-71 - COMPARING MAXIMUM LIKELIHOOD AND LEAST SQUARES AND MOMENTS METHOD FOR TAS DISTRIBUTION

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COMPARING MAXIMUM LIKELIHOOD AND LEAST SQUARES AND MOMENTS METHOD FOR TAS DISTRIBUTION

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ABSTRACT

This paperdeals with the newly developed distribution which called Transmuted Arcsine Distribution (TAS) for Salma and Arwa (2018). Classical estimation methods such as Maximum Likelihood (ML), Least Squares (LS) and the Moments Method (MOM) are used to estimate the unknown parameter and the reliability function. Moreover, the comparing between the classical methods are presented in terms of estimated quality using MSE of the estimator and MSE of the reliability function. Finally, three real data sets are analyzed to illustratesthe proposed methods.

Keywords: *ArcSine Distribution; Maximum Likelihood Method; Least Square Methods; Moments Method; Reliability Function; Estimated of Parameters.*

1. INTRODUCTION

ArcSine distribution is an extensive model of data analysis for multiple situations and it is of great importance in probabilistic applications. It can be used effectively in the field of research to study the behavior of random variables, genetics, statistical communication theory, in economic modeling and genetic divisions.Salma &Arwa (2018) developed the distribution of Arcsine to another new distribution using the proposed QRTM method of Shaw & Buckley(2007). They called it the Transmuted Arcsine (TAS) distribution. The cumulative distribution function F(x) and the probability density function f(x) of the TAS distribution are given by:

$$F(x) = \left[\left(\frac{\pi + 2\arcsin x}{2\pi}\right)\right]\left[1 + \lambda\left(\frac{\pi - 2\arcsin x}{2\pi}\right)\right], -1 \leq x \leq 1, \quad \lambda > 0 \quad (1)$$

$$f(x) = \frac{1}{\pi\sqrt{1-x^2}} \left(1 - \frac{2\lambda}{\pi}\arcsin x\right), -1 \leq x \leq 1, \lambda > 0 \qquad (2)$$

Where$\lambda > 0$is the shape parameter for the TAS distribution?The reliability function$RF_{TAS}(x)$of the TAS distribution is defined as follows:

$$\boldsymbol{RF_{TAS}(x) = \left[\frac{\pi - 2\arcsin x}{2\pi}\right]\left[1 - \lambda\left(\frac{\pi + 2\arcsin x}{2\pi}\right)\right]}$$

2. MAXIMUM LIKELIHOOD METHOD

The ML method is the most important and widely used estimation method in statistical estimates of the probability distribution. It is a simple method used to find the estimation of unknown parameters $\underline{\boldsymbol{\lambda}} = (\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2}, \dots \dots .. \boldsymbol{\lambda_k})$, which makes the ML function at the maximum limit. The maximum likelihood estimation (MLE) of the ML method is efficient and has a less contrasting property, as well as a very important characteristic of stability and consistency. Moreover, this method is more accurate than other estimation methods, especially when increasing the sample size.If$\underline{\boldsymbol{x}} = (\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n})$ a random sample selected from a probability distribution $\boldsymbol{f}(\boldsymbol{x}; \underline{\boldsymbol{\lambda}})$, [1], then the joint distribution function of this sampleis

$$\boldsymbol{f(\underline{x}; \underline{\lambda}) = f(x_1, x_2, \dots \dots . x_n, \underline{\lambda}) = \prod_{i=1}^{n} f(x_i ; \underline{\lambda})}$$

Then the ML function is written as follows:

$$L = \prod_{i=1}^{n} f(x,\lambda) = \frac{1}{\pi^n}\prod_{i=1}^{n}\left\{(1+x_i^2)^{\frac{-1}{2}}\left(1-\frac{2\lambda}{\pi}\arcsin x_i\right)\right\} \qquad (4)$$

From equation (4), the log-likelihood equation for estimating (λ) is given by

$$\frac{-2}{\pi}\sum_{i=1}^{n}\frac{\arcsin(x_i)}{1-\frac{2\hat{\lambda}}{\pi}\arcsin(x_i)} = 0 \qquad (5)$$

Note that equation (5) is a nonlinear equation that is difficult to solve bythe ordinary ways, so the MathCad15 program will be used to solve it.

3. LEAST SQUARES METHOD

The mathematician Karl Friedrich Gauss studied the least squares method as early as (1794), but Gauss did not publish this technique until (1809). In the meantime, the French scientist Légendre discovered this method and published it in (1806), which led to the quarrels of the French scientist Légendre with the scientist Gauss about who first discovered the method. Gauss proved that reducing the sum of squares of deviations from actual values leads to finding the best linear regression model, while Légendre succeeded in linking the smaller squares method with the principles of probabilities. This method states that in any linear regression model where the probability of errors is zero, and are not related to each other and their differences are equal, the best unbiased linear estimate of the parameters can be obtained using the least squares method, [5]. The least squares method is one of the most important methods in the estimation process. It has many characteristics, including unbiased, efficiency, and linearity, and in which the distribution parameters are estimated, which makes the sum of the

square errors as small as possible.
In (1988) Swain & Wilson proposed the least squares method for estimating the unknown parameters of the Beta distribution, where $\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n}$ was a random sample with a cumulative distribution function F(x) that $\boldsymbol{x_1} < x_2 < \dots < x_n$, then

$$\hat{F}(x_i) = E[F(x_i)] = \frac{i}{n+1}, i = 1, 2, \dots, n \qquad (6)$$

Note that $\left(\frac{i}{n+1}\right)$ is a non-parametric estimation of the cumulative distribution function $F(x_i)$, so the LS method can be obtained by reducing the amount of p (λ) for the unknown parameters $\underline{\lambda}$, [5]. Where

$$p(\underline{\lambda}) = \sum_{i=1}^{n} \left[F(x_i) - \hat{F}(x_i)\right]^2 \qquad (7)$$

Now, by differential Eq.(7) with respect to (λ),the estimationof the TAS parameter (λ), is obtained as follows:

Since

$$F(x) = \left(\frac{\pi + 2\arcsin x}{2\pi}\right)\left[1 + \lambda\left(\frac{\pi - 2\arcsin x}{2\pi}\right)\right], -1 \le x \le 1, \quad \lambda > 0$$

And from equation (6), we have

$$\hat{F}(x_i) = E[F(x_i)] = \frac{i}{n+1}$$

And so, the

$$p(\lambda) = \sum_{i=1}^{n} \left\{ \left(\frac{\pi + 2\arcsin x_i}{2\pi} \right) \left[1 + \lambda \left(\frac{\pi - 2\arcsin x_i}{2\pi} \right) \right] - \frac{i}{n+1} \right\}^2 \quad (8)$$

Bydifferentialthe Eq.(8) with respect to the parameter (λ), we get

$$\sum_{i=1}^{n} \left[\lambda \left(\frac{1}{4} - \frac{arcsinx_i^2}{\pi^2} \right)^2 - \left(\frac{i}{n+1} - \left(\frac{1}{2} + \frac{\arcsin x_i}{\pi} \right) \right) \left(\frac{1}{4} - \frac{arcsinx_i^2}{\pi^2} \right) \right] = 0$$

Therefore

$$\lambda^o = \frac{\sum_{i=1}^{n} \left(\frac{1}{4} - \frac{arcsinx_i^2}{\pi^2} \right) \left\{ \frac{i}{n+1} - \left(\frac{1}{2} + \frac{\arcsin x_i}{\pi} \right) \right\}}{\sum_{i=1}^{n} \left(\frac{1}{4} - \frac{arcsinx_i^2}{\pi^2} \right)^2}$$

Thus(λ^o) o is the LS parameter of the unknown parameter (λ).

4. MOMENTSM ETHOD

Historically, it is one of the oldest and most commonly used methods of estimation, which has been known by Carl Pearson since (1894). This method often provides the estimates of the unknown parameters when other methods fail to find them or when they are difficult to obtain as in the distribution of Gamma. However, when using this method, it is possible that the best estimators are not available with the required efficiency or in some cases may not be used when the required moments are not present such as the distribution of Cauchy.If $x_1, x_2, \ldots, x_n$is an independent identically random sample, then theunknown parameters can be estimated using the inverse method by equating the population moments $M_k$with the corresponding samplemoments m_k, [5].

Where$M_k = E(X^k)$ and $m_k = \frac{1}{n}\sum_{i=1}^{n} {x_i}^k$, $k = 1, 2, \ldots$

Themoments' estimatorof the unknown parameter $(\lambda)$of the TAS distribution can be obtained as follows:

Since $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\mu = \frac{4\lambda}{\pi^2} = \bar{X}$, then$\tilde{\lambda} = \frac{\bar{X}\pi^2}{4}$

Thus, $\tilde{\lambda}$ is the moments' estimatorof the unknown parameter $(\lambda)$of the TAS distribution.

5. PRACTICAL APPLICATION

In this section, the methodology discussed in the previous sections will be applied using three sets of real data. The results of the practical application were obtained based on a program written in the Math language (MathCad15).The first data represent the failure times of the air-conditioning system (days) and presented by Linhart& Zucchini (1986):

0.87 0.46 0.46 0.46 0.5 0.58 0.58 0.58 0.67 0.67 0.83

The second data represents the life of the epileptic kylver threads (in days)and presented by Tipagornet. al. (2015):

0.79 0.6 0.35 0.12 0.06 0.79 0.6 0.34 0.1 0.05

Kylver Epoxy is a synthetic fiber of the Para-Aramid, strong and light, resembling aramid fibers such as Nomex and Technora. It is used as an alternative to steel in racing car tires. Kevlar fibers are used in the manufacture of various applications, for example: bicycle tires, to strengthen the structures of objects, because of their distinctive characteristics compared to their weight, which is five times stronger than steel with the same weight.Third data represent the time of survival of eleven infected with toxic tubercles (month)andreported by Haq (2016):

0.03 0.03 0.03 0.0 0.02 0.02 0.02 0.01 0.01 0.00 0.00
8 2 1 3 9 5 3 8 4 4 3

TABLE 1: THE RESULTS OF THE K-S TEST

Data Sets	calculated K-S values	calibrated values
1	0.4670	0.4677
2	0.4870	0.4889
3	0.4542	0.4889

The Kolmogrov-Smirnov test was used to assess the goodness fit of the data sets, and the results in Table.1 shows that all the data sets follow to the TAS distribution.

TABLE 2: ESTIMATION OF THE RELIABILITY FUNCTION OF THE DATA SETS

1st DATA				2nd Data				3rd Data			
λ_0	$\hat{\lambda}$	(t)	R(t)	λ_0	$\hat{\lambda}$	(t)	R(t)	λ_0	$\hat{\lambda}$	(t)	R(t)
1	1.004	3.5	0.265	0.36	0.369	3.5	0.424	0.30	0.31	3.5	0.438
		11.5	0.255			11.5	0.414			11.5	0.429
		32.5	0.243			32.5	0.401			32.5	0.416
		44.29	0.231			44.29	0.39			44.29	0.404
1.25	1.249	3.5	0.204	0.761	0.764	3.5	0.325	0.5	0.657	3.5	0.352
		11.5	0.194			11.5	0.315			11.5	0.342
		32.5	0.182			32.5	0.303			32.5	0.329
		44.29	0.17			44.29	0.291			44.29	0.318
0.9	0.830	3.5	0.291	0.523	0.369	3.5	0.424	1.5	1.385	3.5	0.17
		11.5	0.281			11.5	0.414			11.5	0.16
		32.5	0.269			32.5	0.401			32.5	0.148
		44.29	0.257			44.29	0.39			44.29	0.136
1.15	1.249	3.5	0.204	0.62	0.764	3.5	0.325	1.7	0.657	3.5	0.352
		11.5	0.194			11.5	0.315			11.5	0.342
		32.5	0.182			32.5	0.303			32.5	0.329
		44.29	0.17			44.29	0.291			44.29	0.318
1.32	1.344	3.5	0.18	0.123	0.124	3.5	0.485	1.15	0.657	3.5	0.352
		11.5	0.17			11.5	0.475			11.5	0.342
		32.5	0.158			32.5	0.463			32.5	0.329
		44.29	0.146			44.29	0.451			44.29	0.318

The results in Table2 indicates that the reliability function decreases when the mission time (t) increases. As we know that no unit can survive forever, its age will decrease over time. To compare the estimation of the proposed methods, and for the purpose of reaching the best estimate, the mean square error (MSE) of the parameter and the mean squareerror $\left[MSE\hat{R}(x)\right]$ of the reliability functionwas used, [7].

Where $MSE = \frac{\sum_{i=1}^{n}(\lambda_i-\hat{\lambda})^2}{n}$ and $MSE\hat{R}(x) = \sum_{i=1}^{n}\frac{[R(x_i)-\hat{R}(x_i)]^2}{n}$

TABLE 3: THE MSEOF THEML ESTIMATORS

	$\boldsymbol{\lambda_0}$	$\hat{\boldsymbol{\lambda}}$	$\mathbf{MSE\hat{R}(x)}$	MSE
1st Data	0.9	0.830	9.47210E-5	4.86110E-5
	1.02	1.004	4.9410E-6	2.53510E-4
	1.15	1.249	1.91910E-4	9.84910E-3
	1.20	1.249	1.91910E-4	9.84910E-3
	1.32	1.344	1.15310E-5	5.91710E-4
2nd Data	0.123	0.124	1.04410E-7	1.70710E-6
	0.36	0.369	4.74210E-6	7.75210E-5
	0.523	0.369	1.45410E-3	0.024
	0.62	0.764	1.26210E-3	0.021
	0.761	0.764	4.210E-7	6.86510E-6
3rd Data	0.30	0.31	1.23810E-4	1.02610E-4
	0.50	0.642	0.024	0.02
	1.15	0.31	0.851	0.705
	1.50	1.38	0.017	0.014
	1.70	0.624	1.351	1.119

TABLE 4: THE MSEOF THELS ESTIMATORS

	$\boldsymbol{\lambda_0}$	$\tilde{\boldsymbol{\lambda}}$	$\mathbf{MSE\hat{R}(x)}$	MSE
1st Data	0.90	4.90410E3	4.68510E5	2.40410E7
	1.02	4.90410E3	4.68510E5	2.40410E7
	1.15	4.90410E3	4.68410E5	2.40410E7
	1.20	4.90410E3	4.68410E5	2.40410E7
	1.32	4.90410E3	4.68410E5	2.40410E7
2nd Data	0.123	3.07810E3	0.682	3.92210E-3
	0.360	3.07810E3	0.63	0.03
	0.523	3.07810E3	0.599	0.114
	0.620	3.07810E3	0.582	0.189
	0.761	3.07810E3	0.559	0.331

3rd Data	0.30	181.882	0.244	0.202
	0.50	181.882	0.075	0.062
	1.15	181.882	0.194	0.161
	1.50	181.882	0.68	0.563
	1.70	181.882	1.091	0.904

TABLE 5: THE MSEOF THEMOM ESTIMATORS

	λ_0	λ_{EM}	$\mathbf{MSE}\widehat{\mathbf{R}}(\mathbf{x})$	MSE
1st Data	0.90	0.225	8.87810E-3	0.456
	1.02	0.225	0.011	0.585
	1.15	0.287	0.014	0.744
	1.20	0.287	0.014	0.744
	1.32	0.33	0.019	0.98
2nd Data	0.123	0.04	2.41110E9	9.47310E6
	0.360	0.09	2.41110E9	9.47210E6
	0.523	0.131	2.41110E9	9.47110E6
	0.620	0.155	2.41110E9	9.4710E6
	0.761	0.19	2.41110E9	9.46910E6
3rd Data	0.30	0.013	5.03110E11	3.29710E4
	0.50	0.013	5.03110E11	3.2910E4
	1.15	0.013	5.03110E11	3.26610E4
	1.50	0.013	5.03110E11	3.25410E4
	1.70	0.013	5.03110E11	3.24710E4

From the results in the previous tables, we note that the ML method is the best method to estimate the parameter of the TAS distribution, where it was found that their estimates have the smallest MSE and this tends to the convergence.

6. CONCLUSION

In this paper, the unknown parameter of the TAS distribution was estimated using the ML, LS, and MOM method. The comparing between the classical methods were presented in terms of estimated quality using MSE of the estimator and MSE of the reliability function. Three sets of real data were dealt with for practical purposes, and the results showed that ML method gave a good estimate of the shape parameter with good characteristics compared to LS method and the MOM method.

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