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Error Analysis of the High Order Newton Cotes Formulas

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Abstract- The importance of numerical integration may be appreciated by noting how frequently the formulation of problems in applied analysis involves derivatives. It is then natural to anticipate that the solutions of such problems will involve integrals. For most integrals no representation in terms of elementary functions is possible, and approximation becomes necessary.
As with any approximate method, the utility of polynomial interpolation cannot be stretched too far. In this paper we shall quantify the errors that can occur in polynomial interpolation and develop techniques to calculate such errors.

Index Terms- Newton's Forward Interpolation, Numerical integration , Error , Polynomial , Newton Cotes .

I. Introduction

The usual error sources are present. However, input errors in the data values $y_0, y_1, \ldots, y_n$ are not magnified by most integration formulas, so this source of error is not nearly so troublesome as it is in numerical differentiation. The truncation error, which is

$$\int_a^b [y(x) - P(x)]\, dx$$

A wide variety of efforts to estimate this error have been made. A related question is that of *convergence*. This asks whether, as continually higher degree polynomials are used, or as continually smaller intervals h_m between data points are used with $\lim h_m = 0$, a sequence of approximations is produced for which the limit of truncation error is zero. In many cases, the trapezoidal and Simpson rules being excellent examples, convergence can be proved. Round off errors also have a strong effect. A small interval h means substantial computation and much rounding off. These algorithm errors ultimately obscure the convergence which should theoretically occur, and it is found in practice that decreasing h below a certain level leads to larger errors rather than smaller. As truncation error becomes negligible, round off errors accumulate, limiting the accuracy obtainable by a given method.

This paper constructs of:

II. PRELIMINARIES

We first introduce the Weierstrass Approximation Theorem as one of the motivations for the use of polynomials. Here are the details.

Theorem

Let $f \epsilon C[a , b]$.and $\epsilon > 0$. Then there exists a polynomial P of sufficiently high degree such that
$|f(x) - P(x)| < \epsilon$, for all $x \epsilon [a , b]$.

Statement (Newton's Forward Interpolation Formula)

If $x_0, x_1 , x_{2,} \ldots , x_n$ are given set of observations with common difference h and let $y_0, y_1 , y_{2,} \ldots , y_n$are their corresponding values, where $y = f(x)$ be the given function then

$$f(x) = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)(p-2)\ldots(p-(n-1))}{n!}\Delta^n y_0$$

Where $p = \frac{x - x_0}{h}$

Integration is a summing process. Thus virtually all numerical approximations can be represented by

$$I = \int_a^b f(x)dx = \sum_{i=1}^{n} w_i f(x_i) + E_t$$

Where : w_i = weights , x_i = nodes , E_t = truncation error .

Theorem

If $f \epsilon c^n[\, a, b\,]$ *and* $f^{(n+1)}$ exists on $(\, a \,,\, b\,)$, then for any $x \epsilon [a , b]$.

$$f(x) = \Bigg[\underbrace{\sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k}_{Taylor\ polynomial}\Bigg] + \Bigg[\underbrace{\frac{1}{(n+1)!} f^{(n+1)}(\xi_x)(x - x_0)^{n+1}}_{error\ term}\Bigg]$$

Where ξ_x is same point between *x* and x_0.

Theorem

Let $f(x)$ be a real-valued function defined on [a , b] and *n+1* times differentiable on (a , b) .If $P_n(x)$ is the polynomial of degree $\leq n$ which interpolates $f(x)$ at the (*n+1*) distinct points $x_0, x_1, \ldots . x_n \epsilon [a\,, b]$, then for all $x \epsilon [a\,, b]$, there exists point $\xi = \xi(x) \epsilon (a\,, b)$ such that .

$$E_n(x) = f(x) - P_n(x) = \frac{f^{n+1}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

$E_n(x)$ is called the Interpolation error .

Example

If *P(x)* is the polynomial that interpolates the function $f(x) = \sin(x)$ at 12 points on the interval [0, 1], what is the greatest possible error?

• In this example, we have *n +1* = 12 and

$$f^{(n+1)}(x) = f^{(12)}(x) = \sin(x)$$

So the largest possible error would be the maximal value of

$$max \left| \frac{f^{12}(\xi)}{(12)!} \prod_{i=0}^{11} (x - x_i) \right|$$

For $x_0, x_1, \ldots . x_n, \xi \in [1\,, 0\,]$ .Clearly, on the interval [0, 1]

$$max|x - x_i| = 1$$

$$max\left|f^{(n+1)}(\xi)\right| = max|-\sin(\xi)| = 1$$

And the maximal error would be

$$\frac{1}{12!}(1)(1)^{12} \approx \frac{1}{479001600} = 2.08767569878681 \times 10^{-9}$$

Proposition

Let $[a, b\,]$ be a real interval, a < b, and let I_n be the quadrature rule based on interpolating polynomials for the distinct points $x_0, x_1, \ldots, x_n \epsilon [a, b\,]$ .If $\omega_{n+1}(x) = \prod_{i=0}^{n}(x - x_i)$ has only one sign on $[a, b\,], (i.e. \omega_{n+1} \geq 0 \; or \; \omega_{n+1} \leq 0 \; on \; [a\,, b])$, then, for each $f \epsilon C^{n+1}[\,a, b\,]$, there exists $\tau \epsilon [\,a, b\,]$ such that

$$\int_a^b f(x)dx - I_n(f) = \frac{f^{(n+1)}(\tau)}{(n+1)!} \int_a^b \omega_{n+1}(x)\, dx$$

In particular, if $f^{(n+1)}$ has just one sign as well, then one can infer the sign of the error term (i.e. of the right-hand side).

III. ERROR ANALYSIS OF NEWTON COTES FORMULAS

Theorem

Suppose that $\sum_{i=0}^{n}\{a_i f(x_i)\}$ is the *n+1* point closed Newton-Cotes formula with $a = x_0\,, b = x_n \; and \; h = \frac{b-a}{n}$. There exists $\xi \epsilon (a\,, b)$ for which $n \; is \; odd$ we get :

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+2} f^{(n+1)}(\xi)}{(n+1)!} \int_0^n t(t-1) \ldots (t-n)dt$$

Then we get the **Estimation error At** $\boldsymbol{n} \in \mathbb{N}$:

$$E_n = \frac{h^{n+2} f^{(n+1)}(\xi)}{(n+1)!} \int_0^n t(t-1) \ldots (t-n)dt \,. \to \langle \mathbf{1} \rangle$$

Estimation error At *n = 1* (Trapezoidal Rule)

Form equation $\langle \mathbf{1} \rangle$ we get

$$E_1 = \frac{h^3 f^{(2)}(\xi)}{2!} \int_0^1 t(t-1)dt = \frac{h^3 f^{(2)}(\xi)}{2!} \int_0^1 (t^2 - t)dt = \frac{h^3 f^{(2)}(\xi)}{2!} \left[\left(\frac{t^3}{3} - \frac{t^2}{2} \right) \right]_0^1 = -\frac{h^3 f^{(2)}(\xi)}{12}$$

Example

Find the approximate value of $\int_0^1 e^{-x^2} dx$.

$$\int_0^1 e^{-x^2} dx = \frac{h}{2}[f(0) + f(1)] - \frac{h^3 f^{(2)}(\xi)}{12}$$

$$h = \frac{(1-0)}{1} = 1$$

$$\therefore \int_0^1 e^{-x^2} dx = \frac{1}{2}[1 + e^{-1}] = -\frac{f^{(2)}(\xi)}{12}$$

Here, $f(x) = e^{-x^2}$ is particularly simple, we can calculate and bound the second derivative, obtaining

$$f'(x) = -2xe^{-x^2}$$

$$f''(x) = (-2 + 4x^2)e^{-x^2}$$

To find the value of ξ we need to get

$$f(\xi) = \max_{0 \leq x \leq 1} |f''(x)|\,.$$

We have $f'''(x) = (12 + 48x^2 + 16x^4)e^{-x^2}$

Then we need to find the value of x , when $f'''(x) = 0$

$$\to (12 + 48x^2 + 16x^4)e^{-x^2} = 0$$

Give us $x = \left\{ \infty\,, \pm \frac{1}{2}\sqrt{6 - 2\sqrt{6}}\,, \pm \frac{1}{2}\sqrt{6 + 2\sqrt{6}} \right\}$.

The only value that`s in (0 , 1) is $\frac{1}{2}\sqrt{6 - 2\sqrt{6}}$,

That is $\xi = \frac{1}{2}\sqrt{6 - 2\sqrt{6}}$

Then

$$\frac{f^{(2)}(\xi)}{12} = \frac{(-2 + 4\xi^2)e^{-\xi^2}}{12}$$

$$\therefore \frac{f^{(2)}\left(\frac{1}{2}\sqrt{6 - 2\sqrt{6}}\right)}{12} = -0.036292412368924601147568642826 22$$

And by *Trapezoidal Rule* we get

$$\int_0^1 e^{-x^2} dx = \frac{1}{2}[1 + e^{-1}] = 0.68393972058572116079776188508073$$

$$\therefore \int_0^1 e^{-x^2} dx =$$

0.68393972058572116079776188508073
+ 0.03629241236892460114756864282622
=0.72023213295464576194533052790695

Estimation error At *n = 2* ($\frac{1}{3}$ Simpson rule)

Form equation $\langle \mathbf{1} \rangle$ we get

$$E_2 = \frac{h^4 f^{(3)}(\xi)}{(3)!}\int_0^2 t\,(t-1)(t-2)dt = o$$

This doesn't mean that the error is zero. It simply means that the cubic term is identically zero. The error term can be obtained from the next term in the Newton Polynomial , obtaining

$$E_2 = \frac{h^4 f^{(3)}(\xi)}{(3)!}\int_0^2 t\,(t-1)(t-2)dt.$$

$$becames \rightarrow E_2 = \frac{h^5 f^{(4)}(\xi)}{(4)!}\int_0^2 t\,(t-1)(t-2)(t-3)dt$$

$$\therefore E_2 = \frac{h^5 f^{(4)}(\xi)}{(4)!}\int_0^2 t(t-1)(t-2)(t-3)dt$$

$$= \frac{h^5 f^{(4)}(\xi)}{(4)!}\int_0^2 (t^4 - 6t^3 + 11t^2 - 6t)\,dt$$

$$= \frac{h^5 f^{(4)}(\xi)}{(4)!}\left[\left(\frac{t^5}{5} - \frac{3t^4}{2} + \frac{11t^2}{2} - 3t^2\right)\right]_0^2$$

$$\therefore E_2 = -\frac{h^5 f^{(4)}(\xi)}{90}$$

Estimation error of Newton Cotes formula of degree 4

Form eq. $\langle \mathbf{1} \rangle$ we get

$$E_4 = \frac{h^7 f^{(6)}(\xi)}{(6)!}\int_0^4 t(t-1)(t-2)(t-3)(t-4)dt = 0$$

Again, that doesn't mean that the error is zero. The error term can be obtained from the next term in the Newton Polynomial. i.e.

$$E_4 = \frac{h^7 f^{(6)}(\xi)}{(6)!}\int_0^4 t(t-1)(t-2)(t-3)(t-4)(t-5)dt =$$

$$\frac{h^7 f^{(6)}(\xi)}{(6)!}\int_0^4 (t^6 - 15t^5 + 85t^4 - 225t^3 + 274t^2 - 120t)dt$$

$$= \frac{h^7 f^{(6)}(\xi)}{(6)!}\left[\left(\frac{t^7}{7} - \frac{5t^6}{6} + 17t^5 - \frac{255t^4}{4} - \frac{274t^3}{3} - 105t^2\right)\right]_0^4$$

$$= -\frac{8}{945}h^7 f^{(6)}(\xi)$$

Estimation error of Newton Cotes formula of degree 12

Form eq. $\langle \mathbf{1} \rangle$ we get

$$E_{12} = \frac{h^{15} f^{(14)}(\xi)}{(14)!}\int_0^{12}\begin{pmatrix} t(t-1)(t-2)(t-3)(t-4) \\ (t-5)(t-6)(t-7)(t-8) \\ (t-9)(t-10)(t-11)(t-12)\end{pmatrix} dt = 0$$

And , that doesn't mean that the error is zero. The error term can be obtained from the next term in the Newton Polynomial. i.e.

$$E_{12} = \frac{h^{15} f^{(14)}(\xi)}{(14)!}\int_0^{12}\begin{pmatrix} t(t-1)(t-2)(t-3)(t-4)(t-5) \\ (t-6)(t-7)(t-8)(t-9) \\ (t-10)(t-11)(t-12)(t-13)\end{pmatrix}$$

$$= \frac{h^{15} f^{(14)}(\xi)}{(14)!}\int_0^{12} (t^{14} - 91\,t^{13} + 3731\,t^{12} - 91091\,t^{11} + 1474473\,t^{10} - 16669653\,t^9 + 135036473t^8 - 790943153\,t^7 + 3336118786\,t^6 - 9957703756\,t^5 + 20313753096\,t^4 - 26596717056\,t^3 + 19802759040\,t^2 - 6227020800\,t)dt$$

$$= \frac{h^{15} f^{(14)}(\xi)}{(14)!}\left[\left(\frac{t^{15}}{15} - \frac{t^{13}}{2} + 287\,t^{13} - 91091\frac{t^{12}}{12} + 134043\,t^{11} - 16669653\frac{t^{10}}{10} + 135036473\,\frac{t^9}{9} - 790943153\frac{t^8}{8} + 476588398\,t^7 - 4978851878\,\frac{t^6}{3} + 20313753096\,\frac{t^5}{5} - 6649179264\,t^4 + 6600919680\,t^3 - 3113510400\,t^2\right)\right]_0^{12}$$

$$= -\frac{1498963968}{5} \times \frac{h^{15} f^{(14)}(\xi)}{(14)!}$$

$$\therefore E_{12} = -\frac{3012}{875875}h^{15} f^{(14)}(\xi) \rightarrow \langle \mathbf{2} \rangle$$

Estimation error of Newton Cotes formula of degree 13

$$E_{13} = \frac{h^{15} f^{(14)}(\xi)}{(14)!}\int_0^{13}\begin{pmatrix} t(t-1)(t-2)(t-3)(t-4(t-5) \\ (t-6)(t-7)(t-8)(t-9) \\ (t-10)(t-11)(t-12)(t-13)\end{pmatrix} dt$$

$$= \frac{h^{15} f^{(14)}(\xi)}{(14)!}\int_0^{13} (t^{14} - 91\,t^{13} + 3731\,t^{12} - 91091\,t^{11} + 1474473\,t^{10} - 16669653\,t^9 + 135036473t^8 - 790943153\,t^7 + 3336118786\,t^6 - 9957703756\,t^5 + 20313753096\,t^4 - 26596717056\,t^3 + 19802759040\,t^2 - 6227020800\,t)dt$$

$$= \frac{h^{15} f^{(14)}(\xi)}{(14)!}\left[\left(\frac{t^{15}}{15} - \frac{t^{13}}{2} + 287\,t^{13} - 91091\frac{t^{12}}{12} + 134043\,t^{11} - 16669653\frac{t^{10}}{10} + 135036473\,\frac{t^9}{9} - 790943153\frac{t^8}{8} + 476588398\,t^7 - 4978851878\,\frac{t^6}{3} + 20313753096\,\frac{t^5}{5} - 6649179264\,t^4 + 6600919680\,t^3 - 3113510400\,t^2\right)\right]_0^{13}$$

$$E_{13} = -\frac{2639651053}{344881152000}\,h^{15}\,f^{(14)}(\xi)$$

Note that we use the same polynomial in the last two equations . And we can see that , both of them have the same amount of $\rightarrow h^{15}$ and the same number of derivatives $\rightarrow f^{(14)}(\xi)$. Here we clearly find out that $E_{12} < E_{13}$ .i.e.

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$$E_{12} - E_{13} = \frac{132282840127}{31384184832000} = -0.0076537991064138$$

IV. Report

In Newton Cotes formula , when ***n*** is **even** then we have .

Report 1)-

$$\int_0^n t(t-1)\dots(t-n)dt = 0 .$$

Example

■At $n = 2$

$$\int_0^2 t(t-1)(t-2)dt = 0 \rightarrow \int_0^2 (t^3 - 3t^2 + 2t)dt = 0$$

■At $n = 4$

$$\int_0^4 t(t-1)(t-2)(t-3)(t-4)dt = 0$$

$$\rightarrow \int_0^4 (t^5 - 10t^4 + 35t^3 - 50t^2 + 24t)dt = 0$$

■At $n = 12$

$$\int_0^{12} t(t-1)(t-2)(t-3)(t-4(t-5)(t-6)(t-7)(t-8)(t-9)(t-10)(t-11)(t-12)dt$$

$$\rightarrow \int_0^{12} (t^{13} - 78\,t^{12} + 2717\,t^{11} - 55770\,t^{10} + 749463\,t^9 - 6926634t^8 + 44990231\,t^7 - 2060150\,t^6 + 657206836\,t^5 - 1414014888t^4 + 1931559552\,t^3 - 1486442880\,t^2 + 479001600\,t)dt = 0$$

Report 2) –

We can find the we can find error term of E_n , from two different mays.

A). From equation ⟨**1**⟩ to be obtained from the next term in the Newton Polynomial. i.e.

$$E_n = \frac{h^{n+2} f^{(n+1)}(\xi)}{(n+1)!} \int_0^n t(t-1)\dots(t-n)(t-(n+1))dt .$$

B). From this equation

$$E_n = \frac{h^{n+3} f^{(n+2)}(\xi)}{(n+2)!} \int_0^n t^2(t-1)\dots(t-n)dt$$

Example

■At $n = 2$ the error coefficient will be

From A).

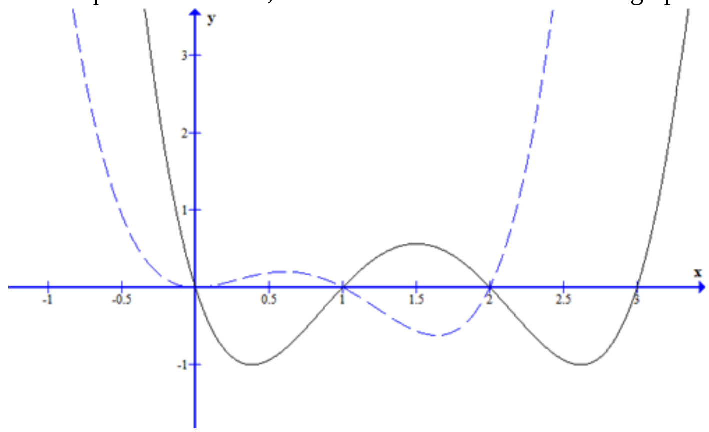
$$\int_0^2 t(t-1)(t-n)(t-3)dt$$
$$= \int_0^2 (t^4 - 6t^3 + 11t^2 - 6t)dt = -\frac{4}{15}$$

From **B).**

$$\int_0^2 t^2(t-1)(t-2)dt$$
$$= \int_0^2 (t^4 - 3t^3 + 2t^2)dt = -\frac{4}{15}$$

Note that

Both equations A & B , are deferent . As we see it in the graph .



Although we have a two deferent graphs

A) $f(t) = t^4 - 6t^3 + 11t^2 - 6t$. → ———————

B) $f(t) = t^4 - 3t^3 + 2t^2$. → – – – – – –

But they both gives the same amount of area form $t = 0$ to $t = 2$.

■At $n = 4$ the error coefficient will be

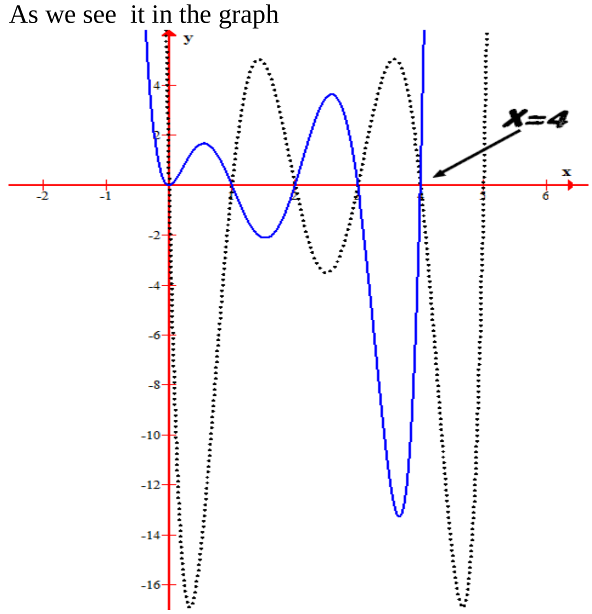
From **A)**.

$$\int_0^4 t(t-1)(t-2)(t-3)(t-4)(t-5)dt$$
$$= \int_0^4 (t^6 - 15t^5 + 85t^4 - 225t^3 + 274t^2 + 210t)dt$$
$$= -\frac{128}{21}.$$

From **B).**

$$\int_0^4 t^2(t-1)(t-2)(t-3)(t-4)dt$$
$$= \int_0^4 (t^6 - 10t^5 + 35t^4 - 50t^3 + 24t^2)dt$$
$$= -\frac{128}{21}.$$

As we see it in the graph



A) $f(t) = t^6 - 15t^5 + 85t^4 - 225t^3 + 274t^2 - 120t$. → …….

B) $f(t) = t^6 - 10t^5 + 35t^4 - 50t^3 + 24t^2$. → – – – – – –

Each one have a deferent graphs , but they both gives the same amount of area form $t = 0$ to $t = 4$.

■ At $n = 12$ the error coefficient will be

From A)

$$\int_0^{12}\begin{pmatrix} t(t-1)(t-2)(t-3)(t-4)(t-5) \\ (t-6)(t-7)(t-8)(t-9) \\ (t-10)(t-11)(t-12)(t-13) \end{pmatrix}$$

$$=\int_0^{12}(t^{14}-91\,t^{13}+3731\,t^{12}-91091\,t^{11}+1474473\,t^{10}-16669653\,t^{9}+135036473t^{8}-790943153\,t^{7}+3336118786\,t^{6}-9957703756\,t^{5}+20313753096\,t^{4}-26596717056\,t^{3}+19802759040\,t^{2}-6227020800\,t)dt=-\frac{1498963968}{5}$$

From B)

$$\int_0^{12}\begin{pmatrix} t^2(t-1)(t-2)(t-3)(t-4) \\ (t-5)(t-6)(t-7)(t-8) \\ (t-9)(t-10)(t-11)(t-12) \end{pmatrix}$$

$$=\int_0^{12}(t^{14}-78\,t^{13}+2717\,t^{12}-55770\,t^{11}+749463\,t^{10}-6926634\,t^{9}+44990231\,t^{8}-206070150\,t^{7}+657206836\,t^{6}-1414014888\,t^{5}+1931559552\,t^{4}-1486442880\,t^{3}+479001600\,t^{2})dt=-\frac{1498963968}{5}$$

V. CONCLUSION

1 – For all $n \in \mathbb{N}$ the estimation error of the Newton Cotes formula will be

$$E_n=\begin{cases} \dfrac{h^{n+2}f^{(n+1)}(\xi)}{(n+1)!}\displaystyle\int_0^n t(t-1)\ldots(t-n)dt & ;(n:odd) \\ \dfrac{h^{n+3}f^{(n+2)}(\xi)}{(n+2)!}\displaystyle\int_0^n t(t-1)\ldots(t-n)(t-(n+1))dt & ;(n:even) \end{cases}$$

Knowing that

$$\int_0^n t(t-1)\ldots(t-n)dt.\ allways=0\,,when\ n\ is\ even$$

2 – When n is even , we can find coefficient of E_n , from two different equations.

$$E_n=\begin{cases} \dfrac{h^{n+3}f^{(n+2)}(\xi)}{(n+2)!}\displaystyle\int_0^n t^2(t-1)\ldots(t-n)dt \\ \dfrac{h^{n+3}f^{(n+2)}(\xi)}{(n+2)!}\displaystyle\int_0^n t(t-1)\ldots(t-n)(t-(n+1))dt \end{cases}$$

3 – In Newton Cotes formulas of high degree , *also for low once* , The even degree formulas are slightly more accurate than the next higher odd degree formulas. For example, although both have $O(h^4)$ accuracy, the 2nd degree formula (Simpson's rule) is slightly more accurate than the 3rd degree formula (Simpson's 3/8 rule) **.**

4 – In the matter of the coefficient of the estimated $\boldsymbol{E_n}$. The accuracy of the high order Newton Cotes , is much more better than the low once .

($E_{12}-E_{10}=-0.0041183035350963\,. \rightarrow E_{12}<E_{10}$)

($E_{14}-E_{16}=-0.0025413328834151\,. \rightarrow E_{16}<E_{14}$)

($E_{16}-E_{20}=-0.001985028856832\,. \rightarrow E_{20}<E_{16}$).

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