Republic of Iraq
Ministry of Higher Education
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Results Of The Factor Group CF(G,Z)/R(G)

A Thesis
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Requirements for the Degree of
Master of Science in Applied Mathematics

By

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الاهداء

الى :

- شمعتيّ التي أضاءت لي الدنيا ...

- صاحبيّ القلب الدافىء ...

والِدَيَّ ... حباً واحساناً

*　*　*

- ينبوع الحنان الذي لا ينضب وارض العطاء التي لا تجدب ...

- سندي وملجأي في الشدائد ...

أخي وأخواتي ... فخراً وامتناناً

*　*　*

- الذين تعلمت منهم في هذه الدنبا ...

أساتنتي ...اعتزازاً واحتراماً

*　*　*

- الذين يساندوا قلبي كلما أحس بالرهبة ...

أصدقائي ... شكراً وعرفاناً

اليهم جميعا أهدي هذا الجهد المتواضع

نيران

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NIRAN

SUPERVISOR'S CERTIFICATION

I certify that this thesis was prepared under my
supervision at the University of Technology
in partial fulfillment of the requirements
for the degree of Master of Science in
Applied Mathematic

Signature :
Name : Dr. Mohammed Serdar I.Kirdar
Date : / /2005

Linguistic Supervisor Certification

This is to certify that I have read the thesis titled

" RESULTS OF THE FACTOR GROUP cf(G,Z) / R(G) "

and corrected any grammatical mistake I found.

The thesis is therefore qualified for debate.

Signature:

Name: ***Ass. Prof. Eyad Shamseldeen***

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EXAMINATION COMMITTEE CERTIFICATION

We " the examination committee ", after reading this thesis entitled " RESULTS OF THE FACTOR GROUP cf(G,Z) / R(G) " and examining the student " NERAN SABAH JASIM " in its contents and in what is related to it, find it adequate as a thesis for the degree of " MASTER OF SCIENCE IN APPLIED MATHEMATICS".

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ABSTRACT

The set of all Z – valued class functions of a finite group G form an abelian group **cf(G,Z)** under point wise addition. Inside this group we have a subgroup of Z – valued generalized characters of G denoted by **R(G)**.

The problem of finding the cyclic decomposition of the factor group **K(G) = cf(G,Z) / R(G)** has been considered in this thesis for **G = the special linear finite group SL(2,*p*)** where *p* = 3, 5, 7, 11, 13, 17, and 19.

LIST OF NOTATIONS

C	Set of complex numbers.
Q	Set of rational numbers.
Z	Set of integers numbers.
GL(n,F)	Group of invertible n × n matrices over the field F.
C_n	Cyclic group of order n.
S_n	Symmetric group of n symbols.
Z_n	Cyclic group of order n.
Tr	Trace of the matrix.
χ	Character of the representation T.
$\langle \chi , \rho \rangle$	Inner product of the characters χ and ρ.
< X >	Cyclic group generated by X.
×	Direct product.
Ker χ	Kernel of χ.
$\lhd$	Normal subgroup.
$\chi\uparrow^{G}$	Induced character.
C_g	Conjugacy class of the element g.
≡ G	Character table of G.
≡*G	Rational valued character table of G.
det(≡*G)	Determinant of the matrix ≡*G.
K(G)	Factor group cf(G,Z) / R(G).
Γ	Galois group.
$\oplus$	Direct sum.
$\otimes$	Tensor product.
SL(n,F)	Group of invertible n × n matrices with determinant 1 over the field F.
SZ(n,F)	Center of SL(n,F).
$C_G(g)$	Centralizer of g.in G.
$N_G(x)$	Normalizer of x in G
(j,n)	Greatest common divisor.

$\|G\|$	Order of the group G.
$o(g)$	Order of the element g.
$\|$	Not divide.
$\sim$	Not equivalent.
$[G: H]$	Index of H over G.
F^*	F – {0}.
$\exp(G)$	Exponent of the group G.
$\Phi(n)$	Eular totient function to the number n.
$\mu(n)$	Möbius function to the number n.
$char(F)$	characteristic of the field F.

Introduction

INTRODUCTION

The importance of representation and character theory for the study of the groups stems on the one hand from the fact that should it be necessary to present a concrete description of a group , this can be achieved with a matrix representation. On the other hand, group theory benefits mainly from the use of representations and characters, when these approaches are employed as an additional means to analyse the structure of a group. Moreover representation and character theory provide varied applications, not only in other branches of mathematics but also in physics and chemistry.

In this thesis our focus will lie on the cyclic decomposition of K(SL(n,*p*)) the factor of Z – valued class functions of the considered group SL(2,*p*), *p* = 3, 5, 7, 11, 13, 17, and 19 module the group of the generalized characters of the same group. The problem of the determination of the cyclic decomposition of K(G) has been considered for many groups and been solved completely for G = S_n, D_n, and elementary abelian groups but for the special linear finite groups it seems to be untouched

<u>In chapter 1</u> The foundations of representation theory are described and illustrated by a number of definitions and basic examples **(section 1.2)**, the notion of a character of a finite group is deduced and considered in detail in **(section 1.3)**. Again great importance is attached to the description of examples to illustrate these concepts, we use that for constructing the ordinary character table of SL(n,p^k) where p is prime and $p \neq 2$ in chapter 2. In **(section 1.4)** we develop the theory necessary to understand the contents of a character table. In **(section 1.5)** we introduce some important definitions of the abelian group and describe the character table of this group. In **(section 1.6)** we describe an additional and very important method which constructs characters of a group from characters of an arbitrary subgroup, this method is used to find some of the irreducible characters of the special linear finite groups in chapter 2. In **(section 1.7)** we study the factor group K(G) also the section includes fundamental concepts and the order of this group.

chapter 2 Is devoted to the some members and an important class of a group the special linear finite groups. After describing important features of this group and investigating their conjugacy classes **(section 2.2)** we move on to constructing the ordinary character table of SL(n,p^k) where p is prime and $p \neq 2$. Great care is taken to illustrate each step and a number of additional calculations **(section 2.3)**.

In chapter 3 This chapter covers the study of the character table of the irreducible rational representations of SL(2,p^k), p is an odd prime, $k > 0$ and odd by using the character table and the Schur indices **(section 3.2)**, then we introduce the diagonalization of the matrix (≡*SL(2,p)) which gives us the cyclic decomposition of K(SL(2,p)), where p = 3, 5, 7, 11, 13, 17, and 19 **(section 3.3)**.

Chapter One

Preliminaries

PRELIMINARIES

1.1. INTRODUCTION:

In this chapter, some definitions and basic concepts of representation theory, character theory, the characters of finite abelian group, induced characters and factor group are introduced.

1.2. REPRESENTATION THEORY:

This section introduces some important definitions of the representation theory of finite groups.

DEFINITION (1.2.1):

The set of all n×n non-singular matrices over the field F this set forms a group under the operation of matrix multiplication. This group is called **the general linear group** of dimension **n** over the field F, denoted by **GL(n,F)**. See [1].

DEFINITION (1.2.2):

Let V be a vector space over the field F and let GL(V) denote the group of all linear isomorphisms of V onto itself.

A **representation** of a group G with representation space V is a homomorphism t: $g \to t(g)$ of G into GL(V). See [1].

DEFINITION (1.2.3):

A **matrix representation** of a group G is a homomorphism T: $g \to T(g)$ of G into GL(n,F), where **n** is called the **degree** of the matrix representation. See [1].

EXAMPLE (1.1):

Consider the symmetric group S_3 of order 6, define T: $S_3 \to GL(3,C)$,as follows:

$$T(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T(13) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$T(23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad T(123) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad T(132) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

It is easy to show that T is a matrix representation of $S_3$of degree 3.

DEFINITION (1.2.4):

Two representations t and t' with representation spaces V and M respectively are said to be **equivalent** if there exists an isomorphism f of V onto M such that for all $g \in G$, $t'(g)\, f = f\, t(g)$.

Similarly, two matrix representations T_1 and T_2 are **equivalent** if they have the same degree, say **n**, and if there exists a fixed invertible matrix A $\in$ GL(n,F) such that for all $g \in G$, $T_2(g) = A^{-1} T_1(g) A$. See [1].

EXAMPLE (1.2):

Consider two representation of degree 2 of the symmetric group

$$S_3 = < r, c: r^2 = c^3 = 1, rc = c^2r > = \{1, (12), (13), (23), (123), (132)\},$$

where r = (12), c = (123). Let $T_1: S_3 \to GL(2,C)$ and $T_2: S_3 \to GL(2,C)$ such that:

$$T_1((12)) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad T_1((123)) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

$$T_2((12)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad T_2((123)) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

T_1 and T_2 are homomorphism, Then T_1 and T_2 are matrix representations, also there exists an invertible matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in GL(2,C) such that

$$T_1(x) = A^{-1} T_2(x) A, \ \forall\ x \in S_3$$

Hence T_1 and T_2 are equivalent.

DEFINITION (1.2.5):

Let T be a representation of a group G such that $T(g) = 1, \forall\ g \in G$. Then t is called the **linear trivial representation**. See [1].

DEFINITION (1.2.6):

A matrix representation T of a group G is called **reducible representation** if it is equivalent to a matrix representation of the form

$$\begin{pmatrix} T_1(x) & * \\ 0 & T_2(x) \end{pmatrix}, \forall\, x \in \mathrm{G}$$

where T_1 and T_2 are representations of G, whose degree is less than the degree of T.

A reducible representation T is called **completely reducible** if it is equivalent to a matrix representation of the form

$$\begin{pmatrix} T_1(x) & 0 \\ 0 & T_2(x) \end{pmatrix}, \forall\, x \in \mathrm{G}$$

If T is not reducible then T is said to be an **irreducible representation**. See [11].

EXAMPLE (1.3):

In symmetric group S_3 define T: $S_3 \rightarrow GL(2,C)$ such that

$$T(x)=\begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } x \text{ is even permutation} \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } x \text{ is odd permutation} \end{cases}$$

Then T is a matrix representation on S_3.

Also there exists a non-singular matrix A = $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in GL(2,C) such that

$$A^{-1}\ T(x)\,A = \begin{pmatrix} T_1(x) & * \\ 0 & T_2(x) \end{pmatrix} \qquad \forall\ x \in S_3$$

Therefore T is reducible.

1.3. <u>CHARACTER THEORY</u>:

This section introduces some important definitions and basic concepts of the character theory of finite groups.

<u>DEFINITION (1.3.1)</u>:

Let T be a matrix representation of a finite group G over the field F.

The character χ of T is the mapping χ: G → F defined by

χ (*g*) = Tr (T(*g*)) ∀ *g* ∈ G, where Tr (T(*g*)) refers to the trace of the matrix T(*g*).

The characters of degree **1** are called **linear characters**. See [4].

<u>EXAMPLE (1.4)</u>:

In symmetric group $S_3 = \langle r, c \mid r^2 = c^3 = 1, rc = c^2 r \rangle$, define the representation T: $S_3 \to$ GL(2,C) such that:

$$r \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad c \to \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \qquad \text{where } \omega = e^{2\pi i/3}$$

The character χ of T is: χ(r) = 0, $\chi(c) = \omega + \omega^2 = -1$.

DEFINITION (1.3.2):

The function 1_G, with constant value 1 on G, is a linear character, it is called **the principle** or sometimes (**unit** or **trivial**) character of G.

The character afforded by irreducible representation is called **irreducible Character,** otherwise it is called **compound Character**. See [4].

EXAMPLE (1.5):

Linear characters are irreducible character.

PROPOSITION (1.3.3):

If χ is the character of a representation T of degree n, we have:

i. $\chi(1) = n$.

ii. $\chi(s^{-1}) = \overline{\chi}(s)$ for $s \in G$ (where the bar denotes the complex conjugate)

PROOF: See [5].

LEMMA (1.3.4):

Let χ_T be the character of a representation T of a group G of degree **n**, if T and T'' are representations of G, then $\chi_T \oplus \chi_{T''} = \chi_T + \chi_{T''}$.

PROOF: See [6].

The matrix of $T \oplus T''$ is $\begin{pmatrix} T & 0 \\ 0 & T'' \end{pmatrix}$. Hence $tr(T \oplus T'') = tr(T) + tr(T'')$.

<u>LEMMA (1.3.5)</u>:

Let χ is the character of a group G and suppose char(F) | |G|. Then there exist irreducible characters $\chi_1, \dots, \chi_k$ of G such that $\chi = \chi_1 + \dots + \chi_k$.

PROOF: See [6].

<u>DEFINITION (1.3.6)</u>:

A **class function** on a group G is a function f: G → C which is constant on conjugacy classes, that is $f(x^{-1}y\,x) = f(y)\quad \forall\, x, y \in G$.

If all values of **f** are in Z, then it is called **Z-valued class function**. See [4].

<u>LEMMA (1.3.7)</u>:

Characters of a group G are class function.

PROOF: See [4].

Let ρ be matrix representation and χ character of ρ

$$\chi(x^{-1}y\,x) = \mathrm{Tr}\,\rho(x^{-1}y\,x) = \mathrm{Tr}\,\rho(x^{-1}).\mathrm{Tr}\,\rho(y).\mathrm{Tr}\,\rho(x) = \mathrm{Tr}\,\rho(y) = \chi(y).$$

<u>DEFINITION (1.3.8)</u>:

Let χ and ψ be characters of the group G. The **inner product** is defined as

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}) \qquad \textbf{...(1.1)}$$

See [4].

<u>EXAMPLE (1.6)</u>:

Let G = C_3 = {1, *a*, a^2} and suppose that θ, ϕ are characters of a group G define from G into C as follows:

	1	***a***	a^2
ϕ	1	1	1
θ	2	i	-1

$$\langle\theta,\phi\rangle=\frac{1}{3}(2\cdot 1+i\cdot 1+(-1)\cdot 1)=\frac{1}{3}(1+i),$$

$$\langle\theta,\theta\rangle=\frac{1}{3}(2\cdot 2+i\cdot\bar{i}+(-1)\cdot(-1))=2$$

<u>THEOREM (1.3.9)</u>:

Let χ and ψ be characters of two non-isomorphic irreducible representation of a group G. Then we have

i. $\langle\chi,\psi\rangle=0$.

ii. $\langle\chi,\chi\rangle=1$.

PROOF: See [5].

<u>Corollary (1.3.10)</u>:

If χ_1, … , χ_k are irreducible characters of a group G, and χ is any character of G, then
$$\chi=\sum_{i=1}^{k}\langle\chi,\chi_i\rangle\chi_i$$

PROOF: See [6].

<u>Corollary (1.3.11)</u>:

If $\chi_1, \ldots, \chi_k$ are irreducible characters of a group G, and $\chi = \sum_{i=1}^{k} n_i \chi_i$, and $\psi = \sum_{i=1}^{k} m_i \chi_i$ are any two characters of G, then

$$\langle \chi, \psi \rangle = \sum_{i=1}^{k} n_i \, m_i$$

PROOF: See [6].

<u>THEOREM (1.3.12)</u>:

Let T and S be representations of G with characters χ and ψ. Then T and S are equivalent if and only if they have the same character.

PROOF: See [4].

<u>PROPOSITION (1.3.13)</u>:

Let χ be a character of G. Then χ is irreducible if and only if $\langle \chi, \chi \rangle = 1$.

PROOF: See [4].

<u>PROPOSITION (1.3.14)</u>:

If $\chi_1, \ldots, \chi_k$ are all irreducible characters of G, then $\sum_{i=1}^{k} \chi_i^{\,2}(1) = |G|$.

This is a convenient criterion for the irreducibility of a character.

PROOF: See [4].

THEOREM (1.3.15):

Sum and product of characters are character.

where if we suppose χ and ψ are characters of a group G, then

1. The sum of characters is defined by

$$(\chi + \psi)(g) = \chi(g) + \psi(g) \qquad \text{for } g \in G$$

2. The multiplication of characters is defined by

$$(\chi \cdot \psi)(g) = \chi(g) \cdot \psi(g) \qquad \text{for } g \in G$$

PROOF: See [4].

THEOREM (1.3.16):

The number **n** of distinct irreducible characters of G is equal to the number of its conjugacy classes.

PROOF: See [5].

PROPOSITION (1.3.17):

The degres $\mathbf{n_i}$ satisfy the relations

1. $\sum_{i=1}^{k} n_i^{\,2} = |G|$.

2. If $1 \neq g \in G$, we have $\sum_{I=1}^{K} n_i \, \chi_i(g) = 0$.

PROOF: See [5].

THEOREM (1.3.18):

Let $\chi_1, \dots, \chi_k$ be all the irreducible characters of a group G and let $g_1, \dots, g_k$ be the representation of the conjugacy classes $C_1, \dots, C_k$ of G. Then we have:

1. The row orthogonality relation:

$$\sum_{\alpha=1}^{k} \frac{\chi_i(g_\alpha)\overline{\chi_j(g_\alpha)}}{|C_G(g_\alpha)|} = \delta_{ij} \qquad \text{for all } i, j = 1, 2, \dots, k$$

2. The column orthogonality relation:

$$\sum_{i=1}^{k} \chi_i(g_\alpha)\overline{\chi_i(g_\beta)} = \delta_{\alpha\beta} \, |C_G(g_\alpha)| \qquad \text{for all } \alpha, \beta = 1, 2, \dots, k$$

where $C_G(g_\alpha)$ denote the centralizer of g_α in G , $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

PROOF: See [6].

EXAMPLE (1.7):

Consider the characters of a group S_3 as follows, where the conjugacy classes of this group are:

$$C_1 = \{1\},\ C_2 = \{(12), (13), (23)\},\ C_3 = \{(123), (132)\}$$

C_g	C_1	C_2	C_3
$\mid C_g \mid$	**1**	**3**	**2**
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$$\langle \chi_3, \chi_3 \rangle = \frac{1}{6}(1.2.2 + 3.0.0 + 2(-1)(-1)) = \frac{1}{6}(4+2) = 1$$

$$\langle \chi_3, \chi_2 \rangle = \frac{1}{6}(1.2.1 + 3.0.(-1) + 2(-1).1) = \frac{1}{6}(2-2) = 0$$

$$\sum_{t=1}^{3} \chi_t(C_2)\,\chi_t(C_3) = 1.1 + (-1).1 + 0.(-1) = 0$$

$$\sum_{t=1}^{3} \chi_t(C_2)\,\chi_t(C_2) = 1.1 + (-1).(-1) + 0.0 = 2$$

DEFINITION (1.3.19):

Let χ be the character of the representation T of degree **n**, define the **kernel of** χ to be ker χ = ker T. See [4].

LEMMA (1.3.20):

If χ be a character of G, then Ker χ = {g ∈ G: χ (g) = χ (1)}

PROOF: See [4].

Corollary (1.3.21):

If $H \lhd G$, then $H = \cap \{ \text{Ker}\, \chi_i : H \leq \text{Ker}\, \chi_i \}$.

PROOF: See [6].

1.4. THE CHARACTER TABLE :

The complete information about the characters of a group G is conveniently displayed in a character table, which lists the values of the k irreducible characters for all elements of G. Since the character is constant on each of the conjugacy classes C_α, $(1 \leq \alpha \leq k)$, thus it is sufficient to record the values

$$\chi_i (g_\alpha),\ i=1, 2, \ldots, k \text{ , if } g_\alpha \in C_\alpha$$

Table (1.1) presents a typical character table, the body of the table is a k by k square matrix whose rows correspond to the different characters while each column contains the values of all irreducible characters for a particular conjugacy classes, denoting the number of elements in C_α by h_α we have the class equation $h_1 + h_2 + \ldots + h_k = |G|$, and the degree of the k distinct representations of G over C by n_i, $i = 1, 2, \ldots, k$.

The size of the centralizer $C_G (C_\alpha) = |G| / h_\alpha = m_\alpha$, although they are not properly speaking, a part of the table. See [1].

C_α	C_1	C_2	...	C_α	...	C_k
$\|C_\alpha\|$	h_1	h_2	...	h_α	...	h_k
$\|C_G(C_\alpha)\|$	m_1	m_2	...	m_α	...	m_k
χ_1	1	1	...	1	...	1
χ_2	n_2	$\chi_2(g_2)$	...	$\chi_2(g_\alpha)$	...	$\chi_2(g_k)$
χ_3	n_3	$\chi_3(g_2)$	...	$\chi_3(g_\alpha)$	...	$\chi_3(g_k)$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
χ_k	n_k	$\chi_k(g_2)$	...	$\chi_k(g_\alpha)$	...	$\chi_k(g_k)$

TABLE (1.1)

EXAMPLE (1.9):

The group S_3 has three conjugacy classes, namely

$$C_1 = \{1\},\ C_2 = \{(12), (13), (23)\},\ C_3 = \{(123), (132)\}$$

Therefore S_3 has three irreducible representations, they are:

1. $\rho_1(g) = 1 \qquad \forall g \in S_3$

2. $\rho_2(g) = \begin{cases} 1 & \text{if } x \text{ is an even permutation} \\ -1 & \text{if } x \text{ is an odd permutation} \end{cases}$

3. $\rho_3(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho_3(12) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \rho_3(13) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$

$\rho_3(23) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \qquad \rho_3(123) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \qquad \rho_3(132) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$

Then the character table of S_3 is:

$\equiv S_3=$

C_α	C_1	C_2	C_3
$\|C_\alpha\|$	**1**	**3**	**2**
$\|C_G(C_\alpha)\|$	**6**	**2**	**3**
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

We check that

$$n_1^2 + n_2^2 + n_3^2 = 1^2 + 1^2 + 2^2 = 6$$

which confirms that we have indeed found all the irreducible characters of S_3.

1.5. CHARACTERS OF FINITE ABELIAN GROUP:

In this section we introduce some important definitions of the abelian group and describe the character table of this group.

DEFINITION (1.5.1):

A group G is called **abelian group** if every pair of elements commutes, that is, if $xy = yx$ for all x, y in G. See [10].

DEFINITION (1.5.2):

A group G is called **finite group** if G is a finite set. In this case the number of elements in G is called the **order of G** and is denoted by **|G|**. This amount to saying that each conjugace class of G consists of single element, also that each function on G is a class function. See [10].

THEOREM (1.5.3):

A finite abelian group G of order **n** has exactly **n** distinct characters.

PROOF: See [10].

1.5.4 The Character Table Of Finite Abelian Group :

For a finite abelian group G of order **n** a complete information about the irreducible characters of G is displayed in a table called **the character table** of G. We list the elements of G in the 1st row, we put

$$\chi_i(x^j) = \chi_i^j \ , 1\leq i \leq n \ , 1\leq j \leq n-1$$

≡ G =

C_α	**1**	**x**	x^2	. . .	x^{n-1}
$\lvert C_\alpha \rvert$	**1**	**1**	**1**	. . .	**1**
$\lvert C_G(C_\alpha) \rvert$	**n**	**n**	**n**	. . .	**n**
χ_1	1	1	1	…	1
χ_2	1	$(\chi_2)^1$	$(\chi_2)^2$	…	$(\chi_2)^{n-1}$
⋮	⋮	⋮	⋮	⋮	⋮
χ_n	1	$(\chi_n)^1$	$(\chi_n)^2$	…	$(\chi_n)^{n-1}$

TABLE (1.2)

If G = Z_n, the cyclic group of order **n**, and let ω = $e^{2\pi i/n}$ be a primitive n-th root of unity then the general formula of the character table of Z_n is:

$\equiv Z_n =$

C_α	1	z	z^2	. . .	z^{n-1}
$\lvert C_\alpha \rvert$	1	1	1	. . .	1
$\lvert C_G(C_\alpha) \rvert$	n	n	n	. . .	n
χ_1	1	1	1	…	1
χ_2	1	ω	ω^2	…	ω^{n-1}
χ_3	1	ω^2	ω^4	…	ω^{n-2}
⋮	⋮	⋮	⋮	⋮	⋮
χ_n	1	ω^{n-1}	ω^{n-2}	…	ω

TABLE (1.3)

See [4].

<u>EXAMPLE (1.10)</u>:

The group Z_5 consists the elements 1, z, z^2, z^3, z^4, ($z^5 = 1$).
Let ω = $e^{2\pi i/5}$. Then the character table of Z_5 is:

$\equiv Z_5 =$

C_α	1	z	z^2	z^3	z^4
$\lvert C_\alpha \rvert$	1	1	1	1	1
$\lvert C_G(C_\alpha) \rvert$	5	5	5	5	5
χ_1	1	1	1	1	1
χ_2	1	ω	ω^2	ω^3	ω^4
χ_3	1	ω^2	ω^4	ω	ω^3
χ_4	1	ω^3	ω	ω^4	ω^2
χ_5	1	ω^4	ω^3	ω^2	ω

For the general case of a finite abelian group G of order **n**, can be written G as a direct product of a cyclic subgroups ,say

$$G=\langle Z_1\rangle \times \langle Z_2\rangle\times \dots \times\langle Z_m\rangle$$

where Z_μ is of order n_μ and $n = n_1.n_2 \dots n_m.$

An arbitrary element x є G is then uniquely expressed as $x = z_1^{a_1} z_2^{a_2} \dots z_m^{a_m}$ where the exponents are subject to the conditions $0 \le a_\mu \le n_\mu$ $(\mu =1, 2, \dots, m)$ In order to construct the irreducible characters of G, we choose for each μ an n_μ-th root of unity

$\xi_\mu = e^{(2\pi i r_\mu / n_\mu)}$ where r_μ is any integer satisfying

$$0 \le r_\mu \le n_\mu, (\mu =1, 2, \dots, m) \qquad \textbf{...(1.2)}$$

Corresponding to each **m**-tuple $[r] = [r_1, r_2, \dots, r_m]$
We define the function

$$\chi_{[r]}(x) = e^{(2\pi i \sum_{\mu=1}^{m} a_\mu r_\mu / n_\mu)} \qquad \textbf{...(1.3)}$$

Then there are **nm**–tuples satisfying (1.2), since distinct **m**–tuples correspond to distinct functions, then all **n** irreducible characters of G are obtained by (1.3). See [4].

EXAMPLE (1.11):

The characters of $G = Z_2\times Z_3$ can be found by applying (1.3), $r = [r_1,r_2]$ and $| G | = 2 \times 3 = 6$.

Let $\omega = e^{2\pi i/3}$, then the character table of G is:

C_α	(1,1)	(x,1)	(1,y)	(x,y)	$(1,y^2)$	(x,y^2)
$\lvert C_\alpha \rvert$	1	1	1	1	1	1
$\lvert C_G(C_\alpha) \rvert$	6	6	6	6	6	6
$\chi_{[0,0]}$	1	1	1	1	1	1
$\chi_{[1,0]}$	1	-1	1	-1	1	-1
$\chi_{[0,1]}$	1	1	ω	ω	ω^2	ω^2
$\chi_{[1,1]}$	1	-1	ω	$-\omega$	ω^2	$-\omega^2$
$\chi_{[0,2]}$	1	1	ω^2	ω^2	ω	ω
$\chi_{[1,2]}$	1	-1	ω^2	$-\omega^2$	ω	$-\omega$

1.6. INDUCED CHARACTERS:

In this section we describe an additional and very important method which constructs characters of a group from characters of an arbitrary subgroup

DEFINITION (1.6.1):

Let H be a subgroup of a group G, and φ be a class function of H. Then $\varphi\uparrow^G$, the **induced class function on G**, is given by

$$\varphi\uparrow(g) = \frac{1}{|H|}\sum_{x\in G}\varphi^{\circ}(xgx^{-1})$$

where φ° is defined by :

$$\varphi^\circ(h) = \varphi(h) \qquad \text{if} \quad h \in H$$

and

$$\varphi^\circ(h) = 0 \qquad \text{if} \quad h \notin H$$

Clearly $\varphi\uparrow^{G}$ is a class function on G and $\varphi\uparrow^{G}(1) = [G:H]\,\varphi(1)$.

Then the character of the induced representation is called **induced character** and can be rewritten as:

$$\varphi\uparrow^{G}(C_\alpha) = \frac{|C_G(C_\alpha)|}{|C_H(C_\alpha)|}\sum_{g\to C_\alpha}\varphi(g) \qquad \textbf{...(1.4)}$$

where $\varphi\uparrow^{G}(C_\alpha) = 0$ if $C_\alpha \not\subset H$. See [4].

EXAMPLE (1.12):

The three conjugacy classes of the symmetric group S_3, are:

$$C_1 = \{1\},\ C_2 = \{(12), (13), (23)\},\ C_3 = \{(123), (132)\}$$

To calculate the induced characters of S_3 from the unit characters of the cyclic subgroups C_i, i = 1, 2, 3, by using the formula (1.4).

The partitions to the order of S_3 equal 3, they are 1^3, 12, 3, while the orders of the three classes of S_3 are 1, 3, 2, and the orders of the centralizer are 6, 2, 3 respectively. Thus:

1- (1^3): $1_{C1}\uparrow^{S_3} = [S_3: C_1]\,\chi_1(1) = 6\,(1) = 6$

$1_{12\ C1}\uparrow^{S_3} = 1_{3\ C1}\uparrow^{S_3} = 0.$ since $(12) \not\subset (1^3)$, and $(123) \not\subset 1^3$.

$$\Phi_1(x) = (\,6 \quad 0 \quad 0\,)$$

2- (12): $1_{C2}\uparrow^{S_3} = \frac{6}{2}\sum 1 = 3.$

$1_{12\ C2}\uparrow^{S_3} = \frac{2}{2}\sum 1 = 1$

$1_{3\ C2}\uparrow^{S_3} = 0.$ since $(123) \notin (12)$.

$\Phi_2(x) = (\,3 \quad 1 \quad 0\,)$

3- (3): $1_{C3}\uparrow^{S_3} = \frac{6}{3}\sum 1 = 2.$

$1_{12\ C3}\uparrow^{S_3} = 0.$ since $(12) \notin (3)$

$1_{3\ C3}\uparrow^{S_3} = \frac{3}{3}\sum 1 + 1 = 2$

$\Phi_3(x) = (\,2 \quad 0 \quad 2\,)$

We declare that in this table:

C_α	1^3	12	3
$\lvert C_\alpha\rvert$	1	3	2
$\lvert C_G(C_\alpha)\rvert$	6	2	3
Φ_1	6	0	0
Φ_2	3	1	0
Φ_3	2	0	2

Corollary (1.6.2):

Let H be a subgroup of G, and φ be a character of H. Then $\varphi\uparrow^G$ is a character of G.

PROOF: See [4].

1.7. THE FACTOR GROUP K(G):

In this section we will study the factor group K(G) of a group G, also this section includes fundamental concepts and the order of K(G).

DEFINITION (1.7.1):

A **rational valued** character θ of G is a character whose values are in Z, that is $\theta(x) \in Z$ for all $x \in G$. See [7].

DEFINITION (1.7.2):

Two elements of G are said to be **Q-conjugate** if the cyclic subgroups they generate are conjugate in G, this defines an equivalence relation on G, its classes are called the **Q-classes of G.** See [7].

Let G be a finite group and let $\chi_1, \chi_2, \ldots, \chi_k$ be its distinct irreducible characters, A class function on G is a character if and only if it is a linear combination of the χ_i's with non-negative integer coefficients. We will denote by **$R^+(G)$** the set of all these functions, the group generated by **R+(G)** is called the **group of the generalized characters of G** and denoted by **R(G)**. We have

$$R(G) = Z\chi_1 \oplus Z\chi_2 \oplus \ldots \oplus Z\chi_k$$

An element of R(G)is called a **virtual character**. Since the product of two characters is character, R(G) is a subring of the ring cf(G) of C-valued class functions on G.

Let **cf (G,Z)** be the **group of all Z-valued class functions of G** which are constant on Q-classes, and let $\overline{R}(G)$ be the **intersection of cf (G,Z) with R(G)** $\overline{R}(G)$ is **a ring of Z-valued generalized characters of G**.

Let ε_m be a complex primitive m-th root of unity. We know that the Galois group $Gal(F(\varepsilon_m) / F)$ is a subgroup of the multiplicative group $(Z / mZ)^*$ of invertible elements of Z / mZ. More precisely, if $\sigma \in Gal(F(\varepsilon_m) / F)$, there exists a unique element $t \in (Z / mZ)^*$ such that

$$\sigma(\varepsilon_m) = \varepsilon^t_m \qquad \text{if } \varepsilon^m = 1$$

We denote by Γ_F the image of $Gal(F(\varepsilon_m) / F)$ in $(Z / mZ)^*$, and if $t \in \Gamma_F$, we let σ_t denote the corresponding element of $Gal(F(\varepsilon_m) / F)$.

Take as ground field F the field Q of rational numbers. The Galois group of $Q(\varepsilon_m)$ over Q is the group denoted by Γ. See [8].

<u>THEOREM (1.7.3):</u> [Gauss Kronecker]:

We have $\Gamma = (Z / mZ)^*$.

PROOF: See [5].

<u>PROPOSITION (1.7.4)</u>:

The characters Φ_1, Φ_2, …, Φ_h form a basis of $\overline{R}(G)$ and their number is equal to the number of conjugacy classes of cyclic subgroups of a group G, where

$$\phi_i = \sum_{\sigma \in Gal(Q(\chi_i)/Q)} \chi_i^{\ \sigma}$$

and χ_i are the irreducible C-characters of G.

PROOF: See [8].

<u>LEMMA (1.7.5)</u>:

The factor group K(G) has a finite exponent equal to the order of G.

PROOF: See [8].

<u>DEFINITION (1.7.6):</u>

Let M be a matrix with entries in a principal domain R. A k- minor of M is the determinant of a k by k submatrix preserving row and column order. See [7].

<u>DEFINITION (1.7.7):</u>

A k-th determinant divisor of a matrix M is the greatest common divisor of all the k-minors of M, and is denoted by $D_k(M)$. See [7].

THEOREM (1.7.8):

Let M, P, Q be matrices with entries in a principal domain R. Let P and Q be invertible matrices. Then $D_k(Q\,M\,P^{-1}) = D_k(M)$.

PROOF: See [7].

THEOREM (1.7.9):

Let M be an m × n matrix with entries in a principal domain R. Then there exist matrices P, Q, D such that:

1. P and Q are invertible.
2. $Q\,M\,P^{-1} = D$.
3. D is diagonal matrix.
4. If we denote D_{ii} by d_i then there exists a natural number r, $0 \leq r \leq \min(m,n)$ such that $j > r$ implies $d_j = 0$ and $j \leq r$ implies $d_j \neq 0$ and $1 \leq j < r$ implies d_j divides d_{j+1}.

PROOF: See [7].

DEFINITION (1.7.10):

Let M be a matrix with entries in a principal domain R, be equivalent to a matrix D = diag $\{ d_1, d_2, \ldots, d_r, 0, \ldots, 0 \}$ such that d_j / d_{j+1} for $1 \leq j < r$, we call **D the invariant factor matrix of M** and **$d_1, d_2, \ldots, d_r$ the invariant factors of M**. See [7].

THEOREM (1.7.11):

If M be a matrix with entries in a principal domain R, then the invariant factors are unique (modulo unit multiples).

PROOF: See [7].

THEOREM (1.7.12):

Let M be a finitely generated module over a principal domain R, then M is the direct sum of cyclic submodules with annihilating ideals

$$< d_1 >, < d_2 >, \ldots, < d_m >, d_j / d_{j+1} \text{ for } j = 1, 2, \ldots, m-1.$$

PROOF: See [7].

Suppose cf(G,Z) is of rank r , and let $(\equiv^* G)$ denote the $r \times r$ matrix whose rows correspond to the Φ_j's and columns correspond to the Γ– classes of G. The matrix expresses the $\overline{R}(G)$ basis in terms of the cf(G,Z) = Z^r basis $(\equiv^* G)$ hence, by theorem(1.7.9), we can find two matrices P and Q with determinant ± 1 such that $Q (\equiv^* G) P^{-1} = \text{diag} \{ d_1, d_2, \ldots, d_r \}$,

$$d_i = \pm D_i(\equiv^* G) / D_{i-1}(\equiv^* G)$$

This yields a new basis for R(G) and cf(G,Z), $\{ v_1, v_2, \ldots, v_r \}$ and $\{u_1, u_2, \ldots, u_r\}$ respectively with the property $v_j = d_j u_j$.

Hence by theorem (1.7.12) the Z–module K(G) is the direct sum of cyclic submodules with annihilating ideals $< d_1 >, < d_2 >, \ldots, < d_r >$.

THEOREM (1.7.13):

K(G) = $\overset{r}{\underset{i=1}{\oplus}} Z_{d_i}$, where $d_i = \pm D_i(\equiv^* G) / D_{i-1}(\equiv^* G)$.

PROOF: See [8].

THEOREM (1.7.14):

$|K(G)| = \det(\equiv^* G)$.

PROOF: See [8].

THEOREM (1.7.15):

Let $\{x_i\}$, $1 \le i \le t$ be the set of representatives of Γ– classes of G and assume each x_i contains n_i classes of G, then

$$|K(G)| = \left[\prod_{i=1}^{t} \frac{n_i\, g}{|(x_i)|} \right]^{1/2}$$

PROOF: See [8].

LEMMA (1.7.16): [8]

If A and B are two matrices of degree n and m respectively, then

$$\det(A \otimes B) = (\det(A))^m . (\det(B))^n$$

<u>LEMMA (1.7.17)</u>: See [8]

Let A and B are two non–singular matrices of degree n and m respectively over a principal domain R, and let

$$P_1\, A\, Q_1 = D(A) = \text{diag}\,\{\, d_1(A), d_2(A), \ldots, d_n(A)\,\},$$

$$P_2\, B\, Q_2 = D(B) = \text{diag}\,\{\, d_1(B), d_2(B), \ldots, d_m(B)\,\}$$

Be the invariant factor matrices of A and B then

$$(P_1 \otimes P_2)\,(A \otimes B)\,(Q_1 \otimes Q_2) = D(A) \otimes D(B),$$

And from this the invariant factor matrix of $A \otimes B$ can be written down.

Let H and L be $P_1$and P_2 –groups respectively, where $P_1$and P_2 are distinct primes. We know that

$$\equiv (H \times L) = \equiv (H) \otimes \equiv (L)$$

since gcd $(P_1,P_2) = 1$, we have

$$\equiv^* (H \times L) = \equiv^*(H) \otimes \equiv^* (L)$$

We consider the case where G is a cyclic P–group, for the cyclic group of prime order, all the non principal irreducible characters are Γ–conjugate.

Hence

$$(\equiv^* G) = \begin{pmatrix} 1 & 1 \\ p-1 & -1 \end{pmatrix} = A$$

det(A) = P = |K(G)| = exp(K(G))

<u>THEOREM (1.7.18):</u>

Let G be a cyclic P–group. Then

$$K(G) = Z_p.$$

PROOF: See [8].

THEOREM (1.7.19):

Let G be a cyclic group of order P^n. Then

$$K(G) = \overset{n}{\underset{i=1}{\oplus}} Z_{P^i}$$

PROOF: See [8].

EXAMPLE (1.13):

The rational valued character of Z_2 and Z_3 are

$$(\equiv^* Z_2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } (\equiv^* Z_3) = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

Let

$P_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $P_2 = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$, $Q_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $Q_2 = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$

Then

$P_1 (\equiv^* Z_2) Q_1 = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$, $P_2 (\equiv^* Z_3) Q_2 = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}$

By lemma (1.7.17), we obtain

$$(P_1 \otimes P_2)((\equiv^* Z_2) \otimes (\equiv^* Z_3))(Q_1 \otimes Q_2) = \begin{pmatrix} -6 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Hence

$K(Z_2 \times Z_3) = Z_6 \oplus Z_3 \oplus Z_2 \oplus Z_1$.

Chapter Two

The Finite Special Linear Groups

THE FINITE SPECIAL LINEAR GROUPS

2.1. INTRODUCTION:

This chapter concerns some members of an important class of groups: the finite linear groups. Important features are described and the conjugacy classes of this group is investigated. We develop character tables for some of the finite special linear group.

2.2. THE GROUPS SL(n, p^k):

In chapter one we already met the general linear group, i.e the group of invertible n×n matrices over a field F denoted by GL(n,F). The determinant of these matrices is a homomorphism from GL(n,F) into F^* and we denote the kernel of this homomorphism by SL(n,F),the special linear group. Thus SL(n,F) is the subgroup of GL(n,F) which contains all matrices of determinant one.

In this chapter we are interested in finite special linear group, and we choose F to be finite, we consider the case when n=2 and F = p^k, where p is prime, $p \neq 2$, k is natural number.

THEOREM (2.2.1):

The order of SL(2,p^k) is | SL(2,p^k) | = p^k (p^{2k} – 1).

PROOF: See [6]

SL(2,p^k) = $\left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} : x_1, x_2, x_3, x_4 \in F \textit{ and } x_1x_4 - x_2x_3 = 1 \right\}$

We now count the elements of SL(2,p^k) by considering two cases, the sum of which will give the required result.

Case I: $x_3 = 0$

Then $x_1x_4 - x_2x_3 = x_1x_4 = 1$. Thus, if we fix $x_1 \neq 0$, then x_4 is determined as the multiplicative inverse of x_1. Hence there are p^k–1 choices for $x_1$and non for x_4.On the other hand x_2 can be chosen arbitrarily, i.e. p^k choices for x_2 In total we have counted p^k (p^k – 1) elements for case I.

Case II: $x_3 \neq 0$

From $x_1x_4 - x_2x_3 = 1$ we deduce $x_2 = (x_1x_4 - 1) / x_3$. Now we have p^k – 1 choices for x_3. We may choose x_1 and x_4 arbitrarily and x_2 is then determined. Hence we have p^k choices for x_1, p^k choices for x_4 and non for x_2

Case II cover p^{2k} (p^k – 1) element of SL(2,p^k).

EXAMPLE (2.1):

The order of SL(2,3) is | SL(2,3) | = 3 (3^2 – 1) = 3 (8) = 24.

Case I gives 3 (3 – 1) = 6 elements.

Case II gives 3^2 (3 – 1) = 18 elements.

$$SL(2,3) = \left\{ \begin{matrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \\ \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} \end{matrix} \right\}$$

PROPOSITION (2.2.2):

The centre of SL(2,p^k) is SZ(2,p^k), where SZ(2,p^k) denotes the subgroup of SL(2,p^k) of all matrices αI, $\alpha \in$ F, such that $\alpha^2 = 1$,

for $p \neq 2$ $\alpha \in \{-1,1\}$. See [6].

THEOREM (2.2.3):

G=SL(2,p^k) has exactly $p^k + 4$ conjugacy classes :

$$1, z, c, d, zc, zd, a, a^2, \ldots, a^{\frac{p^k-3}{2}}, b, b^2, \ldots, b^{\frac{p^k-1}{2}}$$

Let:

- v be the generator of the cyclic multiplicative group F^*,
- $1 \leq \ell \leq (p^k - 3)/2$,
- $1 \leq m \leq (p^k - 1)/2$.

Thus this conjugacy classes is satisfied

g ∈ G	**Notation**	C_g	$\lvert C_g \rvert$	$\lvert C_G(g) \rvert$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1	C_1	1	$p^k (p^{2k} - 1)$
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	z	C_z	1	$p^k (p^{2k}-1)$
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	c	C_c	$(p^{2k} - 1)/2$	$2p^k$
$\begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix}$	d	C_d	$(p^{2k} - 1)/2$	$2p^k$
$\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$	zc	C_{zc}	$(p^{2k} - 1)/2$	$2p^k$
$\begin{pmatrix} -1 & 0 \\ -\nu & -1 \end{pmatrix}$	zd	C_{zd}	$(p^{2k} - 1)/2$	$2p^k$
$\begin{pmatrix} \nu^{\ell} & 0 \\ 0 & \nu^{-\ell} \end{pmatrix}$	a^{ℓ}	$C_{a^{\ell}}$	$p^k (p^k + 1)$	$p^k - 1$
Element of order(p^k+1) m	b^m	C_{b^m}	$p^k (p^k - 1)$	$p^k + 1$

TABLE (2.1)

PROOF: See [6]

Step 1: C_1, C_z, C_c, C_d, C_{zc}, C_{zd}, are as described in table (2.1)

Clearly, the elements 1 and z = -1 both form a conjugacy class of their own, since they lie in the center of G. We have $\lvert G \rvert = p^k (p^{2k} - 1)$, and hence the first two rows of the table.

To find the conjugacy classes of c, d, zc and zd we consider an arbitrary element $g = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in G$ and its inverse $g^{-1} = \begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix}$. Then

$$g^{-1}cg = \begin{pmatrix} 1-x_1x_2 & -x_2^{\ 2} \\ x_1^{\ 2} & 1-x_1x_2 \end{pmatrix} \qquad \textbf{...(2.1)}$$

From (2.1) we deduce that *zc* cannot be conjugate to *c*, since for this we would need $x_2 = 0$ and then 1 = -1, a contradiction.

A similar argument shows that $d \sim zd$

The elements *d* and *zd* cannot be conjugate to *c* either. In this case it would follow that $x_1^2 = \pm v$.If, however *v* is a square in F^*, it will not generate all of F^*, for $|F| = p^k$ is odd. Hence, C_c, C_d, C_{zc} and C_{zd} are all distinct.

To find the size of these conjugacy classes we examine the sizes of the centralizers of *c*, *d*, *zc* and *zd*. We use the element *c* again as an example.
By (2.1) the following must hold for *g* to be in $C_G(c)$:

$$1- x_1 x_2 = 1, - x_2^2 = 0, x_1^2 = 1 \Leftrightarrow x_1 = \pm 1, x_2 = 0.$$

So x_3 can be chosen arbitrarily and x_4 is determined as the multiplicative inverse of x_1, i.e. $x_4 = x_1$.Thus

$$C_G(c) = \left\{ \begin{pmatrix} x_1 & 0 \\ x_3 & x_1 \end{pmatrix} : x_1, x_3 \in F, x_1^{\ 2} = 1 \right\},$$

which is a set of size $2p^k$. It turn out that the sizes of the centralizers of *d*, *zc* and *zd* are exactly the same as $|C_G(c)|$. The next four rows of the table follow.

PROOF: Step 2:

(i) The order of $a = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}$ is $p^k - 1$.

(ii) If $y \in \langle a \rangle$ with $| y | > 2$, then $C_G(y) = \langle a \rangle$.

(iii) If y is conjugate to a power y^r in G, then $y^r \in \{ y, y^{-1} \}$.

(iv) The C_{a^ℓ} are as described in the table. (2.1).

(i) Clearly, since v generates F^*, we have $| \langle v \rangle | =| F^* | = p^k - 1$ and so $| \langle a \rangle | = p^k - 1$.

(ii) Assume that $y \in \langle a \rangle$, i.e $y = a^\ell$ for some ℓ, $< 1 \leq \ell \leq p^k - 1$ and assume that y has order greater than two. If $\left| y = \begin{pmatrix} \nu^\ell & 0 \\ 0 & \nu^{-\ell} \end{pmatrix} \right| > 2$, then $| v^\ell | > 2$. Hence $v^\ell \neq v^{-\ell}$, and so y is a diagonal matrix which is not a scalar multiple of the identity.

We now want to find the elements of the centralizer of y in G. The element y is a power of a, and thus it commutes with all other power of a, i.e $\langle a \rangle \subseteq C_G(y)$. Suppose there is some other $g \in G$, $g \notin \langle a \rangle$, that commutes with y

i.e $$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} \nu^\ell & 0 \\ 0 & \nu^{-\ell} \end{pmatrix} = \begin{pmatrix} \nu^\ell & 0 \\ 0 & \nu^{-\ell} \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x_1\nu^\ell & x_2\nu^{-\ell} \\ x_3\nu^\ell & x_4\nu^{-\ell} \end{pmatrix} = \begin{pmatrix} x_1\nu^\ell & x_2\nu^\ell \\ x_3\nu^{-\ell} & x_4\nu^{-\ell} \end{pmatrix}$$

For $g \notin \langle a \rangle$ this is the case if and only if $v^\ell = v^{-\ell}$, which contradicts our initial assumption that $| y | > 2$.Thus $\langle a \rangle = C_G(y)$.

(iii) Consider the normalizer of $\langle y \rangle$ in G, $N_G(\langle y \rangle)$. Then g $\in$G lies in $N_G(\langle y \rangle)$ if and only if $g^{-1}cg$ is diagonal. For $g = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ this means that

$$g^{-1}yg = \begin{pmatrix} x_1x_4\nu^{\ell} - x_2x_3\nu^{-\ell} & x_2x_4\nu^{\ell} - x_2x_4\nu^{-\ell} \\ -x_1x_3\nu^{\ell} + x_1x_3\nu^{-\ell} & -x_2x_3\nu^{\ell} + x_1x_4\nu^{-\ell} \end{pmatrix}$$

has to be diagonal. Thus we have $x_2x_4 = 0$ and $x_1x_3 = 0$, because $\nu^{\ell} \neq \nu^{-\ell}$. Since g is assumed to be invertible, we are left with two cases: $x_2 = x_3 = 0$ or $x_1 = x_4 = 0$. Then $N_G(\langle y\rangle)$ is the following set:

$$\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \beta \\ -\beta^{-1} & 0 \end{pmatrix} : \alpha, \beta \in F^* \right\}.$$

In other words, $N_G(\langle y\rangle) = \left\langle C_G(y), \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$. We observe that $N_G(\langle y\rangle)$ contains all the power of y and that $[N_G(\langle y\rangle) : C_G(y)] = 2$. Thus the cojugacy class of y in $N_G(\langle y\rangle)$ has size 2, i.e. it contains only one element other than y. It is easy to check that if $g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $g^{-1}yg = y^{-1}$. Hence if y is conjugate to a power y^r in $N_G(\langle y\rangle)$, then $y^r \in \{ y, y^{-1} \}$. As $N_G(\langle y\rangle)$ contains all the powers of y in G, the result follows.

(iv) First consider a^{ℓ} for $1 \leq \ell \leq (p^k - 3)/2$ .In this case $| a^{\ell} | > 2$ and (iii) tells us that the only powers of a^{ℓ} conjugate to it are a^{ℓ} and $a^{-\ell}$. Now $a^{-\ell} = a^{(p^k-1)-\ell}$ and.$(p^k + 1) / 2 \leq (p^k - 1) - \ell \leq p^k - 2$. We deduce that the conjugacy classes of a^{ℓ} are all distinct for $1 \leq \ell \leq (p^k - 3)/2$.The size of these conjugacy classes is $|G| / |C_G(a^{\ell})|$ and by (i) and (ii), we have $\left| C_{a^{\ell}} \right| = p^k(p^k+1)$.

Now we examine the remaining powers of a.For $\ell = (p^k - 1) / 2$ we have $| a^{\ell} | = 2$. Also, $a^{(p^k-1)/2} \neq 1$, since $\nu^{(p^k-1)/2} \neq 1$. We deduce $a^{(p^k-1)/2} = z$.

Clearly, for $\ell = p^k - 1$, $a^{\ell} = 1$ and we have cover all powers of a.

PROOF: Step 3:

(i) G contains an element, b say, of order. $p^k + 1$.

(ii) If $y \in \langle b \rangle$ with $| y | > 2$, then $C_G(y) = \langle b \rangle$.

(iii) If y is a conjugate to a power y^r in G, then $y^r \in \{ y, y^{-1} \}$.

(iv) The C_{b^m} are as described in the table (2.1).

We will omit the proofs of (i)-(iii), since they do not provide insights which could be helpful for the construction of the character table of SL(2,p^k), and we omit the proof of (iv) since it is practically identical to the one in the proof of (iv) in step2.

Now to show that the number of conjugacy classes is $p^k + 4$ and that these conjugacy classes are all disjoint. If we add up the elements contained in those conjugacy classes we get:

$$1+1+4\frac{p^{2k}-1}{2}+\frac{p^k-3}{2}p^k(p^k+1)+\frac{p^k-1}{2}p^k(p^k-1)=p^k(p^{2k}-1).$$

As $p^k (p^{2k} - 1) = | G |$, so this theorem gives all conjugacy classes of SL(2,p^k).

<u>EXAMPLE (2.2)</u>:

To compute the conjugacy classes of the group G = SL(2,5).

$| G | = | SL(2,5) | = 5(5^2 - 1) = 5 (24) = 120.$

This group has exactly 5 + 4 = 9 conjugacy classes.

$$v = 2, \quad 1 \le \ell \le 1, \quad 1 \le m \le 2$$

So these conjugacy classes are:

$$1, z, c, d, zc, zd, a, b, b^2$$

We can table these conjugacy classes as a table (2.1):

$g \in G$	Notation	C_g	$\|C_g\|$	$\|C_G(g)\|$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1	C_1	1	120
$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	z	C_z	1	120
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	c	C_c	12	10
$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$	d	C_d	12	10
$\begin{pmatrix} 4 & 0 \\ 4 & 4 \end{pmatrix}$	zc	C_{zc}	12	10
$\begin{pmatrix} 4 & 0 \\ 3 & 4 \end{pmatrix}$	zd	C_{zd}	12	10
$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$	a	C_a	30	4
$\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$	b	C_b	20	6
$\begin{pmatrix} 2 & 2 \\ 4 & 2 \end{pmatrix}$	b^2	C_{b^2}	20	6

2.3. IRREDUCIBLE CHARACTERS OF SL(2,p^k), $p\neq 2$

In this section we are able to construct the irreducible characters of this group since we know the conjugacy classes of SL(2,p^k).

THEOREM (2.3.1):

Let $\varepsilon = (-1)^{(p^k-1)/2}$, let $\rho \in$ C be a (p^k − 1)-th root of unity and $\sigma \in$ C be a (p^k + 1)-th root of unity. Note that the character values for C_{zc} and C_{zd} can be derived from the following relations for all irreducible characters χ of SL(2,p^k):

$$\chi(zc) = \frac{\chi(z)}{\chi(1)}\chi(c) \quad , \quad \chi(zd) = \frac{\chi(z)}{\chi(1)}\chi(d)$$

Then the ordinary character table for G = SL(2,p^k), $p\neq 2$ is:

C_g	**1**	z	c	d	a^{ℓ}	b^m
$\| C_g \|$	**1**	***1***	$(p^{2k}-1)/2$	$(p^{2k}-1)/2$	$p^k(p^k+1)$	$p^k(p^k-1)$
$\| C_G(g) \|$	$p^k(p^{2k}-1)$	$p^k(p^{2k}-1)$	$2p^k$	$2p^k$	p^k-1	p^k+1
1_G	1	1	1	1	1	1
ψ	p^k	p^k	0	0	1	-1
χ_i	(p^k+1)	$(-1)^i(p^k+1)$	1	1	$\rho^{i\ell}+\rho^{-i\ell}$	0
θ_j	(p^k-1)	$(-1)^j(p^k-1)$	-1	-1	0	$-(\sigma^{jm}+\sigma^{-jm})$
ξ_1	$(p^k+1)/2$	$\varepsilon(p^k+1)/2$	$\frac{1}{2}(1+\sqrt{\varepsilon p^k})$	$\frac{1}{2}(1-\sqrt{\varepsilon p^k})$	$(-1)^{\ell}$	0
ξ_2	$(p^k+1)/2$	$\varepsilon(p^k+1)/2$	$\frac{1}{2}(1-\sqrt{\varepsilon p^k})$	$\frac{1}{2}(1+\sqrt{\varepsilon p^k})$	$(-1)^{\ell}$	0
η_1	$(p^k-1)/2$	$-\varepsilon(p^k-1)/2$	$\frac{1}{2}(-1+\sqrt{\varepsilon p^k})$	$\frac{1}{2}(-1-\sqrt{\varepsilon p^k})$	0	$(-1)^{m+1}$
η_2	$(p^k-1)/2$	$-\varepsilon(p^k-1)/2$	$\frac{1}{2}(-1-\sqrt{\varepsilon p^k})$	$\frac{1}{2}(-1+\sqrt{\varepsilon p^k})$	0	$(-1)^{m+1}$

TABLE (2.2)

For $1 \le i \le (p^k-3)/2$, $1 \le j \le (p^k-1)/2$, $1 \le \ell \le (p^k-3)/2$, $1 \le m \le (p^k-1)/2$.

Columns for *zc* and *zd* are missing in this table.These values are computed from the relations:

$$\chi(zc) = \frac{\chi(z)}{\chi(1)}\chi(c)\ , \chi(zd) = \frac{\chi(z)}{\chi(1)}\chi(d)$$

PROOF: See [6]

Step 1: $1_G, \psi, \chi_1, \ldots, \chi_{(p^k-3)/2}$ are irreducible characters of G=SL(2,p^k) are described in the table (2.2).

Clearly, 1_G is the linear trivial character. To construct the remaining characters of this step of the proof, we use the technique of induced characters as described in chapter one .

Consider the following subgroup of G:

$$H = \left\{ \begin{pmatrix} x_1 & 0 \\ x_3 & x_1^{-1} \end{pmatrix} : x_1 \in F^*, x_3 \in F \right\}.$$

To examine the structure of H further, we, in turn, consider a subgroup S of H:

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ x_3 & 1 \end{pmatrix} : x_3 \in F \right\}.$$

Thus |S| = p^k, S $\triangleleft$ H, H = S·$\langle a \rangle$ and S $\cap$ $\langle a \rangle$ = 1. Then |H| = |S| | $\langle a \rangle$ | = $p^k (p^k - 1)$.

Define linear characters for H by:

$$\lambda_i : \begin{pmatrix} \nu^t & 0 \\ \beta & \nu^{-t} \end{pmatrix} \rightarrow \rho^{it} \qquad \textbf{...(2.2)}$$

where $\beta \in$ F, $\rho \in$ C is a (p^k – 1)-th root of and $0 \le i \le p^k - 1$.We observe that, S $\subseteq$ Ker λ_i, and that $\lambda_{p^k-1} = \lambda_0 = 1_H$, the linear trivial character of H.

Let $\lambda_i'' = \lambda_i$ for g $\in$ H and λ_i''= 0 for $g \notin$ H. We use formula (1.4) to derive the induced characters $\lambda_i\uparrow^G$of G:

$$\lambda_i\uparrow^G(g) = \frac{[G:H]}{|C_g|} \sum_{y \in C_g} \lambda_i''(y) \qquad \textbf{...(2.3)}$$

To compute the actual values of $\lambda_i\uparrow^G(g)$ it is useful to know which of the elements of each conjugacy class C_g lie in the subgroup H, since they can be ignored otherwise. Let F_s^* be the set of non-zero squares in F, then $|F_s^*| = (p^k - 1)/2$.We will now consider C_c and C_{a^ℓ} in detail. From (2.1) we derive for C_c and

$$g = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

$$g^{-1}cg = \begin{pmatrix} 1 - x_1x_2 & -x_2^2 \\ x_1^2 & 1 - x_1x_2 \end{pmatrix} \in H \Leftrightarrow x_2^2 = 0,\ i.e.\ x_2 = 0 \text{ , thus } C_c \cap H = \left\{ \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} : \beta \in F_s^* \right\}.$$

For C_{a^ℓ} we have from the proof of step 2 (iii) in theorem (2.2.3) that

$$g^{-1}a^\ell g = \begin{pmatrix} x_1x_4\nu^\ell - x_2x_3\nu^{-\ell} & x_2x_4\nu^\ell - x_2x_4\nu^{-\ell} \\ -x_1x_3\nu^\ell + x_1x_3\nu^{-\ell} & -x_2x_3\nu^\ell + x_1x_4\nu^{-\ell} \end{pmatrix}.$$

For $1 \le \ell \le (p^k - 3)/2$ we have seen that $\nu^\ell \ne \nu^{-\ell}$, and hence

$$g^{-1} a^\ell g \in H \Leftrightarrow x_2 x_4 = 0$$

1) Let $x_2 = 0$. Then $g^{-1}a^\ell g = \begin{pmatrix} x_1x_4\nu^\ell & 0 \\ x_1x_3(\nu^{-\ell} - \nu^\ell) & x_1x_4\nu^{-\ell} \end{pmatrix}$, where $x_1x_4 = 1$, since g has determinant 1.

2) Let $x_4 = 0$. Then $g^{-1}a^\ell g = \begin{pmatrix} -x_2x_3\nu^{-\ell} & 0 \\ x_1x_3(\nu^{-\ell} - \nu^\ell) & -x_2x_3\nu^\ell \end{pmatrix}$, where $x_2x_3 = -1$, since g has determinant 1

Hence $C_{a^\ell} \cap H = \left\{ \begin{pmatrix} \nu^\ell & 0 \\ \beta & \nu^{-\ell} \end{pmatrix}, \begin{pmatrix} \nu^{-\ell} & 0 \\ \beta & \nu^\ell \end{pmatrix} : \beta \in F \right\}.$

g	$\mid C_g \mid$	$\lambda_i''(g)$ for $g \in (C_g \cap H)$	$\mid C_g \cap H \mid$	$\lambda_i\uparrow^G(g)$

Clearly, we have $| C_{a^\ell} \cap H | = 2p^k$, since β can be chosen arbitrarily and $v^\ell \neq v^{-\ell}$. In a similar way to the above the following can be deduced:

$$C_{zc} \cap H = \left\{ \begin{pmatrix} -1 & 0 \\ -\beta & -1 \end{pmatrix} : \beta \in F_s^* \right\},$$

$$C_d \cap H = \left\{ \begin{pmatrix} 1 & 0 \\ \nu\beta & 1 \end{pmatrix} : \beta \in F_s^* \right\},$$

$$C_{zd} \cap H = \left\{ \begin{pmatrix} -1 & 0 \\ -\nu\beta & -1 \end{pmatrix} : \beta \in F_s^* \right\}.$$

We have $| C_c \cap H | = | C_{zc} \cap H | = | C_d \cap H | = | C_{zd} \cap H | = (p^k - 1) / 2$. If we sum all the elements of H over all the conjugacy classes we get:

$$1 + 1 + 4\frac{p^k - 1}{2} + \frac{p^k - 3}{2} 2p^k = p^k(p^k - 1) = |H|.$$

Thus we have already accounted for all elements of H, and hence no element of the conjugacy classes C_{b^m} can lie in H.

Now with [G:H]= $p^k + 1$ and using (2.2) and (2.3) we can compute $\lambda_i\uparrow^G(g)$ for $g \in G$:

1	1	1	1	$(p^k + 1)$
z	1	$\rho^{i(p^k-1)/2} = (-1)^i$	1	$(-1)^i(p^k + 1)$
c	$(p^{2k} - 1)/2$	1	$(p^k - 1)/2$	1
zc	$(p^{2k} - 1)/2$	$\rho^{i(p^k-1)/2} = (-1)^i$	$(p^k - 1)/2$	$(-1)^i$
d	$(p^{2k} - 1)/2$	1	$(p^k - 1)/2$	1
zd	$(p^{2k} - 1)/2$	$\rho^{i(p^k-1)/2} = (-1)^i$	$(p^k - 1)/2$	$(-1)^i$
a^ℓ	$p^k(p^k + 1)$	(*)	$2p^k$	$\rho^{i\ell} + \rho^{-i\ell}$
b^m	$p^k(p^k - 1)$	-	0	0

TABLE (2.3)

Note that $\rho^{(p^k-1)/2} = -1$, since $p^k - 1$ is even.

(*) For $a^\ell \in (C_{a^\ell} \cap H)_1 = \left\{ \begin{pmatrix} \nu^\ell & 0 \\ \beta & \nu^{-\ell} \end{pmatrix} : \beta \in F \right\}$ we have $\lambda_i''(a^\ell) = \rho^{i\ell}$ and

$|(C_{a^\ell} \cap H)_1| = p^k$

For $a^\ell \in (C_{a^\ell} \cap H)_2 = \left\{ \begin{pmatrix} \nu^{-\ell} & 0 \\ \beta & \nu^{\ell} \end{pmatrix} : \beta \in F \right\}$ we have $\lambda_i''(a^\ell) = \rho^{-i\ell}$ and

$|(C_{a^\ell} \cap H)_2| = p^k$.

Examining this table we see that not all of the computed characters $\lambda_i\uparrow^G$ are distinct: $\rho^{(-i)\ell} = \rho^{(p^k-1-i)\ell}$, since the roots of unity occur in conjugate pairs.

Hence for $(p^k + 1)/2 \le i \le (p^k - 1)$ we get the same values for $\rho^{i\ell} + \rho^{-i\ell}$ as in the cases of $1 \le i \le (p^k - 3)/2$. Thus we only need to consider $\lambda_i\uparrow^G$ for $0 \le i \le (p^k - 1)/2$. For these values of i let $\chi_i = \lambda_i\uparrow^G$.

The next task is to investigate whether the characters $\chi_0, \ldots, \chi_{(p^k-1)/2}$ are irreducible. We have seen that a suitable irreducibility criterion is the value of the direct product of a character with itself (proposition 1.3.13).

Now, using formula (1.1), we have

$$\langle \chi_0, \chi_0 \rangle = \frac{1}{p^k(p^{2k}-1)}\left[1\cdot(p^k+1)^2 + 1\cdot(p^k+1)^2 + \frac{p^{2k}-1}{2}\cdot 1 + \frac{p^{2k}-1}{2}\cdot 1\right.$$

$$\left. + \frac{p^{2k}-1}{2}\cdot 1 + \frac{p^{2k}-1}{2}\cdot 1 + \frac{p^k-3}{2}.p^k(p^k+1)\cdot 4\right] = 2$$

Thus χ_0 is reducible, i.e., by (lemma 1.3.5), the sum of irreducible characters of G. From (corollary 1.3.10) we know how to calculate the multiplicities of the constituents of χ_0. It is straightforward to check that $\langle \chi_0, 1_G \rangle = 1$.

We can deduce from (corollary 1.3.11) and $\langle \chi_0, \chi_0 \rangle = 2$ that χ_0 is the sum of the linear trivial character and one further irreducible character of G, ψ say, also with multiplicity 1. The degree of χ_0 is $p^k + 1$, and so deg(ψ) must be p^k. The values $\psi(g)$ can be computed using $\psi(g) = \chi_0(g) - 1_G(g)$.

Now we examine the characters χ_i for $1 \le i \le (p^k - 3)/2$. In these cases

$$\langle \chi_i, \chi_i \rangle = \frac{1}{p^k(p^{2k}-1)}\left[1\cdot(p^k+1)^2 + 1\cdot(p^k+1)^2 + \frac{p^{2k}-1}{2}\cdot 1 + \frac{p^{2k}-1}{2}\cdot 1 + \frac{p^{2k}-1}{2}\cdot 1\right.$$

$$+\frac{p^{2k}-1}{2}\cdot 1+\sum_{\ell=1}^{(p^k-3)/2} p^k(p^k+1)\left|\rho^{i\ell}+\rho^{-i\ell}\right|^2\Bigg]=1$$

The only term in this calculation which yields some difficulty is

$\sum_{\ell=1}^{(p^k-3)/2}\left|\rho^{i\ell}+\rho^{-i\ell}\right|^2$ We use:

$$\sum_{\ell=1}^{(p^k-3)/2}\left|\rho^{i\ell}+\rho^{-i\ell}\right|^2=\sum_{\ell=1}^{(p^k-3)/2}\rho^{2i\ell}+\sum_{\ell=1}^{(p^k-3)/2}2+\sum_{\ell=1}^{(p^k-3)/2}\rho^{-2i\ell}$$

$$=(p^k-3)/2\cdot 2+\sum_{\ell=1}^{(p^k-3)/2}\rho^{2i\ell}+\sum_{\ell=1}^{(p^k-3)/2}\rho^{-2i\ell}$$

Now the sum of the roots of unity ρ^i is zero, independently of the value of *i*,

i.e. $\rho^i+\rho^{2i}+\cdots+\rho^{(p^k-3)i}+\rho^{(p^k-1)i}=0$ for all *i*. Consequently, $\sum_{\ell=1}^{(p^k-1)/2}\rho^{2i\ell}=0$

Hence $\sum_{\ell=1}^{(p^k-3)/2}\rho^{2i\ell}=-(\rho^{2i\frac{p^k-1}{2}})=-1$.Similarly, $\sum_{\ell=1}^{(p^k-3)/2}\rho^{-2i\ell}=-1$.

Thus we have shown that the characters $\chi_1, \ldots, \chi_{(p^k-3)/2}$ are irreducible.

In step 2 of this proof we will consider the remaining character $\theta_{(p^k-1)/2}$.

PROOF:

Step 2: $\theta_1,\cdots,\theta_{(p^k-1)/2}$ are irreducible characters of G = SL(2,p^k) as described in the table of theorem(2.3.1).

As in the first step of this proof we use an induced character to derive the required irreducible characters. In this case let L be the subgroup of G generated by the element b, L = ⟨b⟩. Then define the following linear characters for L:

$$\varphi_j : b^t \rightarrow \sigma^{jt} \qquad \textbf{(2.4)}$$

where σ is defined as the $(2^k + 1)$-th root of unity, $1 \leq j \leq (p^k + 1)$.

Again, let $\varphi_j''(g) = \varphi(g)$ if $g \in$ L and $\varphi_j''(g) = 0$ if $g \notin$ L. From step 3 (i) of theorem (2.2.3) we have | L | = $p^k + 1$. Clearly, the only elements of those in G that lie in L are 1, and ⟨b⟩. Thus, using (2.3) and (2.4) as in step 1, we can derive the following table for the induced characters $\varphi_j\uparrow^G$:

TABLE (2.4)

(*) We saw in step 3 (iii) of theorem (2.2.3) that the only powers of b^m conjugate to b^m are $\{b^m, b^{-m}\}$. Thus these two elements of C_{b^m} are the only ones

g	$\lvert C_g \rvert$	$\varphi_j''(g)$ for $g \in (C_g \cap H)$	$\lvert C_g \cap H \rvert$	$\varphi_j\uparrow^G(g)$
1	1	1	1	$p^k(p^k - 1)$
z	1	$\sigma^{j(p^k+1)/2} = (-1)^j$	1	$(-1)^j\, p^k(p^k - 1)$
c	$(p^{2k} - 1)/2$	-	0	0
zc	$(p^{2k} - 1)/2$	-	0	0
d	$(p^{2k} - 1)/2$	-	0	0
zd	$(p^{2k} - 1)/2$	-	0	0
a^ℓ	$p^k(p^k + 1)$	-	0	0
b^m	$p^k(p^k - 1)$	σ^{jm}	2(*)	$\sigma^{jm} + \sigma^{-jm}$

that lie in L.

The characters $\varphi_j{\uparrow}^G$ are not irreducible, since $\deg(\varphi_j{\uparrow}^G) = p^k(p^k - 1)$ and $| p^k(p^k - 1)|^2 > |G|$, which contradicts proposition (1.3.14).

To construct irreducible characters we use a trick: for $1 \le j \le (p^k + 1)/2$, let $\theta_j = \psi\, \lambda_j{\uparrow}^G - \lambda_j{\uparrow}^G - \varphi_j{\uparrow}^G$. As all terms in this expression are characters, θ_j is again a character.

The values for θ_j are easy to compute and are listed in the following table:

TABLE (2.5)

g	$\theta_j(g)$
1	$p^k - 1$
z	$(-1)^j p^k - 1$
c	-1
zc	$(-1)^{j+1}$
d	-1
zd	$(-1)^{j+1}$
a^ℓ	0
b^m	$-\sigma^{jm} - \sigma^{-jm}$

To see whether the characters θ_j are irreducible, we examine once more the inner product $\langle\theta_j,\theta_j\rangle$. A similar calculation as the one in step 1 of this proof shows that for $1 \le j \le (p^k - 1)/2$, $\langle\theta_j,\theta_j\rangle = 1$ and the θ_j are irreducible characters of G.

Again, $\theta_{(p^k+1)/2}$ will be considered in the next step of this proof.

PROOF:

Step 3 : The irreducible characters of G = SL(2,p^k) are

$$1_G, \psi, \chi_1, \ldots, \chi_{(p^k-3)/2}, \theta_1, \cdots, \theta_{(p^k-1)/2} \text{ and } \xi_1, \xi_2, \eta_1, \eta_2$$

as described in the table of theorem (2.3.1).

An easy calculation shows that $\left\langle \chi_{(p^k-1)/2}, \chi_{(p^k-1)/2} \right\rangle = 2$. Thus we conclude as in step 1 that $\chi_{(p^k-1)/2}$ is the sum of two irreducible characters, ξ_1 and ξ_2 say, both with multiplicity 1. We have the following values for $\chi_{(p^k-1)/2} = \xi_1 + \xi_2$

TABLE (2.6)

g	$\xi_1 + \xi_2(g)$
1	$p^k + 1$
z	$\varepsilon p^k + 1$
c	1
zc	ε
d	1
zd	ε
a^ℓ	$2(-1)^\ell$
b^m	0

where $\varepsilon = (-1)^{(p^k-1)/2}$. We also know that tr (A + B) = tr (A) + tr (B), and so

$$(\xi_1 + \xi_2)(g) = \xi_1(g) + \xi_2(g)$$

Similarly, $\left\langle \theta_{(p^k+1)/2}, \theta_{(p^k+1)/2} \right\rangle$ = 2. Let $\theta_{(p^k+1)/2} = \eta_1 + \eta_2$ say, where η_1 and η_2 are irreducible characters of G:

TABLE (2.7)

g	$\eta_1 + \eta_2(g)$
1	$p^k - 1$
z	$\varepsilon p^k - 1$
c	-1
zc	ε
d	-1
zd	ε
a^ℓ	0
b^m	$2(-1)^{m+1}$

Again, $(\eta_1 + \eta_2)(g) = \eta_1(g) + \eta_2(g)$.

Counting the irreducible characters that we have found so far we get $p^k + 4$ characters, which are precisely the number we need, because G has $p^k + 4$ conjugacy classes (theorem 1.3.16). However, we will still need to show that these irreducible characters are all distinct. Since we know the values of 1_G, ψ, χ_i, θ_j and $(\xi_1 + \xi_2)$, $(\eta_1 + \eta_2)$, we can compute the inner products among them.

For example, using formula (1), we have:

$$\left\langle (\xi_1+\xi_2), 1_G \right\rangle = \frac{1}{p^k(p^{2k}-1)} \Big[(p^k+1)\cdot 1 + \varepsilon(p^k+1)\cdot 1 + \frac{p^{2k}-1}{2}\cdot 1\cdot 1 + \frac{p^{2k}-1}{2}\cdot 1\cdot 1$$

$$\frac{p^{2k}-1}{2}\cdot\varepsilon\cdot 1+\frac{p^{2k}-1}{2}\cdot\varepsilon\cdot 1+\left.\sum_{\ell=1}^{(p^k-3)/2} p^k(p^k+1)\cdot 2(-1)^{\ell}\cdot 1\right]=0$$

Similarly, all these inner products are zero. Hence all these characters are distinct, except possibly the ξ_i and η_i , i = 1, 2. However, as we can see from the tables $\xi_1(z) + \xi_2(z) = \varepsilon(\xi_1(1) + \xi_2(1))$, which forces $\xi_i(z) = \varepsilon\,\xi_i(1)$, whereas $\eta_i(z) = -\varepsilon\,\eta_i(1)$. Thus the ξ_i and η_i are different.

To complete the table of theorem (2.3.1) it remains to determine the values of ξ_1, ξ_2, η_1 and η_2. We assume that **ε = +1**, i.e. that $\boldsymbol{p^k \equiv 1}$ (mod 4). The proof in the case of **ε = -1** is in large parts very similar and, therefore, omitted.

If **ε = +1**, then $\xi_i(z) = \xi_i(1)$ and $\eta_i(z) = -\eta_i(1)$, i = 1, 2. Consequently, from lemma (1.3.20) we have that ker η_i does not contain $\langle z\rangle$, but ker ξ_i does. We also see that, for $N \lhd G$, the irreducible characters of G/N are all those irreducible characters of G whose kernels contain N (proof Corollary 1.3.21).Using this information and (lemma 1.3.20) we deduce that for N = $\langle z\rangle$ the irreducible characters of G/N = SL(2,p^k) / $\langle z\rangle$ = PSL(2,p^k) are

$$1_G,\ \psi,\ \chi_2,\ \dots,\ \chi_{(p^k-5)/2},\ \theta_2,\cdots,\theta_{(p^k-1)/2}\ \text{and}\ \xi_1,\ \xi_2.$$

By proposition (1.3.14) the sum of the squares of the degrees of these characters must equal $|G/N| = p^k(p^{2k}-1)/2$, i.e.

$$1 + p^{2k} + \frac{p^k - 5}{4}(p^k + 1)^2 + \frac{p^k - 1}{4}(p^k - 1)^2 + \xi_1(1)^2 + \xi_2(1)^2 = \frac{p^k(p^{2k} - 1)}{2}$$

$$\Leftrightarrow \ \xi_1(1)^2 + \xi_2(1)^2 = \frac{p^k + 1}{2}$$

Together with equation $\xi_1(1) + \xi_2(1) = p^k + 1$we conclude that $\xi_1(1) + \xi_2(1) = (p^k + 1) / 2$. From this it follows directly that $\xi_i(1) = \xi_i(z) = (p^k + 1) / 2.$, $i = 1, 2$.

To find the remaining values of ξ_1 and ξ_2 we use the column orthogonality relations (theorem 1.3.18 (2)) for the elements $\langle z \rangle$ g in G/N:

$$\begin{aligned} \left|C_{G/N}(\langle z\rangle g)\right| &= \left|1_G(\langle z\rangle g)\right|^2 + \left|\psi(\langle z\rangle g)\right|^2 + \left|\chi_2(\langle z\rangle g)\right|^2 + \left|\chi_4(\langle z\rangle g)\right|^2 \\ &+ \cdots + \left|\chi_{(p^k-5)/2}(\langle z\rangle g)\right|^2 + \left|\theta_2(\langle z\rangle g)\right|^2 + \left|\theta_4(\langle z\rangle g)\right|^2 \\ &+ \cdots + \left|\theta_{(p^k-1)/2}(\langle z\rangle g)\right|^2 + \left|\xi_1(\langle z\rangle g)\right|^2 + \left|\xi_2(\langle z\rangle g)\right|^2 \end{aligned} \quad \text{…(2.5)}$$

The only irreducible characters of G/N which are not fully known are ξ_1 and ξ_2 and we will use (2.5) to deduce the values for $\xi_1^2 + \xi_2^2$. Since we also know the values of $\xi_1 + \xi_2 = \chi_{(p^k-1)/2}$, these two equations combined and the comments in the proof of (Corollary 1.3.21) will enable us to complete the table for ξ_1 and ξ_2 .

To be able to use (2.5) in the way described, we need to determine

$| C_{G/N}(\langle z\rangle g) |$ for $\langle z\rangle g \in G/N$. Thus we consider how the sizes of the conjugacy classes of G/N differ from the ones in G: if $\langle z\rangle g \notin C_g$, then the size of the conjugate class remains unchanged, i.e. $| C_{\langle z\rangle g} | = | C_g |$, and so

$$| C_{G/N}(\langle z\rangle g) | = | G/N | / | C_{\langle z\rangle g} | = (1/2) | G | / | C_g | = (1/2) | C_G(g) |$$

whereas, if $\langle z\rangle g \in C_g$, then $| C_{G/N}(\langle z\rangle g) | = | C_G(g) |$ and $| C_{\langle z\rangle g} | = (1/2) | C_g |$.

From theorem (2.2.3) we have that $c \sim zc$ and $d \sim zd$, and hence $\langle z\rangle c \in C_c$ and $\langle z\rangle d \in C_d$. We also see that the only powers of a^{ℓ} conjugate to a^{ℓ} are $\{ a^{\ell}, a^{-\ell} \}$. Now $a^{(p^k-1)/2} = z$ and $a^{-\ell} = a^{(p^k-1)-\ell}$. Thus

$$a^{\ell} \sim za^{\ell} \Leftrightarrow a^{-\ell} = za^{\ell} \Leftrightarrow a^{(p^k-1)-\ell} = a^{(p^k-1)/2} a^{\ell}.$$

Thus for $\ell \neq (p^k - 1)/4$, $\langle z\rangle a^{\ell} \notin C_{a^{\ell}}$, and for $\ell = (p^k - 1)/4$, $\langle z\rangle a^{\ell} \in C_{a^{\ell}}$.

A similar argument for the elements b^m shows that $\langle z\rangle b^m \notin C_{b^m}$.

The orthogonality relation (2.5) for $\langle z\rangle c$, for example, leads to:

$$p^k = 1 + 0 + \frac{p^k - 5}{4}\cdot 1 + \frac{p^k - 1}{4}\cdot 1 + |\xi_1(\langle z\rangle c)|^2 + |\xi_2(\langle z\rangle c)|^2$$

$$\Leftrightarrow |\xi_1(\langle z\rangle c)|^2 + |\xi_2(\langle z\rangle c)|^2 = \frac{p^k + 1}{2}$$

One can show that for $p^k \equiv 1 \pmod 4$ all ordinary characters of $G = SL(2,p^k)$ are real. Thus $|\xi_1|^2 + |\xi_2|^2 = \xi_1^2 + \xi_2^2$.

Using the comments in the proof of (Corollary 1.3.21) we deduce from

$$\xi_1(c)^2 + \xi_2(c)^2 = \xi_1(\langle z\rangle c)^2 + \xi_2(\langle z\rangle c)^2 = \frac{p^k + 1}{2}$$

and

$$\xi_1(\langle z\rangle c)+\xi_2(\langle z\rangle c)=\xi_1(c)+\xi_2(c)=(\xi_1+\xi_2)(c)=1$$

that

$$\xi_1(c)=\frac{1}{2}(1\pm\sqrt{p^k}) \text{ and } \xi_2(c)=\frac{1}{2}(1\mp\sqrt{p^k}).$$

The signs of the square roots must be chosen such that the row orthogonality relation hold. To ensure this we can use, for example, the equations $\langle \xi_i, 1_G\rangle = 0$, $i = 1, 2$. The values of ξ_i for the elements d, a^ℓ and b^m are derived correspondingly and we can use $\xi_i(c) = \xi_i(zc)$ and $\xi_i(d) = \xi_i(zd)$ to find $\xi_i(zc)$ and $\xi_i(zd)$.

The rows for the characters η_1 and η_2 are filled in a very similar way as the ones of ξ_1 and ξ_2: the characters η_1 and η_2 are now the only characters of G not fully known and we use proposition (1.3.14) as above to deduce the values of $\eta_1(1)$ and $\eta_2(1)$. The remaining values are again computed by using first the column and then the row orthogonality relations in combination with $\eta_i(g) = -\eta_i(zg)$ for $g = 1, c, d$, $i = 1, 2$. This finishes the proof of theorem (2.3.1).

EXAMPLE (2.3):

To compute the ordinary character table for SL(2,7).

$|SL(2,7)| = 7(7^2 - 1) = 7(48) = 336$

$\varepsilon = -1$, $1 \le i \le 2$, $1 \le j \le 3$, $1 \le \ell \le 2$, $1 \le m \le 3$,

$\rho \in$ C is a primitive 6-th root of unity, to find the 6-th roots of unity in C:

$$\rho = re^{i\theta} = e^{\frac{2k\pi}{6}} \qquad k = 0,1, \dots ,5$$

$\rho_0 = e^{i0} = \cos(0) + i\sin(0) = 1,$ $\rho_1 = e^{i\frac{\pi}{3}} = \cos(60) + i\sin(60) = \frac{1}{2} + i\frac{\sqrt{3}}{2},$

$\rho_2 = e^{i\frac{2\pi}{3}} = \cos(120) + i\sin(120) = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$ $\rho_3 = e^{i\pi} = \cos(180) + i\sin(180) = -1,$

$\rho_4 = e^{i\frac{4\pi}{3}} = \cos(240) + i\sin(240) = -\frac{1}{2} - i\frac{\sqrt{3}}{2},$ $\rho_5 = e^{i\frac{5\pi}{3}} = \cos(300) + i\sin(300) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

$\rho^{1} + \rho^{-1} = \rho_1 + \rho_5 = 1, \rho^{2} + \rho^{-2} = \rho_2 + \rho_4 = -1, \rho^{4} + \rho^{-4} = \rho_4 + \rho_2 = -1$

$\sigma \in C$ is a primitive 8-th root of unity, to find the 8-th roots of unity in C:

$$\sigma = re^{i\theta} = e^{\frac{2k\pi}{8}} \qquad k = 0,1,\dots,7$$

$\sigma_0 = e^{i0} = 1,$ $\sigma_1 = e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}},$ $\sigma_2 = e^{i\frac{\pi}{2}} = i,$

$\sigma_3 = e^{i\frac{3\pi}{4}} = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}},$ $\sigma_4 = e^{i\pi} = -1,$ $\sigma_5 = e^{i\frac{5\pi}{4}} = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}},$

$\sigma_6 = e^{i\frac{3\pi}{2}} = -i,$ $\sigma_7 = e^{i\frac{7\pi}{4}} = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

$\sigma^{1} + \sigma^{-1} = \sigma_1 + \sigma_7 = \sqrt{2}$, $\sigma^{2} + \sigma^{-2} = \sigma_2 + \sigma_6 = 0$, $\sigma^{3} + \sigma^{-3} = \sigma_3 + \sigma_5 = -\sqrt{2}$,

$\sigma^{4} + \sigma^{-4} = \sigma_4 + \sigma_4 = -2$, $\sigma^{6} + \sigma^{-6} = \sigma_6 + \sigma_2 = 0$, $(\sigma^{9} + \sigma^{-9})_{\text{mod } 8} = \sigma^{1} + \sigma^{-1} = \sqrt{2}$,

This group has 7 + 4 = 11 conjugacy classes:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, z = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}, c = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, d = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, zc = \begin{pmatrix} 6 & 0 \\ 6 & 6 \end{pmatrix}, zd = \begin{pmatrix} 6 & 0 \\ 4 & 6 \end{pmatrix}, a = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix},$$

$$a^2 = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, b = \begin{pmatrix} 0 & 6 \\ 1 & 3 \end{pmatrix}, b^2 = \begin{pmatrix} 6 & 4 \\ 3 & 1 \end{pmatrix}, b^3 = \begin{pmatrix} 4 & 6 \\ 1 & 0 \end{pmatrix}$$

The orders of these conjugacy classes are:

o(z) = 2, o(c) = o(d) = 7, o(zc) = o(zd) = 14, o(a) = 6, o(a^2) = 3,

o(b) = o(b^3) = 8, o(b^2) = 4

The ordinary character table for SL(2,7) is:

C_g	1	z	c	d	zc	zd	a	a^2	b	b^2	b^3
$\lvert C_g \rvert$	1	1	24	24	24	24	56	56	42	42	42
$\lvert C_G(g) \rvert$	336	336	14	14	14	14	6	6	8	8	8
1_G	1	1	1	1	1	1	1	1	1	1	1
ψ	7	7	0	0	0	0	1	1	-1	-1	-1
χ_1	8	-8	1	1	-1	-1	1	-1	0	0	0
χ_2	8	8	1	1	1	1	-1	-1	0	0	0
θ_1	6	-6	-1	-1	1	1	0	0	$-\sqrt{2}$	0	$\sqrt{2}$
θ_2	6	6	-1	-1	-1	-1	0	0	0	2	0
θ_3	6	-6	-1	-1	1	1	0	0	$\sqrt{2}$	0	$-\sqrt{2}$
ξ_1	4	-4	$\frac{1}{2}(1+\sqrt{-7})$	$\frac{1}{2}(1-\sqrt{-7})$	$\frac{1}{2}(-1-\sqrt{-7})$	$\frac{1}{2}(-1+\sqrt{-7})$	-1	1	0	0	0
ξ_2	4	-4	$\frac{1}{2}(1-\sqrt{-7})$	$\frac{1}{2}(1+\sqrt{-7})$	$\frac{1}{2}(-1+\sqrt{-7})$	$\frac{1}{2}(-1-\sqrt{-7})$	-1	1	0	0	0
η_1	3	3	$\frac{1}{2}(-1+\sqrt{-7})$	$\frac{1}{2}(-1-\sqrt{-7})$	$\frac{1}{2}(-1+\sqrt{-7})$	$\frac{1}{2}(-1-\sqrt{-7})$	0	0	1	-1	1
η_2	3	3	$\frac{1}{2}(-1-\sqrt{-7})$	$\frac{1}{2}(-1+\sqrt{-7})$	$\frac{1}{2}(-1-\sqrt{-7})$	$\frac{1}{2}(-1+\sqrt{-7})$	0	0	1	-1	1

Chapter Three

The Cyclic Decomposition Of K(sl(2,p))

THE CYCLIC DECOMPOSITION OF K(SL(2,P))

3.1. INTRODUCTION:

This chapter is devoted to study the character table of the irreducible rational representations of SL(2,p^k), p is an odd prime, k > 0 and odd, then we introduce the diagonalization of the matrix (≡*SL(2,p)) which gives us the cyclic decomposition of K(SL(2,p)), where k = 1, and p = 3, 5, 7, 11, 13, 17,and 19.

3.2.THE CHARACTER TABLE OF IRREDUCIBLE RATIONAL REPRESENTATIONS OF SL(n, p^k):

In this section we will give the character table of the irreducible rational representations of SL(2,p^k), p is an odd prime, k > 0 and odd by using the character table and the Schur indices of SL(2,p^k).

THEOREM (3.2.1): See [2]

Let G = SL(2,p^k), then the Schur indices of the irreducible characters of G over the rational numbers Q are as follows:

	$p^k \equiv 1 \bmod 4$	$p^k \equiv 3 \bmod 4$
1_G	1	1
ψ	1	1
χ_i	2 (*i* odd)	2 (*i* odd)
	1 (*i* even)	1 (*i* even)
θ_j	2 (*j* odd)	2 (*j* odd)
	1 (*j* even)	1 (*j* even)
ξ_1	1	1
ξ_2	1	1
η_1	2	1
η_2	2	1

TABLE (3.1)

LEMMA (3.2.2):

Let ζ be a primitive n-th root of unity. Then $\zeta + \zeta^{-1}$ is rational if and only if n = 1, 2, 3, 4, 6. The values which occur are as follows:

n	1	2	3	4	6
$\zeta + \zeta^{-1}$	2	-2	-1	0	1

TABLE (3.2)

PROOF: See [2].

The result is clear for n = 1 or 2 so that we may assume that $n \geq 3$.

As $x^2 - (\zeta + \zeta^{-1})\, x + 1 = (x - \zeta)\,(x - \zeta^{-1})$, the index $(Q(\zeta):Q(\zeta + \zeta^{-1})) = 2$ unless $\zeta \in Q$, that is, unless n =1or 2.It follows that $\zeta + \zeta^{-1} \in Q$ if and only if $\Phi(n) = (Q(\zeta):Q) = 2$.

Examination of the possibilities shows that $\Phi(n) = 2$ if and only if n = 3, 4 or 6.

Recall that $\Phi(n)$ is the Eular totient function which is defined by the number of positive integers not exceeding *n* which are relatively prime to *n*.

Corollary (3.2.3):

Let ζ be a primitive n-th root of unity and $m \in Z$. If $\zeta + \zeta^{-1} \in Q$, then so is $\zeta^m + \zeta^{-m}$.

PROOF: See [3].

This follows from lemma (3.2.2).

Corollary (3.2.4):

Let ζ be a primitive n-th root of unity, let $1 \le j \le n$. Then $\zeta^j + \zeta^{-j}$ is rational if and only if $n = j, 2j, 3j, 4j, 6j, \frac{3}{2}j, \frac{4}{3}j, \frac{6}{5}j.$

PROOF: See [2].

Let (j,n) denote the greatest common divisor of j and n. Write $j = a\,(j,n)$ and $n = b\,(j, n)$ so that a and b are coprime and $0 \le \frac{a}{b} \le 1.$

As ζ^j is a primitive b-th root of unity, lemma (3.2.2) shows that $\zeta^j + \zeta^{-j}$ is rational if and only if b = 1, 2, 3, 4 or 6. For these values of b, the corresponding possibilities for $\frac{a}{b}$ are $1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}\, and\, \frac{5}{6}$. As $j = \frac{a}{b}n$, the result follows.

LEMMA (3.2.5):

Let σ be a primitive $(p^k + 1)$ -th root of unity where p is odd prime. Suppose that $p^k \equiv 7 (\mathrm{mod}\ 8)$ and that $j = 1, 3, \cdots, \frac{p^k - 1}{2}$ then $\sigma^j + \sigma^{-j}$ is not rational.

PROOF: See [3].

Suppose that $\sigma^{j} + \sigma^{-j} \in$ Q. As $1 \le j \le \frac{p^k - 1}{2}$, Corollary (3.2.4) implies that $j = \frac{p^k + 1}{d}$ for d = 3, 4 or 6, by hypothesis , 8 / (p^k + 1) so that $\frac{p^k + 1}{d}$ is even for d = 3, 4 or 6. This contradicts the assumption that j is odd.

LEMMA (3.2.6):

Let ζ be a primitive (p^k + 1)-th root of unity where p is odd prime. If $p^k \equiv 3 \pmod 8$ and ℓ is a positive integer, then $\zeta^{\frac{p^k+1}{4}\ell} + \zeta^{-\frac{p^k+1}{4}\ell}$ is rational.

PROOF: See [3].

This follows from Corollary (3.2.4) and Corollary (3.2.3).

Corollary (3.2.7):

Let ζ be a primitive n-th root of unity and n ≠ 2. Then

$(Q(\zeta):Q(\zeta + \zeta^{-1})) = 2$ and $(Q(\zeta + \zeta^{-1}):Q) = \frac{1}{2}\,\Phi(n)$.

PROOF: See [2].

This follows from the fact that $(x - \zeta)(x - \zeta^{-1}) = x^2 - (\zeta + \zeta^{-1})x + 1$ and $(Q(\zeta):Q) = \Phi(n)$.

LEMMA (3.2.8):

Let ζ be a primitive n-th root of unity, $i \in Z$ and $d_i = (i,n)$. If $n > 2d_i$, then

$$(Q(\zeta^i + \zeta^{-i}):Q) = \frac{1}{2}\,\Phi(\frac{n}{d_i}).$$

PROOF: See [2].

Corollary (3.2.9):

Let ζ be a primitive n-th root of unity and $1 \le i \le \frac{n}{2}$. Then

$$(Q(\zeta^i + \zeta^{-i}):Q) = \frac{1}{2}\,\Phi(\frac{n}{d_i}).$$

where $d_i = (i,n)$

PROOF: See [2].

This follows from lemma (3.2.8).

Let M be a field of characteristic zero and let K be a subfield of M. suppose that M is a finite and normal extension of K with Galois group $\Gamma = \Gamma$ (M:K). For any $a \in M$ define the trace

$$Tr_{M\to K}(a) = \sum_{\alpha \in \Gamma} a^{\alpha}.$$

LEMMA (3.2.10):

Let K ≤ L ≤ M be fields and let M be a finite and normal extension of K. Then:

$Tr_{L\to K}(Tr_{M\to L}(x)) = Tr_{M\to K}(x)$ where $x \in$ M.

PROOF: See [2].

LEMMA (3.2.11):

Let ζ be a primitive n-th root of unity, $i \in$ Z and $d_i = (i,n)$. and let n ≠ d_i, $2d_i$.Then:

$$\sum_{\alpha \in \Gamma_i} (\xi^i + \xi^{-i})^\alpha = \mu(\frac{n}{d_i}).$$

where $\Gamma_i = \Gamma(Q(\zeta^i + \zeta^{-i}):Q)$ and μ is the Möbius function.

Recall that μ function defined by:

$$\mu(n) = \begin{cases} 1 & if\ n=1 \\ 0 & if\ a^2/n\ for\ some\ a>1 \\ (-1)^K & if\ n = p_1 p_2 \cdots p_K,\ p_i\ are\ distinct\ primes. \end{cases}$$

See [9]

PROOF: See [2].

LEMMA (3.2.12):

Let ζ be a primitive n-th root of unity, $i \in$ Z and d_i = (i,n). and let n ≠ d_i, 2d_i.Let Γ = Γ(Q(ζ + ζ^{-1}):Q). Then

$$\sum_{\alpha \in \Gamma} (\xi^i + \xi^{-i})^{\alpha} = \frac{\Phi(n)}{\Phi(\frac{n}{d_i})} \mu(\frac{n}{d_i}).$$

PROOF: See [2].

Corollary (3.2.13):

Let ζ be a primitive n-th root of unity and $1 \le i \le \frac{n}{2}$. Let

Γ = Γ(Q(ζ + ζ^{-1}):Q). Then

$$\sum_{\alpha \in \Gamma} (\xi^i + \xi^{-i})^{\alpha} = \frac{\Phi(n)}{\Phi(\frac{n}{d_i})} \mu(\frac{n}{d_i}).$$

where d_i = (i,n).

PROOF: See [2].

LEMMA (3.2.14):

Let χ be a rational valued character of G and let x, y ϵ G with

$\langle x \rangle = \langle y \rangle$. Then χ($x$) = χ($y$).

PROOF: See [4].

LEMMA (3.2.15):

Let G = SL(2,p^k), where p is an odd prime. Then $< c > = < d >$ if and only if k is odd.

PROOF: See [2].

NOTATION:

Let G = SL(2,p^k) for some prime $p \neq 2$, e and e' denote divisors of $p^k - 1$ such that $e < \frac{p^k-1}{2}$ and $e' < \frac{p^k-1}{2}$, f and f' denote divisors of $p^k + 1$ such that $f < \frac{p^k+1}{2}$ and $f' < \frac{p^k+1}{2}$, ρ_e is a primitive $(\frac{p^k-1}{e})$-th root of unity, σ_f is a primitive $(\frac{p^k+1}{f})$-th root of unity, 1, z, c, d, a, b are as in theorem (2.2.3), ε, ρ and σ are as in theorem (2.3.1).

$$B(k)=\begin{cases}1 & \text{if } k \text{ is even}\\ 2 & \text{otherwise}\end{cases} \qquad E(p^k)=\begin{cases}1 & \text{if } p^k \equiv 3 \bmod 4\\ 2 & \text{otherwise}\end{cases}$$

$$A(e)=\frac{1}{2}\Phi(\frac{p^k-1}{e}) \qquad C(f)=\frac{1}{2}\Phi(\frac{p^k+1}{f})$$

$$\tau_1(e,e') = \sum_{\alpha\in\Gamma}(\rho_e^{e'}+\rho_e^{-e'})^{\alpha} = \frac{\Phi(\frac{p^k-1}{e})}{\Phi(\frac{p^k-1}{ee'})}\,\mu(\frac{p^k-1}{ee'})$$

by lemma (3.2.11) where $\Gamma = \Gamma(Q(\chi_e):Q)$. [Note that $\Gamma = \Gamma(Q(\rho_e + \rho_e^{-1}):Q)$].

$$\tau_2(f,f') = \sum_{\alpha\in\Gamma_1}(\sigma_f^{\ f'} + \sigma_f^{\ -f'})^{\alpha} = \frac{\Phi(\frac{p^k+1}{f})}{\Phi(\frac{p^k+1}{f\,f'})}\,\mu(\frac{p^k+1}{f\,f'})$$

where $\Gamma_1 = \Gamma(Q(\theta_f):Q)$. [Note that $\Gamma = \Gamma(Q(\sigma_f + \sigma_f^{-1}):Q)$].

χ_i, θ_j are irreducible characters of G as in theorem (2.3.1). Then $\sum_{\alpha\in\Gamma}\chi_i^{\alpha}$, where $\Gamma = \Gamma(Q(\chi_i):Q)$, and $\sum_{\alpha\in\Gamma_1}\theta_j^{\alpha}$, where $\Gamma_1 = \Gamma(Q(\theta_j):Q)$, are rational valued characters of G.

$\chi_e = B(e)\sum_{\alpha\in\Gamma}\chi_i^{\alpha}$ where $e = (i,p^k - 1)$.

$\theta_j = B(f)\sum_{\alpha\in\Gamma_1}\theta_j^{\alpha}$ where $f = (i,p^k + 1)$.

ξ' and η' denote the irreducible characters of the rational representations of G arising from ξ_1 (or ξ_2) and η_1 (or η_2) respectively where k is odd.

Also we know that the column for the class zc is obtained from the relation $\chi(zc) = \frac{\chi(z)}{\chi(1)}\chi(c)$ where χ is an irreducible character of G.

So the character table of rational representations of SL(2,p^k), p an odd prime, k odd is described in table (3.3). See [2].

Cg	**1**	***z***	***c***	***zc***	$a^{e'}$	$b^{f'}$
| Cg |	**1**	***1***	$(p^{2k}-1)/2$	$(p^{2k}-1)/2$	$p^{k}(p^{k}+1)$	$p^{k}(p^{k}-1)$
| CG(g) |	$p^{k}(p^{2k}-1)$	$p^{k}(p^{2k}-1)$	$2p^{k}$	$2p^{k}$	$p^{k}-1$	$p^{k}+1$
1_G	1	1	1	1	1	1
ψ	p^{k}	p^{k}	0	0	1	-1
χ_e	$(p^{k}+1)A(e)B(e)$	$(-1)^{e}(p^{k}+1)A(e)B(e)$	$A(e)B(e)$	$(-1)^{e}A(e)B(e)$	$B(e)\tau_1(e,e')$	0
θ_f	$(p^{k}-1)C(f)B(f)$	$(-1)^{f}(p^{k}-1)C(f)B(f)$	$-C(f)B(f)$	$-(-1)^{f}C(f)B(f)$	0	$-B(e)\tau_2(f,f')$
$\zeta_1+\zeta_2$	$(p^{k}+1)$	$\varepsilon(p^{k}+1)$	1	ε	$(-1)^{e'}2$	0
$\eta_1+\eta_1$	$(p^{k}-1)E(p^{k})$	$-\varepsilon(p^{k}-1)E(p^{k})$	-1	ε	0	$(-1)^{f'+1}2E(p^{k})$

TABLE (3.3)

<u>EXAMPLE (3.1)</u>:

To find the character table of rational representations of SL(2,5). *e* and *e´* are divisors of 4 which are 1, 2, 4, such that $e < 2$, and $e' < 2$, so $\boldsymbol{e = e' = 1}$, *f* and *f ´* are divisors of 6 which are 1, 2, 3, 6, such that $f < 3$, and $f' < 3$, so

$\boldsymbol{f = f' = 1, 2}$,

$\boldsymbol{\varepsilon = 1}$, from that:

the conjugacy classes are **1, *z*, *c*, *zc*, *a*, *b*,** and $\boldsymbol{b^2}$

the characters are $\mathbf{1_G}$, **ψ**, $\boldsymbol{\chi_1}$, $\boldsymbol{\theta_1}$, $\boldsymbol{\theta_2}$, $\boldsymbol{\xi_1 + \xi_2 = \xi'}$, and $\boldsymbol{\eta_1 + \eta_2 = \eta'}$

$$A(1) = \frac{1}{2}\Phi(4) = 1,\ B(1) = 2,\ \tau_1(1,1) = \frac{\Phi(4)}{\Phi(4)}\mu(4) = 0,\ C(1) = 1,$$

$$\tau_2(1,1) = \frac{\Phi(6)}{\Phi(6)}\mu(6) = 1,\ \tau_2(1,2) = \frac{\Phi(6)}{\Phi(\frac{6}{2})}\mu(\frac{6}{2}) = \frac{6}{2}(-1) = -1,\ C(2) = \frac{1}{2}\Phi(\frac{6}{2}) = \frac{1}{2}(2) = 1,$$

$B(2) = 1$, $\tau_2(2,1) = \frac{\Phi(3)}{\Phi(3)}\mu(3) = -1$, we find $\tau_2(2,2)$ by (lemma 3.2.12) and

(corollary 3.2.13) so $\tau_2(2,1) = \ = \frac{\Phi(3)}{\Phi(3)}\mu(3) = -1,\ E(5) = 2.$

by Schur indices(theorem 3.2.1) we divide $\boldsymbol{\chi_1}$, $\boldsymbol{\theta_1}$,and **η´** by 2, and others by 1

The character table of rational representation of SL(2,5) is:

C_g	1	*z*	*c*	*zc*	*a*	*b*	b^2
$\lvert C_g \rvert$	1	1	12	12	30	20	20
$\lvert C_G(g) \rvert$	120	120	10	10	4	6	6
1_G	1	1	1	1	1	1	1
ψ	5	5	0	0	1	-1	-1
χ_1	6	-6	1	-1	0	0	0
θ_1	4	-4	-1	1	0	-1	1
θ_2	4	4	-1	-1	0	1	1
ξ´	6	6	1	1	-2	0	0
η´	4	-4	-1	1	0	2	-2

3.3. THE CYCLIC DECOMPOSITION OF K(SL(2,*p*)) WHERE *p* = 3, 5, 7, 11, 13, 17, AND 19

In this section we will introduce the diagonalization of the matrix (≡*SL(2,*p*)) by row and column operations with the condition (when we multiply row or column by a number the number must be integer).which gives us the cyclic decomposition of K(SL(2,*p*)), where *p* = 3, 5, 7, 11, 13, 17, and 19.

If we suppose that the diagonalization of the matrix (≡*SL(2,*p*)) is

$$\begin{pmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & m_n \end{pmatrix}$$

Then the cyclic decomposition for the group K(SL(2,*p*)) is:

$$K(SL(2,p)) = Z_{m_1} \oplus Z_{m_2} \oplus Z_{m_3} \oplus \ldots \oplus Z_{m_n}$$

EXAMPLE (3.2):

To find the cyclic decomposition of the group K(SL(2,3)).

The character table of rational representations of SL(2,3) is :

C_g	1	*z*	*c*	*zc*	*b*
$\lvert C_g \rvert$	**1**	**1**	**4**	**4**	**6**
$\lvert C_G(g) \rvert$	**24**	**24**	**6**	**6**	**4**
1_G	1	1	1	1	1
ψ	3	3	0	0	-1
θ_1	2	-2	-1	1	0
$\xi_1 + \xi_2$	4	-4	1	-1	0
$\eta_1 + \eta_2$	2	2	-1	-1	2

The diagonalization of the matrix (≡*SL(2,3)) is:

$$\begin{pmatrix} 24 & 0 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Then the cyclic decomposition for the group K(SL(2,3)) is:

$$K(SL(2,3)) = Z_{24} \oplus Z_6 \oplus Z_2 \oplus Z_1 \oplus Z_1$$

<u>EXAMPLE (3.3)</u>:

To find the cyclic decomposition of the group K(SL(2,5)).

The character table of rational representations of SL(2,5) is :

C_g	1	*z*	*c*	*zc*	*a*	*b*	b^2
$\|C_g\|$	1	1	12	12	30	20	20
$\|C_G(g)\|$	120	120	10	10	4	6	6
1_G	1	1	1	1	1	1	1
ψ	5	5	0	0	1	-1	-1
χ_1	6	-6	1	-1	0	0	0
θ_1	4	-4	-1	1	0	-1	1
θ_2	4	4	-1	-1	0	1	1
ξ'	6	6	1	1	-2	0	0
η'	4	-4	-1	1	0	2	-2

The diagonalization of the matrix (≡*SL(2,5)) is:

$$\begin{pmatrix} 120 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -30 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Then the cyclic decomposition for the group K(SL(2,5)) is:

$$K(SL(2,5)) = Z_{120} \oplus Z_{30} \oplus Z_2 \oplus Z_1 \oplus Z_1 \oplus Z_1 \oplus Z_2$$

<u>EXAMPLE (3.4)</u>:

To find the cyclic decomposition of the group K(SL(2,7)).

The character table of rational representations of SL(2,7) is :

C_g	**1**	z	c	zc	a	a^2	b	b^2
$\lvert C_g \rvert$	**1**	**1**	**24**	**24**	**56**	**56**	**42**	**42**
$\lvert C_G(g) \rvert$	**336**	**336**	**14**	**14**	**6**	**6**	**8**	**8**
1_G	1	1	1	1	1	1	1	1
ψ	7	7	0	0	1	1	-1	-1
χ_1	8	-8	1	-1	1	-1	0	0
χ_2	8	8	1	1	-1	-1	0	0
θ_1	12	-12	-2	2	0	0	0	0
θ_2	6	6	-1	-1	0	0	0	2
$\xi_1 + \xi_2$	8	-8	1	-1	-2	2	0	0
$\eta_1 + \eta_2$	6	6	-1	-1	0	0	2	-2

The diagonalization of the matrix (≡*SL(2,7)) is:

$$\begin{pmatrix} 336 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -84 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

Then the cyclic decomposition for the group K(SL(2,7)) is:

$$K(SL(2,7)) = Z_{336} \oplus Z_{84} \oplus Z_{2} \oplus Z_{1} \oplus Z_{1} \oplus Z_{2} \oplus Z_{1} \oplus Z_{2}$$

EXAMPLE (3.5):

To find the cyclic decomposition of the group K(SL(2,11)).

The character table of rational representations of SL(2,11) is :

C_g	1	z	c	zc	a	a^2	b	b^2	b^3	b^4
$\vert C_g \vert$	1	1	60	60	132	132	110	110	110	110
$\vert C_G(g) \vert$	1320	1320	22	22	10	10	12	12	12	12
1_G	1	1	1	1	1	1	1	1	1	1
ψ	11	11	0	0	1	1	-1	-1	-1	-1
χ_1	24	-24	2	-2	1	-1	0	0	0	0
χ_2	24	24	2	2	-1	-1	0	0	0	0
θ_1	20	-20	-2	2	0	0	0	-2	0	2
θ_2	10	10	-1	-1	0	0	-1	1	2	1
θ_3	10	-10	-1	1	0	0	0	2	0	-2
θ_4	10	10	-1	-1	0	0	1	1	-2	1
$\xi_1 + \xi_2$	12	-12	1	-1	-2	2	0	0	0	0
$\eta_1 + \eta_2$	10	10	-1	-1	0	0	2	-2	2	-2

The diagonalization of the matrix (≡*SL(2,11)) is:

$$\begin{pmatrix} 1320 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -330 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Then the cyclic decomposition for the group K(SL(2,11)) is:

$$K(SL(2,11)) = Z_{1320} \oplus Z_{330} \oplus Z_2 \oplus Z_1 \oplus Z_1 \oplus Z_2 \oplus Z_2 \oplus Z_6 \oplus Z_2 \oplus Z_1$$

<u>EXAMPLE (3.6)</u>:

To find the cyclic decomposition of the group K(SL(2,13)).

The character table of rational representations of SL(2,13) is :

C_g	**1**	***z***	***c***	***zc***	***a***	a^2	a^3	a^4	***b***	b^2
$\lvert C_g \rvert$	1	1	84	84	182	182	182	182	156	156
$\lvert C_G(g) \rvert$	2184	2184	26	26	12	12	12	12	14	14
1_G	1	1	1	1	1	1	1	1	1	1
ψ	13	13	0	0	1	1	1	1	-1	-1
χ_1	28	-28	2	-2	0	2	0	-2	0	0
χ_2	14	14	1	1	1	-1	-2	-1	0	0
χ_3	14	-14	1	-1	0	-2	0	2	0	0
χ_4	14	14	1	1	-1	-1	2	-1	0	0
θ_1	36	-36	-3	3	0	0	0	0	-1	1
θ_2	36	36	-3	-3	0	0	0	0	1	1
$\xi_1 + \xi_2$	14	14	1	1	-2	2	-2	2	0	0
$\eta_1 + \eta_2$	12	-12	-1	1	0	0	0	0	2	-2

The diagonalization of the matrix (≡*SL(2,13)) is:

$$\begin{pmatrix} 2184 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -546 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Then the cyclic decomposition for the group K(SL(2,13)) is:

$$K(SL(2,13)) = Z_{2184} \oplus Z_{546} \oplus Z_1 \oplus Z_1 \oplus Z_1 \oplus Z_2 \oplus Z_6 \oplus Z_4 \oplus Z_2 \oplus Z_1$$

EXAMPLE (3.7):

To find the cyclic decomposition of the group K(SL(2,17)).

The character table of rational representations of SL(2,17) is :

C_g	1	z	c	zc	a	a^2	a^4	b	b^2	b^3	b^6
$\mid C_g \mid$	1	1	144	144	306	306	306	272	272	272	272
$\mid C_G(g)\mid$	4896	4896	34	34	16	16	16	18	18	18	18
1_G	1	1	1	1	1	1	1	1	1	1	1
ψ	17	17	0	0	1	1	1	-1	-1	-1	-1
χ_1	72	-72	4	-4	0	0	0	0	0	0	0
χ_2	36	36	2	2	0	0	-4	0	0	0	0
χ_4	18	18	1	1	0	-2	2	0	0	0	0
θ_1	48	-48	-3	3	0	0	0	0	0	-3	3
θ_2	48	48	-3	-3	0	0	0	0	0	3	3
θ_3	16	-16	-1	1	0	0	0	-1	1	2	-2
θ_6	16	16	-1	-1	0	0	0	1	1	-2	-2
$\xi_1 + \xi_2$	18	18	1	1	-2	2	2	0	0	0	0
$\eta_1 + \eta_2$	16	-16	-1	1	0	0	0	2	-2	2	-2

The diagonalization of the matrix (≡*SL(2,17)) is:

$$\begin{pmatrix} 4896 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1224 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

Then the cyclic decomposition for the group K(SL(2,17)) is:

$$K(SL(2,17)) = Z_{4896} \oplus Z_{1224} \oplus Z_2 \oplus Z_1 \oplus Z_1 \oplus Z_4 \oplus Z_2 \oplus Z_1 \oplus Z_2 \oplus Z_3 \oplus Z_6$$

EXAMPLE (3.8):

To find the cyclic decomposition of the group K(SL(2,19)).

The character table of rational representations of SL(2,19) is :

C_g	**1**	*z*	*c*	*zc*	*a*	a^2	a^3	a^6	*b*	b^2	b^4
$\vert C_g \vert$	1	1	180	180	380	380	380	380	342	342	342
$\vert C_G(g)\vert$	6840	6840	38	38	18	18	18	18	20	20	20
1_G	1	1	1	1	1	1	1	1	1	1	1
ψ	19	19	0	0	1	1	1	1	-1	-1	-1
χ_1	60	-60	3	-3	0	0	3	-3	0	0	0
χ_2	60	60	3	3	0	0	-3	-3	0	0	0
χ_3	20	-20	1	-1	1	-1	-2	2	0	0	0
χ_6	20	20	1	1	-1	-1	2	2	0	0	0
θ_1	72	-72	-4	4	0	0	0	0	0	-2	2
θ_2	72	72	-4	-4	0	0	0	0	0	2	2
θ_4	18	-18	-1	1	0	0	0	0	0	2	-2
$\xi_1 + \xi_2$	20	-20	1	-1	-2	2	-2	2	0	0	0
$\eta_1 + \eta_2$	18	18	-1	-1	0	0	0	0	2	-2	-2

The diagonalization of the matrix (≡*SL(2,19)) is:

$$\begin{pmatrix} 6840 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1710 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

Then the cyclic decomposition for the group K(SL(2,19)) is:

$$K(SL(2,19)) = Z_{6840} \oplus Z_{1710} \oplus Z_2 \oplus Z_3 \oplus Z_1 \oplus Z_6 \oplus Z_2 \oplus Z_1 \oplus Z_1 \oplus Z_2 \oplus Z_4$$

3.4. CONCLUSIONS

The diagonalization of the matrix (≡*SL(2,*p*)) is

$$\begin{pmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & m_n \end{pmatrix}$$

Also the cyclic decomposition for the group K(SL(2,*p*)) is:

$$K(SL(2,p)) = Z_{m_1} \oplus Z_{m_2} \oplus Z_{m_3} \oplus \ldots \oplus Z_{m_n}$$

This as result some conclusions can be considered:

1. m_1 = | SL(2,*p*) |.
2. m_2 = | SL(2,*p*) | / 4.
3. We don't find a similar approach to other elements.

3.5. SUGGESTIONS FOR FUTURE WORK

Based on the present work, the following topics are put forword for future work.

1. Generalize the cyclic decomposition of the group SL(2,p), where p is an odd prime ($p \neq 2$).

2. Find the cyclic decomposition of the group SL(2,p^2), where p is an odd prime ($p \neq 2$).

3. Find the cyclic decomposition of the group SL(2,p^k), where p is an odd prime($p \neq 2$), k is even.

4. Find the cyclic decomposition of the group SL(2,2^n).

5. Find the cyclic decomposition of the group PSL(2,p^k), $p \neq 2$.

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المستخلص

ان مجموعة كل دوال الصفوف ذات القيم الصحيحة للزمرة المنتهية G تكون زمرة ابدالية مع الجمع النقطي ويرمز لها بالرمز cf(G,Z) وهنالك زمرة جزئية من تلك الزمرة هي زمرة الشواخص العمومية ذات القيم الصحيحة للزمرة G ويرمز لها بالرمز R(G).

ان مسألة ايجاد التجزئة الدائرية لزمرة القسمة

$$K(G) = cf(G,Z) / R(G)$$

قد اعتبرت في هذه الرسالة للزمرة المنتهية الخطية الخاصة SL(2,p) عندما

$P = 3, 5, 7, 11, 13, 17, 19$

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نتائج حول الزمرة الكسرية

CF(G,Z) / R(G)

رســـــالة

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وهي جزء من متطلبات نيل درجة ماجستير علوم
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نيران صباح جاسم

بإشــــراف

الاستاذ الدكتور محمد سردار اسماعيل قيردار

صفر **اذار**

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