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Wolfe E-duality for E-differentiable E-invex vector optimization problems with inequality and equality constraints

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***Abstract*—In this paper, the class of E-differentiable vector optimization problems with both inequality and equality constraints is considered. The so-called vector Wolfe E-dual problem is defined for the considered E-differentiable multiobjective programming problem with both inequality and equality constraints and several E-dual theorems are established under (generalized) E-invexity hypotheses.**

***Index Terms*—E-invex set, E-invex function, E-differentiable function, Wolfe E-duality.**

AMS Classification: 90C26, 90C30, 90C46, 90C47.

I. Introduction

In the classical theory of duality, the theorems on duality in various senses are based on convexity assumptions. Many attempts have been made to weaken these assumptions by introducing various generalized convexity concepts. One of such important generalizations of the convexity notion is the concept of invexity introduced by Hanson [14]. In the case of differentiable scalar optimization problems. Namely, Hanson showed that, instead of the usual convexity assumption, if all functions are assumed to be invex (with respect to the same function η), then the sufficient optimality conditions and weak duality can be proved. Jeyakumar and Mond [15] generalized Hanson's definition to the vectorial case. They defined V-invexity of differentiable vector-valued functions which preserve the sufficient optimality conditions and duality results as in the scalar case and avoid the major difficulty of verifying that the inequality holds for the same function η for invex functions in multiobjective programming problems. Ben-Isreal and Mond [6] have defined quasi-invex function as a generalization of invex functions. Luc and Malivert [17] have extended the study of invexity to set-valued maps and vector optimization problems with set-valued data. Bazaraa et al. [7] have studied necessary conditions for optimality in a nonlinear vector optimization problem. Jeyakumar [16] defined generalized invexity for nonsmooth scalar-valued functions, established an equivalence of saddle points and optima, and studied duality results for nonsmooth problems. The concept of invexity for multiobjective nonlinear programming problems have been introduced and studied extensively in the literature (see, for example, [6], [9], [10], [13], [14], [17], and others).

Recently, the concepts of E-convex sets and E-convex functions were introduced by Youness [22]. This kind of generalized convexity is based on the effect of an operator $E : R^n \to R^n$ on the sets and the domains of functions. However, some results and proofs presented by Youness [22] were incorrect as it was pointed out by Yang [21]. Further, Megahed et al. [19] presented the concept of an E-differentiable convex function which transforms a (not necessarily) differentiable convex function to a differentiable function based on the effect of an operator $E : R^n \to R^n$.

Later, Abdulaleem [1] introduced a new concept of generalized convexity as a generalization of the notion of E-differentiable E-convexity. Namely, he defined the concept of E-differentiable E-invexity in the case of (not necessarily) differentiable vector optimization problems with E-differentiable functions.

In this paper, a class of nonconvex E-differentiable vector optimization problems with both inequality and equality constraints is considered in which the involved functions are E-invex. For such a (not necessarily differentiable) multiobjective programming problem, its Wolfe vector E-dual problem is defined. Then, several Wolfe E-duality results are established between the considered E-differentiable multicriteria optimization problem and its vector E-dual under appropriate E-invexity hypotheses.

II. Preliminaries

Let R^n be the n-dimensional Euclidean space and R^n_+ be its nonnegative orthant. The following convention for equalities and inequalities will be used in the paper. For any vectors $x = (x_1, x_2, ..., x_n)^T$ and $y = (y_1, y_2, ..., y_n)^T$ in R^n, we define:

1) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, ..., n$;
2) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, ..., n$;
3) $x \geqq y$ if and only if $x_i \geqq y_i$ for all $i = 1, 2, ..., n$;
4) $x \geq y$ if and only if $x \geqq y$ and $x \neq y$.

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Definition 1: [1] Let $E : R^n \to R^n$. A set $M \subseteq R^n$ is said to be an E-invex set if and only if there exists a vector-valued function $\eta : M \times M \to R^n$ such that the relation

$$E(u) + \lambda\eta(E(x), E(u)) \in M$$

holds for all $x, u \in M$ and any $\lambda \in [0, 1]$.

Remark 2: If η is a vector-valued function defined by $\eta(z, y) = z - y$, then the definition of an E-invex set reduces to the definition of an E-convex set (see Youness [9]).

Remark 3: If $E(a) \equiv a$, then the definition of an E-invex set with respect to the function η reduces to the definition of an invex set with respect to η (see Mohan and Neogy [22]).

Definition 4: [8] Let $E : R^n \to R^n$ and $f : M \to R$ be a (not necessarily) differentiable function at a given point $u \in M$. It is said that f is an E-differentiable function at u if and only if $f \circ E$ is a differentiable function at u (in the usual sense) and, moreover,

$$(f \circ E)(x) = (f \circ E)(u) + \nabla(f \circ E)(u)(x - u)$$
$$+\theta(u, x - u)\|x - u\|, \tag{1}$$

where $\theta(u, x - u) \to 0$ as $x \to u$.

Definition 5: [1] Let $E : R^n \to R^n$, $M \subseteq R^n$ be a nonempty open E-invex set with respect to the vector-valued function $\eta : R^n \times R^n \to R^n$ and $f : R^n \to R^k$ be an E-differentiable function on M. It is said that f is a vector-valued E-invex function with respect to η at u on M if, for all $x \in M$,

$$f_i(E(x)) - f_i(E(u)) \geqq \nabla f_i(E(u))\eta(E(x), E(u)),\ i = 1, ..., k. \tag{2}$$

If inequalities (2) hold for any $u \in M$, then f is E-invex with respect to η on M.

Remark 6: From Definition 5, there are the following special cases:

a) If f is a differentiable function and $E(x) \equiv x$ (E is an identity map), then the definition of an E-invex function reduces to the definition of an invex function introduced by Hanson [14].
b) If $\eta : R^n \times R^n \to R^n$ is defined by $\eta(x, u) = x - u$, then we obtain the definition of an E-differentiable E-convex vector-valued function introduced by Megahed et al. [8].
c) If f is differentiable, $E(x) = x$ and $\eta(x, u) = x - u$, then the definition of an E-invex function reduces to the definition of a differentiable convex vector-valued function.
d) If f is differentiable and $\eta(x, u) = x - u$, then we obtain the definition of a differentiable E-convex function introduced by Youness [9].

Definition 7: [1] Let $E : R^n \to R^n$, $M \subseteq R^n$ be an open E-invex set with respect to the vector-valued function $\eta : R^n \times R^n \to R^n$ and $f : R^n \to R^k$ be an E-differentiable function on M. It is said that f is a vector-valued strictly E-invex function with respect to η at u on M if, for all $x \in M$ with $E(x) \neq E(u)$, the inequalities

$$f_i(E(x)) - f_i(E(u)) > \nabla f_i(E(u))\eta(E(x), E(u)),\ i = 1, ..., k, \tag{3}$$

hold. If inequalities (3) are fulfilled for any $u \in M$ ($E(x) \neq E(u)$), then f is strictly E-invex with respect to η on M.

Definition 8: [1] Let $E : R^n \to R^n$, $M \subseteq R^n$ be an open E-invex set with respect to the vector-valued function $\eta : R^n \times R^n \to R^n$ and $f : R^n \to R^k$ be an E-differentiable function on M. It is said that f is a vector-valued pseudo E-invex function with respect to η at u on M if, for all $x \in M$ and $i = 1, ..., k$,

$$f_i(E(x)) < f_i(E(u)) \Longrightarrow \nabla f_i(E(u))\eta(E(x), E(u)) < 0. \tag{4}$$

If (4) holds for any $u \in M$, then f is pseudo E-invex with respect to η on M.

Definition 9: [1] Let $E : R^n \to R^n$, $M \subseteq R^n$ be an open E-invex set with respect to the vector-valued function $\eta : R^n \times R^n \to R^n$ and $f : R^n \to R^k$ be an E-differentiable function on M. It is said that f is a vector-valued quasi E-invex function with respect to η at u on M if, for all $x \in M$ and $i = 1, ..., k$,

$$f_i(E(x)) - f_i(E(u)) \leqq 0 \Rightarrow \nabla f_i(E(u))\eta(E(x), E(u)) \leqq 0. \tag{5}$$

If (5) holds for any $u \in M$, then f is quasi E-invex with respect to η on M.

In this paper, we consider the following (not necessarily differentiable) multiobjective programming problem (VP) with both inequality and equality constraints:

$$\begin{aligned} &\text{minimize } f(x) = (f_1(x), ..., f_p(x)) \\ &\text{subject to } g_j(x) \leqq 0,\ \ j \in J = \{1, ..., m\}, \\ &\qquad h_t(x) = 0,\ \ t \in T = \{1, ..., q\}, \\ &\qquad x \in R^n, \end{aligned} \tag{VP}$$

where $f_i : R^n \to R$, $i \in I = \{1, ..., p\}$, $g_j : R^n \to R$, $j \in J$, $h_t : R^n \to R$, $t \in T$, are real-valued functions defined on R^n. We shall write $g := (g_1, ..., g_m) : R^n \to R^m$ and $h := (h_1, ..., h_q) : R^n \to R^q$ for convenience.

For the purpose of simplifying our presentation, we introduce some notations which will be used frequently throughout this paper. Let

$$\Omega := \{x \in R^n : g_j(x) \leqq 0,\ \ j \in J,$$
$$h_t(x) = 0,\ \ t \in T\}$$

be the set of all feasible solutions of (VP). Further let us denote by $J(x)$ the set of inequality constraint indices that are active at a feasible solution x, that is, $J(x) = \{j \in J : g_j(x) = 0\}$.

For such multicriterion optimization problems, the following concepts of (weak) Pareto optimal solutions are defined as follows:

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Definition 10: A feasible point $\overline{x}$ is said to be a weak Pareto (weakly efficient) solution for (VP) if and only if there exists no other feasible point x such that

$$f(x) < f(\overline{x}).$$

Definition 11: A feasible point $\overline{x}$ is said to be a Pareto (efficient) solution for (VP) if and only if there exists no other feasible point x such that

$$f(x) \leq f(\overline{x}).$$

Let $E : R^n \to R^n$ be a given one-to-one and onto operator. Throughout the paper, we shall assume that the functions constituting the considered multiobjective programming problem (VP) are E-differentiable at any feasible solution.

Now, for the considered multiobjective programming problem (VP), we define its associated differentiable vector optimization problem as follows:

$$\begin{aligned} &\text{minimize } f(E(x)) = (f_1(E(x)), ..., f_p(E(x))) \\ &\text{subject to } g_j(E(x)) \leqq 0, \;\; j \in J = \{1, ..., m\}, \\ &\qquad h_t(E(x)) = 0, \;\; t \in T = \{1, ..., q\}, \\ &\qquad x \in R^n. \end{aligned} \qquad (\mathrm{VP}_E)$$

We call the problem (VP$_E$) an E-vector optimization problem associated to (VP). Let

$$\Omega_E := \{x \in R^n : g_j(E(x)) \leqq 0, \;\; j \in J, \\ h_t(E(x)) = 0, \;\; t \in T\}$$

be the set of all feasible solutions of (VP$_E$). Since the functions constituting the problem (VP) are assumed to be E-differentiable at any feasible solution of (VP), by Definition 4, the functions constituting the E-vector optimization problem (VP$_E$) are differentiable (in the usual sense) at any its feasible solution. Further, by $J_E(x)$, the set of inequality constraint indices that are active at a feasible solution x in (VP$_E$), that is, $J_E(x) = \{j \in J : (g_j \circ E)(x) = 0\}$.

Now, we give the definitions of a weak Pareto (weakly efficient) solution and a Pareto (efficient) solution of the vector optimization problem (VP$_E$), which are, at the same time, a weak E-Pareto solution (weakly E-efficient solution) and an E-Pareto solution (E-efficient solution) of the considered multiobjective programming problem (VP).

Definition 12: A feasible point $E(\overline{x})$ is said to be a weak E-Pareto solution (weakly E-efficient solution) of (VP) if and only if there exists no other feasible point $E(x)$ such that

$$f(E(x)) < f(E(\overline{x})).$$

Definition 13: A feasible point $E(\overline{x})$ is said to be an E-Pareto solution (E-efficient solution) of (VP) if and only if there exists no other feasible point $E(x)$ such that

$$f(E(x)) \leq f(E(\overline{x})).$$

Lemma 14: [3] Let $E : R^n \to R^n$ be a one-to-one and onto. Then $E(\Omega_E) = \Omega$.

Lemma 15: [3] Let $E : R^n \to R^n$ be a one-to-one and onto and $\overline{z} \in \Omega_E$ be a weak Pareto (Pareto) solution of the constrained E-vector optimization problem (VP$_E$). Then $E(\overline{z})$ is a weak E-Pareto solution (E-Pareto solution) of the considered E-differentiable multiobjective programming problem (VP).

Lemma 16: [3] Let $\overline{z} \in \Omega_E$ be a weak Pareto (Pareto) solution of the E-vector optimization problem (VP$_E$). Then $E(\overline{z})$ is a weak Pareto solution (Pareto solution) of the considered multiobjective programming problem (VP).

Definition 17: The tangent cone (also called contingent cone or Bouligand cone) of Ω_E at $\overline{x} \in cl\,\Omega_E$ is defined by

$$T_{\Omega_E}(\overline{x}) = \{d \in R^n : \exists_{\{d_n\} \subset R^n} d_n \to d, \, \exists_{\{t_n\} \subset R} t_n \downarrow 0 \\ \text{s.t. } \overline{x} + t_n d_n \in \Omega_E\}.$$

Definition 18: For the constrained E-vector optimization problem (VP$_E$), the E-linearized cone at $\overline{x} \in \Omega_E$, denoted by $L_E(\overline{x})$, is defined by

$$L_E(\overline{x}) = \{d \in R^n : \nabla g_j(E(\overline{x}))\, d \leqq 0, \, j \in J_E(\overline{x}), \\ \nabla h_t(E(\overline{x}))\, d = 0, \, t \in T\}.$$

Now, we present the E-Guignard constraint qualification which were derived for E-differentiable multiobjective programming problems with both inequality and equality constraints by Abdulaleem [1].

Definition 19: [1] It is said that the so-called E-Guignard constraint qualification (GCQ$_E$) holds at $\overline{x} \in \Omega_E$ for the differentiable constrained E-vector optimization problem (VP$_E$) with both inequality and equality constraints if

$$cl\; conv\; T_{\Omega_E}(\overline{x}) = L_E(\overline{x}). \qquad (6)$$

Now, we present the Karush-Kuhn-Tucker necessary optimality conditions for $\overline{x} \in \Omega_E$ to be a weak Pareto solution of the E-vector optimization problem (VP$_E$). These conditions are, at the same time, the E-Karush-Kuhn-Tucker necessary optimality conditions for $E(\overline{x}) \in \Omega$ to be a weak E-Pareto solution of the considered E-differentiable multiobjective programming problem (VP).

Theorem 20: [1] (E-Karush-Kuhn-Tucker necessary optimality conditions). Let $\overline{x} \in \Omega_E$ be a weak Pareto solution of the E-vector optimization problem (VP$_E$) (and, thus, $E(\overline{x})$ be a weak E-Pareto solution of the considered multiobjective programming problem (VP)). Further, f, g, h be E-differentiable at $\overline{x}$ and the E-Guignard constraint qualification be satisfied at $\overline{x}$. Then there exist Lagrange multipliers $\overline{\lambda} \in R^p$, $\overline{\mu} \in R^m$, $\overline{\xi} \in R^s$ such that

$$\sum_{k=1}^{p} \overline{\lambda}_k \nabla f_k(E(\overline{x})) + \sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(E(\overline{x})) + \sum_{t=1}^{s} \overline{\xi}_t \nabla h_t(E(\overline{x})) = 0, \qquad (7)$$

$$\overline{\mu}_j g_j(E(\overline{x})) = 0, \;\; j \in J(E(\overline{x})), \qquad (8)$$

$$\overline{\lambda} \geq 0, \, \overline{\mu} \geqq 0. \qquad (9)$$

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III. Vector Wolfe E-duality results

In this section, a vector dual problem in the sense of Wolfe is considered for the class of E-invex vector optimization problems with inequality and equality constraints. Let $E : R^n \to R^n$ be a given one-to-one and onto operator. Consider the following dual problem in the sense of Wolfe related to the considered vector optimization problem (VP):

$$\text{maximize } f(E(y)) + \left[\sum_{j=1}^{m} \mu_j g_j(E(y)) + \sum_{t=1}^{q} \xi_t h_t(E(y))\right] e$$

$$\text{s.t. } \sum_{i=1}^{p} \lambda_i \nabla (f_i \circ E)(y) + \sum_{j=1}^{m} \mu_j \nabla (g_j \circ E)(y)$$

$$+ \sum_{t=1}^{q} \xi_t \nabla (h_t \circ E)(y) = 0, \qquad (\text{WD}_E)$$

$$\lambda \in R^p, \lambda \geq 0,\ \lambda e = 1,\ e = (1, 1, ..., 1)^T \in R^p,$$

$$\mu \in R^m, \mu \geqq 0, \xi \in R^q,$$

where all functions are defined in the similar way as for the considered vector optimization problem (VP) and $e = (1, ..., 1) \in R^p$. Further, let

$$\Gamma_E = \Bigg\{ (y, \lambda, \mu, \xi) \in R^n \times R^p \times R^m \times R^q :$$

$$\sum_{i=1}^{p} \lambda_i \nabla (f_i \circ E)(y) + \sum_{j=1}^{m} \mu_j \nabla (g_j \circ E)(y)$$

$$+ \sum_{t=1}^{q} \xi_t \nabla (h_t \circ E)(y) = 0,\ \lambda \geq 0,\ \lambda e = 1,\ \mu \geqq 0 \Bigg\}.$$

be the set of all feasible solutions of the problem (WD_E). Further, $Y_E = \{y \in X : (y, \lambda, \mu, \xi) \in \Gamma_E\}$. We call the vector dual problem (WD_E) Wolfe vector E-dual problem or vector E-dual problem in the sense of Wolfe.

Now, under E-invexity hypotheses, we prove duality results between the E-vector problems (VP_E) and (WD_E) and, thus, E-duality results between the problems (VP) and (WD_E).

Theorem 21: (Wolfe weak duality between (VP_E) and (WD_E)). Let z and (y, λ, μ, ξ) be any feasible solutions of the problems (VP_E) and (WD_E), respectively. Further, assume that at least one of the following hypotheses is fulfilled:

A) each objective function f_i, $i \in I$, is E-invex at y on $\Omega_E \cup Y_E$, each constraint function g_j, $j \in J$, is an E-invex function at y on $\Omega_E \cup Y_E$, the functions h_t, $t \in T^+(E(y))$ and the functions $-h_t$, $t \in T^-(E(y))$, are E-invex at y on $\Omega_E \cup Y_E$.

B) $f(E(y)) + \left[\sum_{j=1}^{m} \mu_j g_j(E(y)) + \sum_{t=1}^{q} \xi_t h_t(E(y))\right] e$ is pseudo E-invex at y on $\Omega_E \cup Y_E$, $\mu_j (g_j \circ E)(y)$ is quasi E-invex at y on $\Omega_E \cup Y_E$, $\xi_t (h_t \circ E)(y)$ is quasi E-invex at y on $\Omega_E \cup Y_E$.

Then

$$f(E(z)) \nless f(E(y)) + \left[\sum_{j=1}^{m} \mu_j g_j(E(y)) + \sum_{t=1}^{q} \xi_t h_t(E(y))\right] e. \tag{10}$$

Proof. Let z and (y, λ, μ, ξ) be any feasible solutions of the problems (VP_E) and (WD_E), respectively.

The proof of this theorem under hypothesis A). By means of contradiction, suppose that

$$f(E(z)) < f(E(y)) + \left[\sum_{j=1}^{m} \mu_j g_j(E(y)) + \sum_{t=1}^{q} \xi_t h_t(E(y))\right] e$$

Thus,

$$f(E(z)) < f(E(y)) + \left[\sum_{j=1}^{m} \mu_j g_j(E(y)) + \sum_{t=1}^{q} \xi_t h_t(E(y))\right],\ i \in I. \tag{11}$$

Multiplying by λ_i and then adding both sides of the above inequalities and taking that $\sum_{i=1}^{p} \lambda_i = 1$, we get the inequality

$$\sum_{i=1}^{p} \lambda_i (f_i \circ E)(z) < \sum_{i=1}^{p} \lambda_i (f_i \circ E)(y) + \sum_{j=1}^{m} \mu_j (g_j \circ E)(y)$$

$$+ \sum_{t=1}^{q} \xi_t (h_t \circ E)(y)$$

holds. From the feasibility of z for the problem (VP_E), it follows that

$$\sum_{i=1}^{p} \lambda_i (f_i \circ E)(z) + \sum_{j=1}^{m} \mu_j (g_j \circ E)(z) + \sum_{t=1}^{q} \xi_t (h_t \circ E)(z)$$

$$< \sum_{i=1}^{p} \lambda_i (f_i \circ E)(y) + \sum_{j=1}^{m} \mu_j (g_j \circ E)(y) + \sum_{t=1}^{q} \xi_t (h_t \circ E)(y). \tag{12}$$

$$\sum_{j=1}^{m} \mu_j (g_j \circ E)(z) \leqq \sum_{j=1}^{m} \mu_j (g_j \circ E)(y), \tag{13}$$

$$\sum_{t=1}^{q} \xi_t (h_t \circ E)(z) = \sum_{t=1}^{q} \xi_t (h_t \circ E)(y). \tag{14}$$

By assumption, z and (y, λ, μ, ξ) are feasible solutions for the problems (VP_E) and (WD_E), respectively. Since the functions f_i, $i \in I$, g_j, $j \in J$, h_t, $t \in T^+$, $-h_t$, $t \in T^-$, are E-invex on $\Omega_E \cup Y_E$, by Definition 5, the inequalities

$$(f_i \circ E)(z) - (f_i \circ E)(y) \geqq$$
$$\nabla (f_i \circ E)(y) \eta (E(z), E(y)),\ i \in I, \tag{15}$$

$$(g_j \circ E)(z) - (g_j \circ E)(y) \geqq$$
$$\nabla (g_j \circ E)(y) \eta (E(z), E(y)),\ j \in J_E(y), \tag{16}$$

$$(h_t \circ E)(z) - (h_t \circ E)(y) \geqq$$
$$\nabla (h_t \circ E)(y) \eta (E(z), E(y)),\ t \in T^+(E(y)), \tag{17}$$

$$-(h_t \circ E)(z) + (h_t \circ E)(y) \geqq$$

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$$-\nabla\left(h_t \circ E\right)(y)\, \eta\left(E(z), E(y)\right), \quad t \in T^{-}\left(E(y)\right) \tag{18}$$

hold, respectively. Multiplying inequalities (15)-(18) by the corresponding Lagrange multiplier and then adding both sides of the resulting inequalities, we obtain that the inequality

$$\sum_{i=1}^{p} \lambda_i \left(f_i \circ E\right)(z) - \sum_{i=1}^{p} \lambda_i \left(f_i \circ E\right)(y) + \sum_{j=1}^{m} \mu_i \left(g_j \circ E\right)(z)$$
$$-\sum_{j=1}^{m} \mu_i \left(g_j \circ E\right)(y) + \sum_{t=1}^{q} \xi_i \left(h_t \circ E\right)(z) - \sum_{t=1}^{q} \xi_i \left(h_t \circ E\right)(y)$$
$$\geqq \left[\sum_{i=1}^{p} \lambda_i \nabla\left(f_i \circ E\right)(y) + \sum_{j=1}^{m} \mu_i \nabla\left(g_j \circ E\right)(y) + \sum_{t=1}^{q} \xi_i \nabla\left(h_t \circ E\right)(y)\right] \eta\left(E(z), E(y)\right)$$

holds. Thus, by (12), it follows that the inequality

$$\left[\sum_{i=1}^{p} \lambda_i \nabla\left(f_i \circ E\right)(y) + \sum_{j=1}^{m} \mu_i \nabla\left(g_j \circ E\right)(y) + \sum_{t=1}^{q} \xi_i \nabla\left(h_t \circ E\right)(y)\right] \eta\left(E(z), E(y)\right) < 0 \tag{19}$$

holds, contradicts the first constraint of the vector Wolfe E-dual problem (WD$_E$). This means that the proof of the Wolfe weak duality theorem between the E-vector optimization problems (VP$_E$) and (WD$_E$) is completed under hypothesis A).

The proof of this theorem under hypothesis B). We proceed by contradiction. Suppose, contrary to the result, that (11) holds. Since the function $\left(f \circ E\right)(\cdot) + \left[\mu_j \left(g_j \circ E\right)(\cdot) + \xi_t \left(h_t \circ E\right)(\cdot)\right] e$ is pseudo E-invex at y on $\Omega_E \cup Y_E$, by Definition 8, the inequality

$$\sum_{i=1}^{p} \lambda_i \nabla\left(f_i \circ E\right)(y) + \sum_{j=1}^{m} \mu_j \nabla\left(g_j \circ E\right)(y) + \sum_{t=1}^{q} \xi_t \nabla\left(h_t \circ E\right)(y) < 0 \tag{20}$$

holds. From $z \in \Omega_E$ and $(y, \lambda, \mu, \xi) \in \Gamma_E$, the relations (13) and (14) are fulfilled. Since $\mu_j \left(g_j \circ E\right)(y)$ and $\xi_t \left(h_t \circ E\right)(y)$ are quasi E-invex at y on $\Omega_E \cup Y_E$, by the foregoing above relations, Definition 9 implies that the inequalities

$$\sum_{j=1}^{m} \mu_j \nabla\left(g_j \circ E\right)(y) \eta\left(E(z), E(y)\right) \leqq 0, \tag{21}$$

$$\sum_{t=1}^{q} \xi_t \nabla\left(h_t \circ E\right)(y) \eta\left(E(z), E(y)\right) \leqq 0. \tag{22}$$

hold, respectively. Combining (20), (21) and (22), it follows that the inequality (19) is fulfilled, contradicting the first constraint of the vector Wolfe E-dual problem (WD$_E$). This means that the proof of the Wolfe weak duality theorem between the E-vector optimization problems (VP$_E$) and (WD$_E$) is completed under hypothesis B). ■

Theorem 22: (Wolfe weak E-duality between (VP) and (WD$_E$)). Let $E(z)$ and (y, λ, μ, ξ) be any feasible solutions of the problems (VP) and (WD$_E$), respectively. Further, assume that all hypotheses of Theorem 21 are fulfilled. Then, Wolfe weak E-duality between (VP) and (WD$_E$) holds, that is,

$$f(E(z)) \nless f(E(y)) + \left[\sum_{j=1}^{m} \mu_j g_j(E(y)) + \sum_{t=1}^{q} \xi_t h_t(E(y))\right] e.$$

Proof. Let $E(z)$ and (y, λ, μ, ξ) be any feasible solutions of the problems (VP) and (WD$_E$), respectively. Then, by Lemma 14. it follows that z is any feasible solution of (VP$_E$). Since all hypotheses of Theorem 21 are fulfilled, the Wolfe weak E-duality theorem between the problems (VP) and (WD$_E$) follows directly form Theorem 21. ■

If some stronger E-invexity hypotheses are imposed on the functions constituting the considered E-differentiable multiobjective programming problem, then the stronger result is true.

Theorem 23: (Wolfe weak duality between (VP$_E$) and (WD$_E$)). Let z and (y, λ, μ, ξ) be any feasible solutions of the problems (VP$_E$) and (WD$_E$), respectively. Further, assume that at least one of the following hypotheses is fulfilled:

A) each objective function f_i, $i \in I$, is strictly E-invex at y on $\Omega_E \cup Y_E$, each constraint function g_j, $j \in J$, is an E-invex function at y on $\Omega_E \cup Y_E$, the functions h_t, $t \in T^{+}\left(E(y)\right)$ and the functions $-h_t$, $t \in T^{-}\left(E(y)\right)$, are E-invex at y on $\Omega_E \cup Y_E$.

B) $f(E(y)) + \left[\sum_{j=1}^{m} \mu_j g_j(E(y)) + \sum_{t=1}^{q} \xi_t h_t(E(y))\right] e$ is strictly pseudo E-invex at y on $\Omega_E \cup Y_E$, $\mu_j \left(g_j \circ E\right)(y)$ is quasi E-invex at y on $\Omega_E \cup Y_E$, $\xi_t \left(h_t \circ E\right)(y)$ is quasi E-invex at y on $\Omega_E \cup Y_E$.

Then

$$\left(f \circ E\right)(z) \nleq f(E(y)) + \left[\sum_{j=1}^{m} \mu_j g_j(E(y)) + \sum_{t=1}^{q} \xi_t h_t(E(y))\right] e. \tag{23}$$

Theorem 24: (Wolfe weak E-duality between (VP) and (WD$_E$)). Let $E(z)$ and (y, λ, μ, ξ) be any feasible solutions of the problems (VP) and (WD$_E$), respectively. Further, assume that all hypotheses of Theorem 23 are fulfilled. Then, Wolfe weak E-duality between (VP) and (WD$_E$) holds, that is,

$$\left(f \circ E\right)(z) \nleq f(E(y)) + \left[\sum_{j=1}^{m} \mu_j g_j(E(y)) + \sum_{t=1}^{q} \xi_t h_t(E(y))\right] e.$$

Theorem 25: (Wolfe strong duality between (VP$_E$) and (WD$_E$) and also strong E-duality between (VP) and (WD$_E$)). Let $\overline{x} \in \Omega_E$ be a (weak) Pareto solution of the E-vector optimization problem (VP) and the E-Guignard constraint qualification (GCQ$_E$) be satisfied at $\overline{x}$.Then there exist $\overline{\lambda} \in R^p$, $\overline{\mu} \in R^m$, $\overline{\xi} \in R^q$ such that $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi})$ is feasible for the problem (WD$_E$) and the objective functions of (VP$_E$)

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and (WD$_E$) are equal at these points. If also all hypotheses of the weak duality theorem (Theorem 21 or Theorem 23) are satisfied, then $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi})$ is a (weak) efficient solution of maximum type for the problem (WD$_E$).
In other words, if, $E(\overline{x}) \in \Omega$ is a (weak) E-Pareto solution of the multiobjective programming problem (VP), then $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a (weak) efficient solution of a maximum type in the dual problem (WD$_E$) in the sense of Wolfe. This means that the strong Wolfe E-duality holds between the problems (VP) and (WD$_E$).

Proof. Since $\overline{x} \in \Omega_E$ is a weak Pareto solution of the problem (VP$_E$) and the E-Guignard constraint qualification (GCQ$_E$) is satisfied at $\overline{x}$, by Theorem 20, there exist $\overline{\lambda} \in R^p$, $\overline{\lambda} \neq 0$, $\overline{\mu} \in R^m$, $\overline{\mu} \geqq 0$, $\overline{\xi} \in R^q$, $\overline{\xi} \geqq 0$ such that $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a feasible solution of the problem (WD$_E$). This means that the objective functions of (VP$_E$) and (WD$_E$) are equal. If we assume that all hypotheses of Wolfe weak duality (Theorem 21 or Theorem 23) are fulfilled, $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a weakly efficient solution of a maximum type in the dual problem (WD$_E$) in the sense of Wolfe.

Moreover, we have by Lemma 14, that $E(\overline{x}) \in \Omega$. Since $\overline{x} \in \Omega_E$ is a weak Pareto solution of the problem (VP$_E$), by Lemma 16, it follows that $E(\overline{x})$ is a weak E-Pareto solution in the problem (VP). Then, by the strong duality between (VP$_E$) and (WD$_E$), we conclude that also the Wolfe strong E-duality holds between the problems (VP) and (WD$_E$). This means that if $E(\overline{x}) \in \Omega$ is a (weak) E-Pareto solution of the problem (VP), there exist $\overline{\lambda} \in R^p$, $\overline{\mu} \in R^m$, $\overline{\mu} \geqq 0$, $\overline{\xi} \in R^q$ such that $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a weakly efficient solution of a maximum type in the Wolfe vector dual problem (WD$_E$). ■

Theorem 26: (Wolfe converse duality between (VP$_E$) and (WD$_E$)). Let $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ be a (weak) efficient solution of a maximum type in the vector E-Wolfe dual problem (WD$_E$) such that $\overline{x} \in \Omega_E$. Moreover, assume that the objective functions f_i, $i \in I$, are (E-invex) strictly E-invex at $\overline{x}$ on $\Omega_E \cup Y_E$, the constraint functions g_j, $j \in J$, are E-invex at $\overline{x}$ on $\Omega_E \cup Y_E$, the functions h_t, $t \in T^+(E(\overline{x}))$ and the functions $-h_t$, $t \in T^-(E(\overline{x}))$, are E-invex at $\overline{x}$ on $\Omega_E \cup Y_E$. Then $\overline{x}$ is a (weak) Pareto solution of the problem (VP$_E$).

Proof. Let $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ be a (weakly) efficient solution of a maximum type in Wolfe E-dual problem (WD$_E$) such that $\overline{x} \in \Omega_E$. By means of contradiction, we suppose that there exists $\widetilde{x} \in \Omega_E$ such that

$$(f \circ E)(\widetilde{x}) < (f \circ E)(\overline{x}) \tag{24}$$

holds. By the feasibility of $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ in the problem (WD$_E$). Hence, by the E-Karush-Kuhn-Tucker necessary optimality conditions, we get

$$(f \circ E)(\widetilde{x}) + \left[\sum_{j=1}^{m} \mu_j (g_j \circ E)(\widetilde{x}) + \sum_{t=1}^{q} \xi_t (h_t \circ E)(\widetilde{x})\right] e <$$
$$(f \circ E)(\overline{x}) + \left[\sum_{j=1}^{m} \mu_j (g_j \circ E)(\overline{x}) + \sum_{t=1}^{q} \xi_t (h_t \circ E)(\overline{x})\right] e. \tag{25}$$

Since $\overline{\lambda}_i \geq 0, i \in I$, then (25) yields

$$\sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(\widetilde{x}) + \left[\sum_{j=1}^{m} \mu_j (g_j \circ E)(\widetilde{x}) + \sum_{t=1}^{q} \xi_t (h_t \circ E)(\widetilde{x})\right] \sum_{i=1}^{p} \overline{\lambda}_i < \sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(\overline{x}) + \left[\sum_{j=1}^{m} \mu_j (g_j \circ E)(\overline{x}) + \sum_{t=1}^{q} \xi_t (h_t \circ E)(\overline{x})\right] \sum_{i=1}^{p} \overline{\lambda}_i. \tag{26}$$

From the feasibility of $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ in the problem (WD$_E$), we have $\sum_{i=1}^{p} \overline{\lambda}_i = 1$. Then, the inequality above implies

$$\sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(\widetilde{x}) + \left[\sum_{j=1}^{m} \mu_j (g_j \circ E)(\widetilde{x}) + \sum_{t=1}^{q} \xi_t (h_t \circ E)(\widetilde{x})\right] < \sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(\overline{x}) + \left[\sum_{j=1}^{m} \mu_j (g_j \circ E)(\overline{x}) + \sum_{t=1}^{q} \xi_t (h_t \circ E)(\overline{x})\right]. \tag{27}$$

Since the functions f_i, $i \in I$, g_j, $j \in J$, h_t, $t \in T^+$, $-h_t$, $t \in T^-$, are E-invex at $\overline{x}$ on $\Omega_E \cup Y_E$, by Definition 5, the following inequalities

$$f_i(E(\widetilde{x})) - f_i(E(\overline{x})) \geqq \nabla f_i(E(\overline{x}))\, \eta(E(\widetilde{x}), E(\overline{x})),\ i \in I, \tag{28}$$

$$g_j(E(\widetilde{x})) - g_j(E(\overline{x})) \geqq \nabla g_j(E(\overline{x}))\, \eta(E(\widetilde{x}), E(\overline{x})),\ j \in J_E(\overline{x}), \tag{29}$$

$$h_t(E(\widetilde{x})) - h_t(E(\overline{x})) \geqq \nabla h_t(E(\overline{x}))\, \eta(E(\widetilde{x}), E(\overline{x})),\ t \in T^+(E(\overline{x})), \tag{30}$$

$$-h_t(E(\widetilde{x})) + h_t(E(\overline{x})) \geqq -\nabla h_t(E(\overline{x}))\, \eta(E(\widetilde{x}), E(\overline{x})),\ \ t \in T^-(E(\overline{x})) \tag{31}$$

hold, respectively. Multiplying inequalities (28)-(31) by the corresponding Lagrange multiplier, respectively, we obtain that the inequality

$$\overline{\lambda}_i f_i(E(\widetilde{x})) - \overline{\lambda}_i f_i(E(\overline{x})) \geqq \overline{\lambda}_i \nabla f_i(E(\overline{x}))\, \eta(E(\widetilde{x}), E(\overline{x})),\ i \in I, \tag{32}$$

$$\overline{\mu}_j g_j(E(\widetilde{x})) - \overline{\mu}_j g_j(E(\overline{x})) \geqq \overline{\mu}_j \nabla g_j(E(\overline{x}))\, \eta(E(\widetilde{x}), E(\overline{x})),\ j \in J_E(\overline{x}), \tag{33}$$

$$\overline{\xi}_t h_t(E(\widetilde{x})) - \overline{\xi}_t h_t(E(\overline{x})) \geqq \overline{\xi}_t \nabla h_t(E(\overline{x}))\, \eta(E(\widetilde{x}), E(\overline{x})),\ t \in T^+(E(\overline{x})), \tag{34}$$

$$-\overline{\xi}_t h_t(E(\widetilde{x})) + \overline{\xi}_t h_t(E(\overline{x})) \geqq -\overline{\xi}_t \nabla h_t(E(\overline{x}))\, \eta(E(\widetilde{x}), E(\overline{x})),\ \ t \in T^-(E(\overline{x})) \tag{35}$$

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hold, respectively. Then adding both sides of (32)-(35), we obtain that the inequality

$$\sum_{i=1}^{p}\overline{\lambda}_i f_i(E(\widetilde{x})) - \sum_{i=1}^{p}\overline{\lambda}_i f_i(E(\overline{x})) + \sum_{j=1}^{m}\overline{\mu}_j g_j(E(\widetilde{x})) - \sum_{j=1}^{m}\overline{\mu}_j g_j(E(\overline{x})) + \sum_{t=1}^{q}\overline{\xi}_t h_t(E(\widetilde{x})) - \sum_{t=1}^{q}\overline{\xi}_t h_t(E(\overline{x})) \geqq \left[\sum_{k=1}^{p}\overline{\lambda}_k \nabla f_k(E(\overline{x})) + \sum_{j=1}^{m}\overline{\mu}_j \nabla g_j(E(\overline{x})) + \sum_{t=1}^{q}\overline{\xi}_t \nabla h_t(E(\overline{x}))\right]\eta\left(E(\widetilde{x}), E(\overline{x})\right) \tag{36}$$

holds. Thus, By (27) and (36), we obtain the following inequality

$$\left[\sum_{i=1}^{p}\overline{\lambda}_i \nabla f_i(E(\overline{x})) + \sum_{j=1}^{m}\overline{\mu}_j \nabla g_j(E(\overline{x})) + \sum_{t=1}^{q}\overline{\xi}_t \nabla h_t(E(\overline{x}))\right]\eta\left(E(\widetilde{x}), E(\overline{x})\right) < 0, \tag{37}$$

contradicting the feasibility of $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ in (WD$_E$). This means that the proof of the converse duality theorem between the E-vector optimization problems (VP$_E$) and (WD$_E$) is completed. ■

Theorem 27: (Wolfe converse E-duality between (VP) and (WD$_E$)). Let $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ be a (weakly) efficient solution of a maximum type in Wolfe vector dual problem (WD$_E$). Further, assume that all hypotheses of Theorem 26 are fulfilled. Then $E(\overline{x}) \in \Omega$ is a (weak) E-Pareto solution of the problem (VP).

Proof. The proof of this theorem follows directly from Lemma 16 and Theorem 26. ■

Theorem 28: (Wolfe restricted converse duality between (VP$_E$) and (WD$_E$)). Let $\overline{x}$ and $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ be feasible solutions for the problems (VP$_E$) and (WD$_E$), respectively, such that

$$(f\circ E)(\overline{x}) < (f\circ E)(\overline{y}) + \left[\sum_{j=1}^{m}\overline{\mu}_j (g_j\circ E)(\overline{y}) + \sum_{t=1}^{q}\overline{\xi}_t (h_t\circ E)(\overline{y})\right]e. \quad (\leq) \tag{38}$$

Moreover, assume that the objective functions f_i, $i \in I$, are (E-invex) strictly E-invex at $\overline{y}$ on $\Omega_E \cup Y_E$, the constraint functions g_j, $j \in J$, are E-invex at $\overline{y}$ on $\Omega_E \cup Y_E$, the functions h_t, $t \in T^+\left(E(\overline{y})\right)$ and functions $-h_t$, $t \in T^-\left(E(\overline{y})\right)$, are E-invex at $\overline{y}$ on $\Omega_E \cup Y_E$. Then $\overline{x} = \overline{y}$, that is, $\overline{x}$ is a (weak) Pareto solution of the problem (VP$_E$) and $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a (weak) efficient point of maximum type for the problem (WD$_E$).

Proof. Note that, by (38), it follows that

$$(f_i\circ E)(\overline{x}) < (f_i\circ E)(\overline{y}) + \sum_{j=1}^{m}\overline{\mu}_j (g_j\circ E)(\overline{y}) + \sum_{t=1}^{q}\overline{\xi}_t (h_t\circ E)(\overline{y}),\ i \in I. \tag{39}$$

Multiplying each inequality (39) by $\overline{\lambda}_i$, $i \in I$, and then adding both sides of the resulting inequalities, we get

$$\sum_{i=1}^{p}\overline{\lambda}_i (f\circ E)(\overline{x}) < \sum_{i=1}^{p}\overline{\lambda}_i (f\circ E)(\overline{y}) + \left[\sum_{j=1}^{m}\overline{\mu}_j (g_j\circ E)(\overline{y}) + \sum_{t=1}^{q}\overline{\xi}_t (h_t\circ E)(\overline{y})\right]\sum_{i=1}^{p}\lambda_i. \tag{40}$$

Since $\sum_{i=1}^{p}\lambda_i = 1$, (40) implies

$$\sum_{i=1}^{p}\overline{\lambda}_i (f_i\circ E)(\overline{x}) < \sum_{i=1}^{p}\overline{\lambda}_i (f_i\circ E)(\overline{y}) + \sum_{j=1}^{m}\overline{\mu}_j (g_j\circ E)(\overline{y}) + \sum_{t=1}^{q}\overline{\xi}_t (h_t\circ E)(\overline{y}). \tag{41}$$

Now, we proceed by contradiction. Suppose, contrary to the result, that $\overline{x} \neq \overline{y}$. By assumption, the functions f_i, $i \in I$, g_j, $j \in J(E(\overline{y}))$, h_t, $t \in T^+\left(E(\overline{y})\right)$, and $-h_t$, $t \in T^-\left(E(\overline{y})\right)$ are E-invex at $\overline{y}$ on $\Omega_E \cup Y$. Then, by Definition 5, the inequalities

$$(f_i\circ E)(\overline{x}) - (f_i\circ E)(\overline{y}) \geqq \nabla(f_i\circ E)(\overline{y})\,\eta\left(E(\overline{x}), E(\overline{y})\right),\ i \in I, \tag{42}$$

$$(g_j\circ E)(\overline{x}) - (g_j\circ E)(\overline{y}) \geqq \nabla(g_j\circ E)(\overline{y})\,\eta\left(E(\overline{x}), E(\overline{y})\right),\ j \in J\left(E(\overline{y})\right), \tag{43}$$

$$(h_t\circ E)(\overline{x}) - (h_t\circ E)(\overline{y}) \geqq \nabla(h_t\circ E)(\overline{y})\,\eta\left(E(\overline{x}), E(\overline{y})\right),\ t \in T^+\left(E(\overline{y})\right), \tag{44}$$

$$-(h_t\circ E)(\overline{x}) + (h_t\circ E)(\overline{y}) \geqq -\nabla(h_t\circ E)(\overline{y})\,\eta\left(E(\overline{x}), E(\overline{y})\right),\ \ t \in T^-\left(E(\overline{y})\right) \tag{45}$$

hold, respectively. Multiplying inequalities (42)-(45) by the corresponding Lagrange multipliers and then adding both sides of the resulting inequalities, we get

$$\sum_{i=1}^{p}\overline{\lambda}_i (f_i\circ E)(\overline{x}) - \sum_{i=1}^{p}\overline{\lambda}_i (f_i\circ E)(\overline{y}) + \sum_{j=1}^{m}\overline{\mu}_i (g_j\circ E)(\overline{x}) - \sum_{j=1}^{m}\overline{\mu}_i (g_j\circ E)(\overline{y}) + \sum_{t=1}^{q}\overline{\xi}_i (h_t\circ E)(\overline{x}) - \sum_{t=1}^{q}\overline{\xi}_i (h_t\circ E)(\overline{y}) \geqq \left[\sum_{i=1}^{p}\overline{\lambda}_i \nabla(f_i\circ E)(\overline{y}) + \sum_{j=1}^{m}\overline{\mu}_i \nabla(g_j\circ E)(\overline{y}) + \sum_{t=1}^{q}\overline{\xi}_i \nabla(h_t\circ E)(\overline{y})\right]\eta\left(E(\overline{x}), E(\overline{y})\right) \tag{46}$$

By (46) and the first constraint of (WD$_E$), it follows that

$$\sum_{i=1}^{p}\overline{\lambda}_i (f_i\circ E)(\overline{x}) + \sum_{j=1}^{m}\overline{\mu}_j (g_j\circ E)(\overline{x}) + \sum_{t=1}^{q}\overline{\xi}_t (h_t\circ E)(\overline{x}) \geqq \sum_{i=1}^{p}\overline{\lambda}_i (f_i\circ E)(\overline{y}) + \sum_{j=1}^{m}\overline{\mu}_j (g_j\circ E)(\overline{y}) + \sum_{t=1}^{q}\overline{\xi}_t (h_t\circ E)(\overline{y}).$$

Hence, by $\overline{x} \in \Omega_E$, we get that the following inequality

$$\sum_{i=1}^{p}\overline{\lambda}_i (f_i\circ E)(\overline{x}) \geqq \sum_{i=1}^{p}\overline{\lambda}_i (f_i\circ E)(\overline{y}) + \sum_{j=1}^{m}\overline{\mu}_j (g_j\circ E)(\overline{y}) +$$

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$$\sum_{t=1}^{q} \overline{\xi}_t \left(h_t \circ E\right)(\overline{y}). \tag{47}$$

holds, contradicting (41). Then, $\overline{x} = \overline{y}$ and this means, by weak duality (Theorem 21 or Theorem 23) that $\overline{x}$ is a weak Pareto solution of the problem (VP$_E$) and $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a weak efficient solution of maximum type for the problem (WD$_E$). Thus, the proof of this theorem is completed. ■

Theorem 29: (Wolfe restricted converse E-duality between (VP) and (WD$_E$)). Let $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ be a feasible solution of the problem (WD$_E$). Further, assume that there exist $E\left(\overline{x}\right) \in \Omega$ such that $\overline{x} = \overline{y}$. If all hypotheses of Theorem 28 are fulfilled, then $E\left(\overline{x}\right)$ is an E-Pareto solution of the problem (VP) and $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a weakly efficient solution of maximum type for the problem (WD$_E$).

Proof. The proof of this theorem follows directly from Lemma 16 and Theorem 28. ■

IV. Concluding remarks

In this paper, the class of E-differentiable vector optimization problems with both inequality and equality constraints has been considered. For such (not necessarily) differentiable vector optimization problems. The so-called vector Wolfe E-dual problem has been defined for the considered E-differentiable E-invexity multiobjective programming problem with both inequality and equality constraints and several E-dual theorems have been established under (generalized) E-invexity hypotheses.

However, some interesting topics for further research remain. It would be of interest to investigate whether it is possible to prove similar results for other classes of E-differentiable vector optimization problems. We shall investigate these questions in subsequent papers.

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