Omega_E \cup Y_E$, by the foregoing above relations, Definition 7 implies that the inequalities

$$\sum_{j \in J_2} \mu_j \nabla (g_j \circ E)(y)(E(x) - E(y)) \leqq 0, \tag{102}$$

$$\sum_{t\in T_2}\xi_t\nabla\left(h_t\circ E\right)(y)\left(E\left(x\right)-E\left(y\right)\right)\leqq 0. \tag{103}$$

hold, respectively. Combining (101), (102) and (103), it follows that the inequality (100) is fulfilled, contradicting the first constraint of the vector mixed E-dual problem (VMD$_E$). This means that the proof of the mixed weak duality theorem between the E-vector optimization problems (VP$_E$) and (VMD$_E$) is completed under hypothesis B). □

Theorem 39. *(Mixed weak E-duality between (MOP) and (VMD$_E$)). Let $E\left(x\right)$ and $\left(y,\lambda,\mu,\xi\right)$ be any feasible solutions of the problems (MOP) and (VMD$_E$), respectively. Further, assume that all hypotheses of Theorem 38 are fulfilled. Then, mixed weak E-duality between (MOP) and (VMD$_E$) holds, that is,*

$$(f\circ E)(x)\nless(f\circ E)(y)+\left[\mu_{J_1}\left(g_{J_1}\circ E\right)(y)+\xi_{T_1}\left(h_{T_1}\circ E\right)(y)\right]e.$$

Proof. Let $E\left(x\right)$ and $\left(y,\lambda,\mu,\xi\right)$ be any feasible solutions of the problems (MOP) and (VMD$_E$), respectively. Then, by Lemma 12. it follows that x is any feasible solution of (VP$_E$). Since all hypotheses of Theorem 38 are fulfilled, the mixed weak E-duality theorem between the problems (MOP) and (VMD$_E$) follows directly form Theorem 38. □

If some stronger E-convexity hypotheses are imposed on the functions constituting the considered E-differentiable multiobjective programming problem, then the stronger result is true.

Theorem 40. *(Mixed weak duality between (VP$_E$) and (VMD$_E$)). Let x and $\left(y,\lambda,\mu,\xi\right)$ be any feasible solutions of the problems (VP$_E$) and (VMD$_E$), respectively. Further, assume that at least one of the following hypotheses is fulfilled:*

A) each objective function f_i, $i\in I$, is strictly E-convex at y on $\Omega_E\cup Y_E$, each constraint function g_j, $j\in J$, is E-convex at y on $\Omega_E\cup Y_E$, the functions h_t, $t\in T^+\left(E\left(y\right)\right)$ and the functions $-h_t$, $t\in T^-\left(E\left(y\right)\right)$, are E-convex at y on $\Omega_E\cup Y_E$.

B) $(f\circ E)(y)+\left[\mu_{J_1}\left(g_{J_1}\circ E\right)(y)+\xi_{T_1}\left(h_{T_1}\circ E\right)(y)\right]e$ is strictly pseudo-E-convex at y on $\Omega_E\cup Y_E$, $\mu_{J_2}\left(g_{J_2}\circ E\right)(y)$ is quasi-E-convex at y on $\Omega_E\cup Y_E$, $\xi_{T_2}\left(h_{T_2}\circ E\right)(y)$ is quasi-E-convex at y on $\Omega_E\cup Y_E$.

Then

$$(f\circ E)(x)\nleq(f\circ E)(y)+\left[\mu_{J_1}\left(g_{J_1}\circ E\right)(y)+\xi_{T_1}\left(h_{T_1}\circ E\right)(y)\right]e. \tag{104}$$

Theorem 41. *(Mixed weak E-duality between (MOP) and (VMD$_E$)). Let $E\left(x\right)$ and $\left(y,\lambda,\mu,\xi\right)$ be any feasible solutions of the problems (MOP) and (VMD$_E$), respectively. Further, assume that all hypotheses of Theorem 40 are fulfilled. Then, mixed weak E-duality between (MOP) and (VMD$_E$) holds, that is,*

$$(f\circ E)(x)\nleq(f\circ E)(y)+\left[\mu_{J_1}\left(g_{J_1}\circ E\right)(y)+\xi_{T_1}\left(h_{T_1}\circ E\right)(y)\right]e.$$

Theorem 42. *(Mixed strong duality between (VP$_E$) and (VMD$_E$) and also strong E-duality between (MOP) and (VMD$_E$)). Let $\overline{x}\in\Omega_E$ be a weak Pareto solution (a Pareto solution) of the E-vector optimization problem (VP$_E$) and the E-Abadie constraint qualification (ACQ$_E$) be satisfied at $\overline{x}$. Then there exist $\overline{\lambda}\in R^p$, $\overline{\lambda}\neq 0$, $\overline{\mu}\in R^m$, $\overline{\mu}\geqq 0$, $\overline{\xi}\in R^q$, $\overline{\xi}\geqq 0$ such that $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ is feasible for the problem (VMD$_E$) and the objective functions of (VP$_E$) and (VMD$_E$) are equal at these points. If also all hypotheses of the mixed weak duality (Theorem 38) Theorem 40 are satisfied, then $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ is a weak efficient (an efficient) solution of a maximum type in problem (VMD$_E$).*

In other words, if $E(\overline{x})\in\Omega$ is a (weak) E-Pareto solution of the multiobjective programming problem (MOP), then $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ is a (weak) efficient solution of a maximum type in the vector mixed dual problem (VMD$_E$). This means that the mixed strong E-duality holds between the problems (MOP) and (VMD$_E$).

Proof. Since $\overline{x}\in\Omega_E$ is a (weak) Pareto solution of the problem (VP$_E$) and the E-Abadie constraint qualification (ACQ$_E$) is satisfied at $\overline{x}$, by Theorem 19, there exist $\overline{\lambda}\in R^p$, $\overline{\lambda}\neq 0$, $\overline{\mu}\in R^m$, $\overline{\mu}\geqq 0$, $\overline{\xi}\in R^q$, $\overline{\xi}\geqq 0$ such that $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ is a feasible solution of the problem (VMD$_E$). This means that the objective functions of (VP$_E$) and (VMD$_E$) are equal. If we assume that all hypotheses of mixed weak duality (Theorem 38 (Theorem 40)) are fulfilled, $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ is a (weak) efficient solution of a maximum type in the dual problem (VMD$_E$) in the sense of mixed.

Moreover, we have by Lemma 12, that $E\left(\overline{x}\right)\in\Omega$. Since $\overline{x}\in\Omega_E$ is a weak Pareto solution of the problem (VP$_E$), by Lemma 14, it follows that $E\left(\overline{x}\right)$ is a weak E-Pareto solution in the problem (MOP). Then, by the strong duality between (VP$_E$) and (VMD$_E$), we conclude that also the mixed strong E-duality holds between the problems (MOP) and (VMD$_E$). This means that if $E\left(\overline{x}\right)\in\Omega$ is a weak E-Pareto solution of the problem (MOP), there exist $\overline{\lambda}\in R^p$, $\overline{\lambda}\neq 0$, $\overline{\mu}\in R^m$, $\overline{\mu}\geqq 0$, $\overline{\xi}\in R^q$, $\overline{\xi}\geqq 0$ such that $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ is a weakly efficient solution of a maximum type in the mixed dual problem (VMD$_E$). □

Theorem 43. *(Mixed converse duality between (VP_E) and (VMD_E)). Let $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ be a (weakly) efficient solution of a maximum type in mixed dual problem (VMD_E) such that $\overline{x}\in\Omega_E$. Moreover, that the objective functions f_i, $i\in I$, are E-convex at $\overline{x}$ on $\Omega_E\cup Y_E$, the constraint functions g_j, $j\in J$, are E-convex at $\overline{x}$ on $\Omega_E\cup Y_E$, the functions h_t, $t\in T^+\left(E\left(\overline{x}\right)\right)$ and the functions $-h_t$, $t\in T^-\left(E\left(\overline{x}\right)\right)$, are E-convex at $\overline{x}$ on $\Omega_E\cup Y_E$. Then $\overline{x}$ is a (weak) Pareto solution of the problem (VP_E).*

Proof. Proof of this theorem follows directly from Theorem 38. □

Theorem 44. *(Mixed converse E-duality between (MOP) and (VMD_E)). Let $\left(\overline{x},\overline{\lambda},\overline{\mu},\overline{\xi}\right)$ be a (weakly) efficient solution of a maximum type in mixed dual problem (VMD_E). Further, assume that all hypotheses of Theorem 43 are fulfilled. Then $E(\overline{x})\in\Omega$ is a (weak) E-Pareto solution of the problem (MOP).*

Proof. The proof of this theorem follows directly from Lemma 14 and Theorem 43. □

6. Conclusion

In this paper, the class of E-differentiable vector optimization problems with both inequality and equality constraints has been considered in which the involved functions are E-convex and/or generalized E-convex. The so-called vector Mond-Weir E-dual problem, vector Wolfe E-dual problem and vector mixed E-dual problem have been formulated for such E-differentiable multiobjective programming problems. Then, various E-duality theorems between the considered E-differentiable vector optimization problem and its Mond-Weir, Wolfe and mixed vector dual problems have been proved under (generalized) E-convexity hypotheses.
However, some interesting topics for further research remain. It would be of interest to investigate whether it is possible to prove similar results for other classes of E-differentiable vector optimization problems. We shall investigate these questions in subsequent papers.

References

[1] Antczak, T., Abdulaleem, N., 2019. *E-optimality conditions and Wolfe E-duality for E-differentiable vector optimization problems with inequality and equality constraints*, Journal of Nonlinear Sciences and Applications, 12, 745–764.

[2] Antczak, T., 2016. *Optimality conditions in quasidifferentiable vector optimization*, J. Optim. Theory Appl., **171** 708-725.

[3] Ahmad, I., 2005. *Sufficiency and duality in multiobjective programming with generalized (F,ρ)-convexity*, J. Appl. Anal., **11** , 19-33.

[4] Bector, C.R., Chandra, S., 2001. *On mixed duality in mathematical programming*, J. Math. Anal. Appl., 259(1), 346-356.

[5] Brumelle, S., 1981. *Duality for multiple objective convex programs*, Math. Oper. Res. 6, 159-172.

[6] Chuong, T.D., 2014. *Optimality conditions and duality in nonsmooth multiobjective optimization problems*, Ann. Oper. Res., **217**, 117-136.

[7] Jahn, J., 1983. *Duality in vector optimization*, Math. Program. 25, 343-353.

[8] Kaul, R.N., Suneja, S.K., Srivastava, M.K., 1994. *Optimality criteria and duality in multiple-objective optimization involving generalized invexity*, J. Optim. Theory Appl., **80**, 465-482.

[9] Nakayama, H., 1989. *Duality theory in vector optimization*: an overview, decision making with multiple objectives. In: Haimes, Y.Y., Chankong, V. (eds.) Lecture Notes in Economics and Mathematical Systems 337, pp. 86-93. Springer-Verlag.

[10] Mond, B., Weir, T., 1981. *Generalized concavity and duality, Generalized concavity in optimization and economics*, Academic Press, New York, 263-279.

[11] Megahed, A.A., Gomma, H.G., Youness, E.A., El-Banna, A.H., 2013. *Optimality conditions of E-convex programming for an E-differentiable function*, J. Inequal. Appl., **2013**, 11 pages.

[12] Syau, Y.R., Lee, E.S., 2005. *Some properties of E-convex functions*, Appl. Math. Lett., **18**, 1074-1080.

[13] Soleimani-Damaneh, M., 2011. *E-convexity and its generalizations*, Int. J. Comput. Math., **88**, 3335–3349.

[14] Yang, X.M., 2001. *On E-convex sets, E-convex functions, and E-convex programming*, J. Optim. Theory Appl., **109**, 699–704.

[15] Youness, E.A., 1999. *E-convex sets, E-convex functions, and E-convex programming*, J. Optim. Theory Appl., **102**, 439–450.

[16] Youness, E.A., 2001. *Quasi and strictly quasi E-convex functions*, J. Stat. Manag. Syst., **4**, 201–210.