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A New Form of Fuzzy Hausdorff Space and Related Topics via Fuzzy Idealization

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Abstract: *In this paper, fuzzy L-open sets due to Abd El-Monsef et al. [4] are used to introduce a new separation axiom and new type of function in fuzzy topological ideals spaces . Some the basic properties of fuzzy L-irresolute functions, as well as the connections between them, are investigated. Possible application to superstrings and* ζ^{∞} *space–time are touched upon.*

I. Introduction.

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh[11]. Subsequently, Chang [6] defined the notion of fuzzy topology. Since then various aspects of general topology were investigated and carried out in fuzzy sense by several authors of this field. The local properties of a fuzzy topological space, which may also be in cretin cases the properties of the whole space, are important field for study in fuzzy topology by introducing the notion of fuzzy ideal and fuzzy local function [7,10]. In 2001, Abd El-Monsef et al. [4] defined and studied the notion of fuzzy L-open set in fuzzy topological space. In the present paper, we introduce and characterize the notion of fuzzy L-Hausdorff spaces which is a generlization of some fuzzy concepts by using a fuzzy L-open sets , we also define the class of fuzzy irresolute functions via fuzzy topological ideals and investigate its relation to fuzzy L-Hausdorff spaces and some topological consepts.

II. Terminologies.

Throughout this paper, by (X,τ) we mean a fts in the sense of Chang's [6]. A fuzzy point in X with support $x \in X$ and value ε (0 < ε ≤ 1) is denoted by x_{ε} .A fuzzy point x_{ε} is said to be contained in a fuzzy set μ in X iff ε ≤ μ(x) and this will denoted by $X_{\varepsilon} \in \mu$ [8]. For a fuzzy set μ in X, $cl(\mu)$, μ^c and $\mathrm{int}(\mu)$ will respectively denote closure , complement and interior of μ .The constant fuzzy sets taking values 0 and 1 on X are denoted by 0_x ,1_x , respectively. A fuzzy set μ is said to be quasi- coincident with a fuzzy set η, denoted by μ q η , if there exists $x \in$ X such that μ(x)+η(x)>1[8]. Obviously, for any two fuzzy set μ and η, μqη will simply η q μ. A fuzzy set ρ in a fts (X,τ) is called a q-nbd of a fuzzy point x_{ε} iff there exists a fuzzy open set ν such that x_{ε}q ν ⊆ρ [6,8]. We will denote the set of all q-nbd of x_{ε} in (X, τ) by N (x_{ε}). A fts (X , τ) is said to be a fuzzy extremely disconnected [1] (F.E.D in short) if the closure of every fuzzy open set in X is fuzzy open set. A fuzzy set μ for a fts (X, τ) is called fuzzy. α-open [1] (resp, β-open [1], preopen [6]) iff μ ≤ int(cl(int(μ))) (resp. μ ≤ cl(int(cl(μ))), μ ≤ int(cl(μ))). A non-empty collection of fuzzy sets L of a set X is called a fuzzy ideal [7,9] iff (i) μ ∈ L and η ⊆ $\mu \Rightarrow$ η∈L (heredity), (ii) μ∈L and η∈L ⇒ μ ∪ η∈ L (finite additivity) Fuzzy closure operator of fuzzy set μ (in short $cl^*(\mu)$) is define $cl^*(\mu)= \mu \vee \mu^*$, and $\tau^*(L)$ be the fuzzy topology generated by cl^* i.e. $\tau^*(L)=\{\mu: cl^*(\mu)^c = \mu^c\}$ [10] .The fuzzy local function [10] μ^* (L,τ) of a fuzzy set μ is the union of all fuzzy points x_{ε} such that if ρ ∈ N(x_{ε}) and λ ∈ L then there is at least one r ∈ X for which ρ(r)+ μ(r) - 1 > λ (r).The fuzzy ideal of fuzzy nowhere dense sets is $L_n = \{\mu \in I^X : \mathrm{int}(cl(\mu)) = O_X\}$.[2,3].

Definitioin2.1. Given *(X, τ) be* a fts with fuzzy ideal L on X , $\mu \in I^X$ and μ is called

i. Fuzzy *-dense in itself if $\mu \leq \mu^*$ [2].
ii. Fuzzy scattered if contains no non empty fuzzy dense –in –itself [4] .
iii. Fuzzy L-open iff $\mu \leq$ int((μ*)) [4].
iv. Fuzzy L-closed set if its complement is fuzzy L-open set [4].

We will denote the family of all fuzzy L-open (resp.L-closed)by FLO(X)(resp.FLC(X)).

Definitioin2.2. [2]. A fts *(X, τ) is said to be fuzzy resolvble space if there is a fuzzy dense subset* $\mu \in I^X$ for wich μ^c is also fuzzy dense equivalently , it can be expressed as the disjoint union of two fuzzy dense subsets.

Lemma 2.1.[2]. A fts *(X, τ) is said to be fuzzy maximally irrresolvble space if it is fuzzy dense in –itself and has the property that every fuzzy dense subset of (X, τ)* is fuzzy open.

Recall that fuzzy topological spaces having the property that their fuzzy dense subsets are fuzzy open are called fuzzy submaximal .

Definitioin2.3 . A fts *(X, τ) is said to be fuzzy* $\beta-$*Hausdorff if for every two different fuzzy points* p, q *of X, there exist disjoint fuzzy* $\beta-$*open sets* μ and ρ *of X such that* $p \in \mu$ and $q \in \rho$.

Definitioin2.3 [4].A function $f:(X,\tau)\to(Y,\sigma)$ with a fuzzy ideal L on X is said to be fuzzy L-continuous if for every $\mu \in \sigma$, $f^{-1}(\mu) \in FLO(X)$.

Definition 2.2 [4]. For a fts *(X, τ)* with fuzzy ideal L on X , said to be compatible with L , denoted by $\tau \sim L,$ if for every fuzzy set ν of X , and for all fuzzy point $x_\varepsilon \in \nu$, there exists a q-nbd μ of x_ε such that $\nu(r)+\mu(r)-1 \le \rho(r)$ hold for every $r \in X$ and for some $\rho \in L$; then $\nu \in L$.

III. Fuzzy L-Hausdorff Space

Definition 3.1. A fts (X,τ) with fuzzy ideals L is called fuzzy L-Hausdorff space if for every pair of fuzzy singletons p, q in X with different supports , there exists $\mu, \rho \in FLO(X)$ with $p \in \mu,\ q \in \rho$ and $\mu \wedge \rho = O_X$.

Then we say that the points p and q are fuzzy separated .

The following theorem gives an equivalent definition for a fuzzy L-Hausdorff Space .

Theorem 3.1.A fts $(X,\tau)$with fuzzy ideals L be L-Hausdorff iff if for each pair of fuzzy singletons p, q in X with different supports , there exists $\mu_p \in FLO(X)$ such that $p \in \mu_p \le cl(\mu_p) \le \{q\}^c$.

Proof Let $(X,\tau)$with fuzzy ideals L is fuzzy L-Hausdorff space and fuzzy singletons p, q with different supports in X . Then there exists a fuzzy L-open set μ such that $p \in \mu, q \notin \rho$ and a fuzzy L-open set ρ such that $\mu \wedge \rho = O_X$ .Then $p \in \mu_p \le cl(\mu_p) \le \rho^c \le \{q\}^c$ .Conversely ,let p, q be two fuzzy singletons fuzzy p, q in X with different supports in X and $\mu \in FLO(X)$such that $p \in \mu_p \le cl(\mu_p) \le \{q\}^c$.Then $q \in (cl(\mu))^c = \rho$ which is fuzzy open, $\mu \wedge \rho = \mu \wedge (cl(\mu))^c = O_X$. Therefore $(X,\tau)$with fuzzy ideal L is fuzzy L-Hausdorff.

Theorem 3.2. Every fuzzy L-Hausdorff space is fuzzy $\beta-$ Hausdorff .

Proof . Let (X,τ)be a fts with fuzzy ideal L is a fuzzy L-Hausdorff space. For every fuzzy L-open subset μ we have $\mu \le \mu^{*\circ} \le cl(\mu^{*\circ})$ .It is clear that $\mu^* \le cl(\mu)$, since $o_x \in L$; hence $\mu \le cl(cl\mu^\circ)$ and thus $(X,\tau)$is fuzzy $\beta-$ Hausdorff .

The following example shows that the reverse of Theorem 3.2 is not generally true .

Example 3.1. Let $X = \{(x,1),(y,1)\}, \tau$ be the fuzzy discrete topology on X and $L = \{I^X\}$.Thus X is fuzzy Hausdorff and consequently fuzzy $\beta-$ Hausdorff , but not fuzzy L-Hausdorff , Since $FLO(X) = O_X$.

Theorem 3.3. For a fts (X,τ) with fuzzy ideal L we have

i. If (X,τ) is a fuzzy Hausdorff and every fuzzy open subspace is fuzzy *-dense in itself , then (X,τ) with L is a fuzzy L-Hausdorff.

ii. If (X,τ) is fuzzy Hausdorff and L is $\tau-$boundary , then (X,τ) with fuzzy ideal L is a fuzzy L-Hausdorff.

Proof

i. Obvious, since for every fuzzy open set $\mu \in I^X$ we have $\mu = \text{int}\,\mu \le \text{int}\,\mu^* \le cl(\mu^{*\circ})$ .For note that $\mu \le \mu^*$ since μ is fuzzy *-dense- in-itself .

ii. Clear from the definition and the fact that $\mu \le \mu^*$ Note that if L , J are two fuzzy ideal on X and $L \le J$, then $\mu^*(J) \le \mu^*(L)$ and the following result holds

Theorem 3.4. If if L , J are two fuzzy ideal on X and $L \le J$, then $FLO(X,\tau,J) \le FJO(X,\tau,L)$, so that (X,τ) with fuzzy ideal L is fuzzy $J-$Hausdorff.

Proof .Obvious

Theorem 3.5 . Let (X,τ) be a fts with fuzzy ideal L is a fuzzy L-Hausdorff space and $\mu \in I^X$ .Then

i. If μ is fuzzy open, then μ is fuzzy L- Hausdorff space .

ii. If μ is fuzzy is α -open, then μ is fuzzy β -Hausdorff.

Proof .

i. Follows directly from Lemma2.1 [3] and the fact that for a fts (X,τ) with fuzzy ideal L, if $\rho \in \tau$ and μ be a subspace of (X,τ) then $\rho \wedge (\rho \wedge \mu)^* \le (\rho \wedge \mu)^*$.

ii. Since every L-open set is fuzzy β -open and the intersection of a fuzzy β -open and fuzzy α -open set is fuzzy β -open in fuzzy α -open set, the theorem is clear .

Thereom3.6. Let with fuzzy ideal be fts with the following property if are two fuzzy singletons with different supports, then there exist such fuzzy Hausdorff space Y and fuzzy L-contiuaus function that .Then is fuzzy L-Hausdorff .

Proof . Obvious

IV. Fuzzy L- irresolute functions

Definition 4.1. A function $f:(X,\tau)\to(Y,\sigma)$ with fuzzy ideal L on X is called fuzzy L- irresolute if $f^{-1}(\mu) \in FLO(X)$ $(resp. FLC(X))$ for every $\mu \in FLO(Y)$ $(resp. FLC(Y))$.

Theorem 4.1 . if a function $f:(X,\tau)\to(Y,\sigma)$ with fuzzy ideal L on X is fuzzy L- irresolute then for any fuzzy L-open set $\mu \in \text{I}^Y$ we have $cl(f^{-1}(\mu^*)) \le f^{-1}(cl(\mu^*))$.

Proof Since $\mu \le \text{int}(\mu^*) \le cl(\mu^*)$ this implies $cl(f^{-1}(\mu^*)) \le f^{-1}(cl(\mu^*))$. and since f is fuzzy L-irresolute the theorem is proved .

Theorem 4.2. For a function $f:(X,\tau)\to(Y,\sigma)$ with fuzzy ideal L and J on X and Y respectively .The following the following are equivalent

i. f fuzzy L- irresolute ;

ii. the inverse image of each fuzzy L-closed in (Y,σ) with fuzzy ideal J is fuzzy L-closed in (X,τ) with fuzzy ideal L;

iii. containing $f(p)$ such that $f(\mu) \le \rho$.

Proof . The proof is obvious and thus omitted .

Theorem 4.3. Let f be a one –to-one fuzzy L- irresolute map from fts (X,τ) with fuzzy ideal L into a fts (Y,σ) with fuzzy ideal J . If (Y,σ) is fuzzy L- Hausdorff , then (X,τ) with fuzzy ideal L is fuzzy L-Hausdorff .

Proof Let p,q be a pair of fuzzy singletons in X with different supports .Then $f(p) \ne f(q)$ and moreover $f(p)$ and $f(q)$ are fuzzy L-separated in (Y,σ) by fuzzy L-open sets μ and ρ , respectively . the disjoint fuzzy sets $f^{-1}(\mu)$ and $f^{-1}(\rho)$ are fuzzy L-open ,since f is fuzzy L- irresolute and contain p,q , respectively. Then , we have that (X,τ) is fuzzy L- Hausdorff

Theorem 4.4. Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\theta)$ be two functions ,where L,J and K are fuzzy ideals on X,Y and Z , respectively .Then $g\circ f$ is fuzzy L-irresolute , if both f and g are fuzzy L-irresolute.

Proof . obvious.

Theorem 4.5 . Let $f:(X,\tau)\to(Y,\sigma)$ where L,J are fuzzy ideals on X and Y respectively. Then

i. If f is fuzzy L-irresolute and each open subspace of Y is fuzzy *-dense-in itself , then f is fuzzy L-continuous .

ii. If f is fuzzy L-continuous and (Y,σ) is fuzzy sub- maximal E.D, then f is fuzzy L- irresolute .

iii. If $L=J=\{O_X\}$, each fuzzy irresolute and β - irresolute is equivalent .

iv. If $L=J=\{I^X\}$, fuzzy L- irresoluteness and fuzzy β -irresoluteness coincide .

Proof .

i. Clear, since each fuzzy open and fuzzy *-dense in-it-self is fuzzy L-open and consequently fuzzy L-open (see [4]).

ii. Clear, since each fuzzy L- open set is fuzzy β -open and thus fuzzy open from Lemma 2.1 [4] .

iii. , iv Follow from Lemma 2.1. [4] .

Theorem 4.6. if $f:(X,\tau)\to(Y,\sigma)$ with fuzzy ideal L is an injective fuzzy L-irresolute map then (Y,σ) is fuzzy L-Hausdorff space and then (X,τ) is fuzzy L- Hausdorff space.

Proof Let ρ_1,ρ_1 be any two distinct fuzzy singletons in X , thus we have $\rho_1=f^{-1}(q_1)$ and $\rho_2=f^{-1}(q_2)$, where q_1 and q_2 are distinct fuzzy singletons in (Y,σ).Since (Y,σ) is fuzzy L-Hausdorff , then there exist $\mu,\rho\in FLO(Y)$ such that $q_1\in\mu, q_2\in\rho$ and $\mu\le Y-\rho$, thus $f^{-1}(q_1)\in f^{-1}(\mu), f^{-1}(q_2)\in f^{-1}(\mu)$ and $f^{-1}(\mu)\le X-f^{-1}(\rho)$ which means that $\mu\wedge\rho=O_X$ this implies (X,τ) is fuzzy L-Hausdorff space .

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Measure and Significance of Association between K Populations: A Non- Parametric Method

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Abstract: *This paper presents a non parametric measure of association between k populations, and a method of testing for its significance. Analysis of variance technique is employed to developa test statistic for the measure of the association. An illustrative example is provided and the method compares equally well with the Friedman's two way analysis of variance by rank.*

I. Introduction:

When assumptions of normality and homogeneity for the use of parametric two way analysis of variance for data analysis are not satisfied, use of a non-parametric equivalence becomes preferable. One of the methods often used is the Friedman's Two Way Analysis by ranks (Gibbons, 1971, Scheaffer and McClave, 1982, Gerald and Warrack, 2003, Zar, 1999, Legendre, 2005,and Sheskin, 1997).

In this paper, we propose to develop a measure of association between populations appropriate for analysis of variance by ranks and to develop an alternative test statistic for the proposed measure.

II. The Proposed Measure

As in Friedman's Test, suppose a random sample of k assessors, judges, observers or teachers are each to observe or assess and rank each of c candidates, patients, conditions , or situations. As in Friedman's test these data if treated as a two-way analysis of variance would correspond to a mixed effects model without replication (Oyeka, 2009). This means that the data are presented in the form of a kxc table with say, the column corresponding to one factor with c treatments or respondents which are considered fixed and the row corresponding to a seccond factor with k blocks, levels or observers which are random and there are only one observation per cell. The data are therefore arranged in a table with c columns and k rows, just as for the corresponding two way analysis of variance with one observation per cell. As in the analogous analysis of variance, the null hypothesis to be tested is that the k judges or assessors are in agreement or do not differ in their assessment of the c conditions or treatments versus the alternative hypothesis that the assessors do not in fact differ. Interest here is also in finding a common measure of association, agreement or concordance between the 'k" assessors in their assessment of the "c' conditions or respondents.

To answer these questions using a non-parametric approach, we first rank the observation in each row (observer) from smallest to the largest or from the largest to the smallest. That is within each row (observer), the rank of 1 is assigned to the smallest or largest value. The rank of 2 is assigned to the next smallest (largest) value, and so on until the rank of "c" is assigned to the largest (smallest) value.

Now let r_{ij} be the rank assigned by the ith observer or assessor to the jth condition, subject, or object, for $i = 1, 2, \ldots, k, j = 1, 2, \ldots, c$. Then the $ith\, row\, is\, a\, permutation$ of the number 1, 2, …, c, and the jth column represents the ranks assigned to the jth subject by the observers. The ranks in each column are then indicative of the agreement between observers since if the jth object has the magnitude relative to all other objects in the opinion of each of the "k" observers, all ranks in the jth column will be the same. Thus if the null hypothesis is true, we would expect the occurrence of the ranks 1, 2, …, c to be equally likely in each column (object) across all rows (observers). This implies that we would expect the column sums of ranks to be the same under the null hypothesis. If the observed sums of column ranks are so discrepant that they are not likely to be as a result of equal probabilities, then this constitutes an evidence against randomness and against the null hypothesis. If however, all the k observers agree perfectly in their ranking of each of the c objects, then the respective column totals $R_1, R_2,, \ldots, R_c$, will be some permutation of the numbers $1k, 2k, \ldots, ck$.

Now since the average column total is $k\left(\frac{c+1}{2}\right)$, for perfect agreement between the k observers in their ranking of the "c" objects, the sum of squares of deviations of column totals from the average column total,S^2_{max} will have maximum value and a constant given as:

$$S^2_{max} = \sum_{j=1}^{c} \left\{jk - k\frac{(c+1)}{2}\right\}^2 = k^2 \sum_{j=1}^{c} \left[j - \frac{(c+1)}{2}\right]^2$$

That is $S^2_{max} = k^2 c \frac{(c^2-1)}{12}$ ………………………………… 1

However in general, the actual sum of squared deviations of observed column totals from the average total,

$$S_{ob}^2 = \sum_{j=1}^{c} \left\{Rj - k\frac{(c+1)}{2}\right\}^2$$

That is $S_{ob}^2 = \sum_{j=1}^{c} R_j^2 - k^2 c\frac{(c+1)^2}{4}$ … ………………………… 2

Note that since S_{max}^2 and S_{ob}^2 are both sums of squares, they are non negative. However since k and c are both positive integers, $S_{max}^2 > 0, (c > 1)$ but $S_{ob}^2 \geq 0$ and is equal to 0 if the ranking of the "c" objects by the k observers are completely at random such that $R_j = \frac{k(c+1),}{2}$, for all $j = 1, 2, \ldots, c$. If the observers are in agreement in their ranking of the "c" objects, then $S_{ob}^2 = S_{max}^2$ hence a good measure W, of agreement between observers in their ranking of the objects is the ratio of these two sums of squares. That is

$W = \frac{S_{ob}^2}{S_{max}^2}$ …………………………………… 3

This is similar to Kendall coefficient of concordance (Gibbon,1971), and hence to Friedman's two- way analysis of variance without replication by ranks. Kendall's coefficient of concordance .and Friedman's two – way analysis of variance are so closely related that they address hypothesis concerning the same data table and use the same χ^2 statistic for testing (Legendry,2005). W ranges between 0 and 1 with 1 designating perfect concordance and 0 indicating no agreement or independence of populations. Usually $0 < W < 1, in\ general.$

Test Statistic for W

We now proceed to develop a test statistic for W, using analysis of variance technique. The total sum of squared deviations of assigned ranks r_{ij} from the mean rank, $\bar{r} = \frac{c+1}{2}, is$

$$SS_{total} = S_t^2 = \sum_{i=1}^{k}\sum_{j=1}^{c}\left[r_{ij} - \bar{r}\right]^2$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{c}\left[r_{ij} - \frac{(c+1)}{2}\right]^2$$
$$= \sum_{i=1}^{k}\sum_{j=1}^{c} r_{ij}^2 - \frac{kc(c+1)^2}{4}$$
$$= \frac{kc(c+1)(2c+1)}{6} - \frac{kc(c+1)^2}{4}$$

That is

$SS_{total} = S_t^2 = \frac{kc(c^2-1)}{12}$ ……………………………. 4

Note from equations 1and 4 that

$S_{max}^2 = kS_t^2$ ……………………………………………… 5

The total sum of squares $SS_{total} = S_t^2$ can be partitioned into three sums of squares that can be shown to be independent (Hogg and Craig, 1971).

Thus $SS_{total} = S_t^2 = \sum_{i=1}^{k}\sum_{j=1}^{c}(r_{ij} - \bar{r})^2$

$$= \sum_{i=1}^{k}\sum_{j=1}^{c}\left[(r_{ij} - \bar{r}_{i.} - \bar{r}_{.j} + \bar{r}) + (\bar{r}_{i.} - \bar{r}) + (\bar{r}_{.j} - \bar{r})\right]^2$$
$$= \sum_{i=1}^{k}\sum_{j=1}^{c}(r_{ij} - \bar{r}_{i.} - \bar{r}_{.j} + \bar{r})^2 + c\sum_{i=1}^{k}(\bar{r}_{i.} - \bar{r})^2 + k\sum_{j=1}^{c}(\bar{r}_{.j} - \bar{r})^2$$

Now $\sum_{i=1}^{k}\sum_{j=1}^{c}(r_{ij} - \bar{r}_{i.} - \bar{r}_{.j} + \bar{r})^2$ $is\ the\ error\ sum\ of\ squared\ deviation\ , SS_E = SS_e$

$c\sum_{i=1}^{k}(\bar{r}_{i.} - \bar{r})^2 = ck\left[\frac{(c+1)}{2} - \frac{(c+1)}{2}\right]^2 = 0$is sum of the squared deviations due to row or observers.

Finally, $k\sum_{j=1}^{c}(\bar{r}_{.j} - \bar{r})^2 = k\left[\sum_{j=1}^{c}\frac{R_{.j}^2}{R^2} - \frac{c(c+1)^2}{4}\right] = \frac{\left\{\sum_{j=1}^{c} R_{.j}^2 - \frac{k^2c(c+1)^2}{4}\right\}}{k}$ is the sum of squared deviations due to column (object),$SS_c = S_c^2$

That is

$SS_c = S_c^2 = \sum_{j=1}^{c} R_{.j}^2 - \frac{\frac{k^2c(c+1)^2}{4}}{k}$ ………………………… 6

In other words, $S_c^2 = \frac{S_{ob}^2}{k}$ ………………………... 7

Therefore, $S_t^2 = \frac{S_{ob}^2}{k} + S_e^2$
8

Now for sufficiently large values of k and c, it is known that the observer or row sum of squares SS_R which is zero has a chi-square distribution with k-1 degrees of freedom, the object or column sum of squares, $SS_c = \frac{S_{ob}^2}{k}$ has a chi-square distribution with c-1 degrees of freedom and the sum of squares error, $SS_E = SS_e$ has a chi-square distribution with $(k-1)(c-1)$ degrees of freedom (Hogg and Craig, 1971). Hence under H_0:

$$F = \frac{SS_c / (c-1)}{SS_e / (k-1)(c-1)} = \frac{(k-1) S_{ob}^2 / k}{S_e^2} \quad ---$$

9

Has an F- distribution with $(c-1)$and $(k-1)(c-1)$ $degrees\, of\, freedom$ or from equation 8, we have that:

$$F = \frac{(k-1) S_{ob}^2 / k}{S_t^2 - \frac{S_{ob}^2}{k}} \quad ---$$

10

has an F-distribution with $(c-1)$and $(k-1)(c-1)$ $degrees\, of\, freedom$.

Using equation 5 in 10, we have that

$$F = \frac{(k-1)S_{ob}^2}{kS_t^2 - S_{ob}^2} = \frac{(k-1)S_{ob}^2}{S_{max}^2 - S_{ob}^2} \quad ---$$

11

Dividing through equation 11 by S_{max}^2 and noting from equation 3 that

$W = \frac{S_{ob}^2}{S_{max}^2}$we have the test statistic

$$F = \frac{(k-1)W}{1-W} \quad ---$$

12

Which has an F- distribution with $(c-1)$and $(k-1)(c-1)$$degrees\, of\, freedom$ which can be used to test our H_0 about W. H_0 $is\ to\ rejected\ at\ \alpha\ level\ of\ significance\ if$

$$F \geq F_{1-} \alpha, (c-1), (k-1)(c-1) \quad ---$$

13

Accept otherwise.

ILLUSTRATIVE EXAMPLE

The percent reduction in heart beat of a random sample of 15 bats of ertain species after the administration of three different dose levels of a certain drug is presented in Table 1.

Table 1. Bats and Dose levels of the Drug

Bat No.				
	A	B	C	Total
1	2	3	1	6
2	2	1	3	6
3	1	2	3	6
4	1	2	3	6
5	2	1	3	6
6	2	3	1	6
7	3	1.5	1.5	6
8	1	3	2	6
9	3	2	1	6
10	1	2.5	2.5	6
11	3	1.5	1.5	6
12	3	1	2	6
13	1	2.5	2.5	6
14	3	1	2	6
15	1.5	1.5	3	6
Total	29.5	28.5	32	90

,

Source: Exercises at the end chapter 14 Question 14.12 (Oyeka, 2009). Interest is in testing at 0.01 level of significance, the null hypothesis of no difference in responses between the three dose levels A, B, C. Or symbolically the null hypothesis of interest is stated thus:

H_0: The locations of all k populations are the same

H_1: At least two populations differ

Then we obtained from computations as follows:

from Equation 4

$$S_t^2 = \frac{15(3)(8)}{12} = 30$$

FromEquation 1, $S_{max}^2 = \frac{15^2(3)(8)}{12} = 45.0$

From equation 2, $S_{ob}^2 = (29.5)^2 + (28.5)^2 + (32)^2 - \frac{15^2 x3x4^2}{4} = 6.5$

From equation 3, W $W = \frac{6.5}{45.0} = 0.014$

From Equation 7, $S_c^2 = \frac{6.5}{15} = 0.433$

$S_e^2 =$ 30-0.433 = 29.567

And from Equation 12, we have

$$F = \frac{14(0.014)}{1-0.014} = 0.199 \quad (pvalue = 0.8207)$$

But $F_{0.99,2,28} = 5.45$

Since $F = 0.199 < 5.45 = F_{0.99,2,28}$, we accept H_0 and conclude that there is no significance difference in responses of the bats to three dose levels of the drug.

Friedman's Two- Way Analysis of Variance by Rank Method

Table 2: Bats, Dose Levels of Drugs and their Ranks

Bat No	Dose Levels			Ranks			
	A	B	C	Rank (A)	Rank (B)	Rank ©	Total
1	5	6	3	2	3	1	6
2	6	4	8	2	1	3	6
3	2	3	8	1	2	3	6
4	2	5	7	1	2	3	6
5	3	2	4	2	1	3	6
6	4	5	3	2	3	1	6
7	12	7	7	3	1.5	1.5	6
8	6	12	7	1	3	2	6
9	7	5	3	3	2	1	6
10	3	4	4	1	2.5	2.5	6
11	4	3	3	3	1.5	1.5	6
12	8	6	7	3	1	2	6
13	2	7	7	1	2.5	2.5	6
14	13	7	8	3	1	2	6
15	5	5	10	1.5	1.5	3	6
Total				29.5	28.5	32	90

To test the null hypothesis that the locations of all k populations are the same against the alternative that at least two populations' locations differ, Friedman's F- ratio (Fr) test statistic is

$Fr = \frac{12}{kc(c+1)} \sum_{j=1}^{c} R_j^2 - 3k(c+1)$ ——— 14

and rejection is given by

$$Fr = \frac{12}{15(3)(4)}(29.5^2 + 28.5^2 + 32) - 3(15)(4)$$
$$= 0.433 \; (p\ Value = 0.8057$$

and$\chi^2_{0.99,2} = 9.21$

Since $Fr = 0.433 < 9.21 = \chi^2_{0.99,2}$, we accept $H_0$and conclude that the responses of the bats to the three dose levels of the drug do not differ.

III. Conclusion

Both the proposed method and Friedman's two way analysis of variance by rank method not only accepted H_0 at 1% significant levels, but also their p-values were almost the same (0.8207, and 0.8057 respectively). Thus one can conclude that the proposed test statistic is as good as the Friedman's test statistic in this case.

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Topic: Ties Adjusted Extended Sign Test For Ordered Data

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Abstract: *This paper developed a Ties adjusted non parametric statistical method for the analysis of ordered repeated measures that are related in time, space or condition that takes account of all possible pairwise combinations of treatment levels. A test statistic is developed to determine whether subjects are increasingly performing better or worse over time or space. The proposed method also enables the researcher to have a bird's eye view of the proportions of subject who are successively improving, experiencing no change or worsening overtime, space or condition to guide the introduction of any desired interventionist measures. The method is illustrated with some data and shown to be more powerful than Friedman test and shown to beeasier to use than the Bartholomew procedure.*

I. Introduction:

If one has repeated measures randomly drawn from a number of related populations that are dependent on some demographic factors or conditions or that are ordered in time or space which do not satisfy the necessary assumptions for the use of parametric tests, then use of nonparametric methods is indicated and preferable. These types of data include subjects' or candidate's scores in examinations or job placement interviews at various points in time; diagnostic test results repeated a certain number of times; commodity prizes at various times, locations or markets; etc.

Statistical analysis of these types of data often require the use of non parametric methods such as Friedman's two way analysis of variance test by ranks or the Cochran's Q test(Gibbons 1971,Gibbons 1993, Oyeka et al, 2010; Siegel, 1956; Hollander and Wolfe, 1999, Freidlin and Gastwirth 2000).

However, a problem with these two methods is that the Friedman's test often tries to adjust for ties that occur in blocks or batches of sample observations by assigning these tied observations their mean ranks a procedure that tends to reduce the power of the test, while the Cochran's Q test requires the observations to be dichotomous assuming only two possible values. Furthermore, if the null hypothesis to be tested is that subjects are increasingly performing better or worse with time or space, then these two statistical procedures may not be readily applicable.

In this case the methods developed by Bartholomew and others (Bartholomew 1959a, 1959b) may then be used. However, some of these methods are rather difficult to apply in practice and the resolution of any ties that may occur within blocks of observations is not often easy.

In this paper, we propose aties adjusted non parametric statistical method for the analysis of ordered repeated measures that are related in time, space or condition, that takes account of all possible pairwise combinations of treatment levels.

II. The Proposed Method

Suppose $(x_{i1}, x_{i2}, \dots, x_{in})$ is the i^{th} batch, set of or match observations randomly drawn from related populations, $X_{1,}, X_2, \dots, X_k$ for $i = 1, 2, \dots, n$ where 'k' may be indexed in time or space. Populations $X_{1,}, X_2, \dots, X_k$ may be measurements on as low as the ordinal scale and need not be continuous.

The problem of research interest here is to determine whether subjects are on the average progressively increasing, experiencing no change or worsening in their scores or performance over time, space or remission of condition. It is quite possible that within any specified time interval say some subject scores at some time in the intervalinstead of as expected being monotone, other increasing or decreasing , may be higher than their scores earlier in the interval which are themselves higher than their scores later in the interval.

Let

$d_{ij} = c$ ………………………….. 1

For $i = 1, 2, \dots, n;\ j = 1, 2, \dots, k-1$

Let

$$u_{ij} = \begin{cases} 1, if\ d_{ij} > 0 \\ 0, if\ d_{ij} = 0 \\ -1\ if\ d_{ij} < 0 \end{cases} \quad \dots\dots\dots\dots\dots\dots 2$$

For $i = 1, 2, \dots, n;\ j = 1, 2, \dots, k-1$

Note that equations 1 and 2 can be combined into one equation as

$$u_{ij} = \begin{cases} 1, if\ x_{ij} > x_{ij+1} \\ 0,\ if\ x_{ij} = x_{ij+1} \\ -1\ if\ x_{ij} < x_{ij+1} \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \text{2b}$$

For $i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, k-1$ which is easier to use when the data being analysed are ordinal scale data that are non- numeric measurements such as letter grades.

Define $\pi_j^+ = P(u_{ij} = 1);\ \pi_j^0 = P(u_{ij} = 0);\ \pi_j^- = P(u_{ij} = -1)$ ………………………… 3

Where

$\pi_j^+ + \pi_j^0 + \pi_j^- = 1$ ……………………………… 4

For $j = 1, 2, \ldots, k-1$

Note that equation2, 4 have structurally and intrinsically provided for the possibility of tied observations between successive pairs of the sampled populations. The model specifications allow ties to occur between these pairs of sampled populations with probability $\pi_j^0,\ j = 1, 2, \ldots, k-1$

Let

$W_j = \sum_{i=1}^{n} u_{ij}$ …………………………….. 5

Also let

$W = \sum_{i=1}^{k-1} w_j = \sum_{j=1}^{k-1}\sum_{i=1}^{n}$ ……………………….6

Now

$E(u_{ij}) = \pi_j^+ - \pi_j^-; Var(u_{ij}) = \pi_j^+ + \pi_j^- - (\pi_j^+ - \pi_j^-)^2$ ……………………. 7

Note that $\pi_j^+,\ \pi_j^- and\ \pi_j^-$ are respectively the probabilities that observations from greater than, equal to, less than observations from population X_{j+1} for $j = 1, 2, \ldots, k-1$. The sample estimates of these probabilities are$\hat{\pi}_j^+ = \frac{f_j^+}{n}; \hat{\pi}_j^0 = \frac{f_j^0}{n}; \hat{\pi}_j^- = \frac{f_j^-}{n}$ …………………………… 8

Where $f_j^+, f_j^0 and\ f_j^-$ are respectively the number of the $1s, 0s\ and - 1s$ in the frequency distribution of these numbers in u_{ij} for $i = 1, 2, \ldots, n, j = 1, 2, \ldots, k-1$

From equation 5 we have that the expected value of $W_j\ is$

$E(w_j) = \sum_{i=1}^{n} E(u_{ij}) = n(\pi_j^+ - \pi_j^-)$ …………………….. 9

And

$Var(w_j) = \sum_{i=1}^{n} Var(u_{ij}) = n\left(\pi_j^+ - \pi_j^- - (\pi_j^+ - \pi_j^-)^2\right)$ ……………………….. 10

Note from equations 9 and 10 that both w_j and its variance are independent of π_j^0 and hence are not affected by any possible ties between successive pairs of sampled populations.

Also, note that $\pi_j^+ - \pi_j^-$ is a measure of the difference between the probabilities that observations from population X_j are on the average greater than observations from population X_{j+1} and the probability that observations from population X_j are on the average less than observations from population X_{j+1} and is estimated by

$\hat{\pi}_j^+ - \hat{\pi}_j^- = \frac{W_j}{n} = \frac{f_j^+ - f_j^-}{n}$ ………………………………… 11

Where

$W_j = f_j^+ - f_j^-$ ……………………………………………12

Using equation $(11)\ in\ equation\ (10)$, we obtain the sample estimate of the variance of W_j as

$Var(W_j) = n(\hat{\pi}_j^+ - \hat{\pi}_j^-) - \frac{W_j^2}{n}$ ………………………………… 13

The null hypothesis that is usually of interest in time or space ordered related populations or several ordered related populations have medians M_j that are successively atmost $(or\ atleast)$ equal to each other. That is, the null hypothesis that is usually of interest is that either $M_j \le M_{j+1}$or $M_j \ge M_{j+1}$,for$j = 1, 2, \ldots, k-1$. Hence the null hypothesis of interest here is

$H_0: \pi_j^+ \le \pi_j^-$say versus $H_1: \pi_j^+ > \pi_j^-$ …………………… 14

Under this null hypothesis, the test statistic

$$\chi_j^2 = \frac{\left(W_j - n(\pi_j^+ - \pi_j^-)\right)^2}{Var\ (W_j)} = \frac{\left(W_j - n(\pi_j^+ - \pi_j^-)\right)^2}{n(\pi_j^+ + \pi_j^-) - \frac{W_j^2}{n}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 15,$$

For $j = 1, 2, \ldots, k-1$ has approximately the chi- square distribution with 1 degree of freedom for sufficiently large n and maybe used to test the null hypothesis of equation 14 where $\pi_j^+ - \pi_j^-$is the hypothesized difference between $\pi_j^+ and\ \pi_j^-, j = 1, 2, \ldots, k-1$. $H_0\ is\ rejected$ atα level of significance if

$c \geq \chi^2_{1-\alpha,1}$ …………………………… 16

Otherwise, $H_0$is accepted.

However to avoid committing a type 11 Error too frequently, it is recommended that the calculated chi-square values of equation 5 be compared with the tabulated chi- square value with k-1 degrees of freedom instead of 1 degree of freedom.

Note that like W_j and its variance, the test statistic χ_j^2 of Equation 15 isnot affected by the presence of any possible ties between populations $\chi_j^2 \, and \, \chi_{j+1}^2, \; j = 1, 2, \dots, k-1$

Of more general interest here however is that 'k' related populations that are ordered in time or space have medians that are atmost(at least) successively equal to one another. In other words, the null hypothesis that may be of interest would be that if $M_1, M_2, \dots M_k \; are$ the medians of 'k' time or space ordered populations, then the expectation would be that M_1 is at most equal to M_2 which is in turn at most equal to M_3 and so on until M_{k-1} is at most equal to M_k say.

The sample estimate of $\sum_{j=1}^{k-1}(\pi_j^+ - \pi_j^-)$ is from equation 17

$$(\hat{\pi}_j^+ - \hat{\pi}_j^-) = \sum_{j=1}^{k-1} \frac{(f_j^+ - f_j^-)}{n} = \frac{\sum_{j=1}^{k-1} W_j}{n} = \frac{W}{n}$$

Note that like W_j and its variance which have been adjusted for the possibility of tied observations, W and its variance have also been similarly ties adjusted and are hence not affected by any ties between $X_j \; and \; X_{j+1}$ for all $j = 1, 2, \dots, k-1$ sampled populations.

Now if pairs of the k populations that are successively ordered in time or space have at most equal medians than the difference between $\pi_j^+ \; and \; \pi_j^-$ would be expected to be equal to some constant θ_0 say which includes zero for all $j = 1, 2, \dots, k-1$, where $\pi^+, \pi^0 \; and \; \pi^-$ are respectively the common values of $\pi_j^+, \pi_j^0 \; and \; \pi_j^- \; under \; H_0$ for $j = 1, 2, \dots, k-1 \; and$ are estimated as

$$\hat{\pi}^+ = \sum_{j=1}^{k-1} \frac{\pi_j^+}{k-1} = \sum_{j=1}^{k-1} \frac{f_j^+}{n(k-1)}; \; \hat{\pi}^0 = \sum_{j=1}^{k-1} \frac{\pi_j^0}{k-1} = \sum_{j=1}^{k-1} \frac{f_j^0}{n(k-1)}; \; \hat{\pi}^- = \sum_{j=1}^{k-1} \frac{\pi_j^-}{k-1} = \sum_{j=1}^{k-1} \frac{f_j^-}{n(k-1)}; \quad ..21$$

Under the null hypothesis H_o, Equation 18 becomes $(k-1)(\hat{\pi}^+ - \hat{\pi}^-) = \frac{W}{n} \; , so \; that$

$W = n(k-1)(\hat{\pi}^+ - \hat{\pi}^-)$ ……………………..22

The sample estimate of the variance of W under H_o is then from Eqn 19

$Var(W) = n(k-1)(\hat{\pi}^+ + \hat{\pi}^- - (\hat{\pi}^+ - \hat{\pi}^-)^2)$ ………………..23

If the null hypothesis of Eqn 20 is true, then the test statistic

$$\chi^2 = \frac{(W - n\theta_0)^2}{Var(W)} = \frac{(W - n\theta_0)^2}{n(k-1)(\hat{\pi}^+ + \hat{\pi}^- - (\hat{\pi}^+ - \hat{\pi}^-)^2)} \quad ………………24$$

has approximately a Chi-Square distribution with k-1 degrees of freedom for sufficiently large n and may be used to test the null hypothesis H_o of Eqn 20. The null hypothesis is rejected at the α level of significance if

$\chi^2 \; \geq \; \chi^2_{1-\alpha, \; k-1}$ ……………………..25

Otherwise $H_o \; is \; accepted.$

Like the test statistic of Eqn 15, the test statistic of Eqn 24 is also unaffected by the presence of any possible ties between success ie pairs of sampled populations. The null hypothesis of Rqn20 is usually tested first. Its rejection would indicate the existence of some difference between the k population medians. In this case one would then proceed to test the null hypothesis of Eqn 14 to determine which paired populations have different medians that may have led to the rejection of the more general null hypothesis of Eqn 20.

III. Illustrative Example

Shown below are data on the letter grades earned by a random sample of 18 undergraduate students of a certain academic program during each of the five years of their studies in a university.

Student's Nos	Year1	Year2	Year3	Year4	Year 5
1	C^+	E	A^+	B	A^-
2	C^+	C	C^+	A^+	A
3	B^+	A^-	B^+	$\boldsymbol{B^+}$	B^-
4	F	C	B	C^-	B^-
5	A^-	C^-	F	F	E
6	B	B^+	B^-	E	E
7	B^-	F	A^+	A^+	B^+
8	E	B^+	A^-	B^-	C^-
9	C	B^+	B	B	F
10	C^-	C^-	E	B^-	C^+
11	C	C^-	F	C^-	A^-

12	A^+	B^+	E	C^+	C
13	F	C^-	A^+	F	B^+
14	B^+	E	B^-	B^+	C
15	C	C^-	B	B	C^-
16	B	A	A^-	B^+	C^+
17	B	A^+	E	A	A
18	A	A^+	A	A	E

To illustrate use of the proposed method, we apply Equation 2b to the above data to obtain values of u_{ij}, results of which are presented in table 1 for $i = 1, 2, ..., 18; j = 1, 2, 3, 4$

Table 1: Tabulation of u_{ij} (Equation 2b) for the illustrative data

Student's S/No	u_{i1}	u_{i2}	u_{i3}	u_{i4}	
1	1	-1	1	-1	
2	1	-1	-1	-1	
3	-1	1	0	1	
4	-1	-1	1	-1	
5	1	1	0	-1	
6	-1	1	1	0	
7	1	-1	0	1	
8	-1	-1	1	1	
9	-1	1	0	1	
10	0	1	-1	1	
11	1	1	-1	-1	
12	1	1	-1	1	
13	-1	-1	1	-1	
14	1	-1	-1	1	
15	1	-1	0	1	
16	-1	1	1	1	
17	-1	1	-1	0	
18	-1	1	0	1	
f_j^+	8	10	6	10	34(= f^+)
f_j^0	1	0	6	2	9(=f^0)
f_j^-	9	8	6	6	29(=f^-)
n	18	18	18	18	72(=n(k-1))
$\hat{\pi}_j^+$	0.444	0.556	0.333	0.556	0.472(=$\hat{\pi}^+$)
$\hat{\pi}_j^0$	0.056	0.000	0.333	0.111	0.125(=$\hat{\pi}^0$)
$\hat{\pi}_j^-$	0.500	0.444	0.333	0.333	0.389(=$\hat{\pi}^-$)
W_j	-1	2	2	4	7(=W)

The values of $f_{j,}^+$, $f_{j,}^0$, $f_{j,}^-$, $\hat{\pi}_j^+$,$\hat{\pi}_j^{0\cdot}$, $\hat{\pi}_j^-$ and W_j for j = 1, 2, 3, 4 are *calculated* as discussed above and shown in Table 1. From Equation 12, we have that W = 34-29 =5. From equation 23, we estimate the variance of 'W' as

$$Var(W) = (18)(4)(0.472 + 0.403 - (0.472 - 0.404)^2)$$

$= (72)(0.870) = 62.64$

Hence to test the null hypothesis of equation 20, we have from Equation 24, with W=5, $\theta_0 = 0$, $\mathrm{Var}(W) = 62.640$ that

$$\chi^2 = \frac{5^2}{62.640} = 0.399, \text{which with } 5 - 1 = 4 \text{ degrees of freedom is not statistically significantlyat}$$

$\alpha = 0.05$. Hence we may conclude that students performance do not seem to be increasing (or decreasing) with time during their five years of study.

It would be instructive to compare the results obtained with the proposed methods with the results that would have been obtained if the Cochran's Q test had been used to analyse the above data. To do this, we first compare the grades of each student during every two successive years assigning the score 1 if the students grade in the past year is greater than the students grade in the immediately succeeding year and 0 otherwise for the five year period.

Application of Cochran test to the Data on letter grades of 18 students

Table 2: Relative order of the grades in Table 1 for use with Cochran Q Test

S/NO	d_{i1}	d_{i2}	d_{i3}	d_{i4}	Total $B_{i.}$
1	1	0	1	0	2
2	1	0	0	1	2
3	0	1	0	1	2
4	0	0	1	0	1
5	1	1	0	0	2
6	0	1	1	0	2
7	1	0	0	1	2
8	0	0	1	1	2
9	0	1	0	1	2
10	0	1	0	0	1
11	1	1	0	0	2
12	1	1	0	1	3
13	0	0	1	0	1
14	1	0	0	1	2
15	1	0	0	1	2
16	0	1	1	1	3
17	0	1	0	0	1
18	0	1	0	1	2
Total $T_{.j}$	8	10	6	10	34

In Table 2 d_{ij} assume the value 1 if the grade earned by the i^{th} student in year 'j' is higher than that in year $j+1$ and assumes the value 0 otherwise for $i = 1, 2, \ldots, 18$; $j = 1, 2, 3, 4$.
Now using the marginal sums $T_{.j}$ and $B_{i.}$ Shown in table 2 in the Cochran's Q test statistic, we have

$$Q = \frac{(3)\,(8^2 + 10^2 + 6^2 + 10^2 - (34)^2)/4}{34 - 61/4}$$

$$= \frac{(3)(300 - 289.0)}{34 - 15.25} = \frac{33}{18.75} = 1.76$$

which with 4 degrees of freedom is not statistically significant at $\alpha = 0.05$ thus the Cochran's Q test like the proposed test statistic is unable to reject the null hypothesis of no successive improvements (or decrease) in students performance during their five years of study. However, the proposed method unlike the Cochran's Q test would enable the researcher to quickly have a birds eye view of the proportions of subject who are successively improving, experiencing no change or worsening overtime, space or condition to guide the introduction of interventionist measures.

Application of Bartholomew test to the Data on letter grades of 18 students

d_{ij}	$n_{.j}$	n	Prop(P)	Revised Prop(P')
1	8	18	0.44	0.56
2	10	18	0.56	0.56
3	6	18	0.33	0.56
4	10	18	0.56	0.56
Total				**0.56**

$$\bar{\chi}^2 = \frac{1}{\bar{p}\bar{q}} \sum_{j=1}^{n} n_{ij} \left(P_j' - \bar{P}\right)^2$$

$$\bar{\chi}^2 = \frac{1}{(0.56)(0.44)} (18(0.56 - 0.56)^2) + 18(0.56 - 0.56)^2 + 18(0.56 - 0.56)^2 + 18(0.56 - 0.56)^2$$

$= 0.000001$

Since all sample sizes are equal, m = 4, and hypothesized ordering not actually obtained in the population, the averaging process necessary before the calculation of $\bar{\chi}^2$ reduces its magnitude to insignificance at α= 0.005 level of significance. Although the proposed method and the Bartholomew approach when applied to the present data both lead to the acceptance of the null hypothesis, nevertheless the relative sizes of the calculated

Chi- square values show that the Bartholomew test statistic is more likely to lead to an acceptance of a false null hypothesis(type 1 error) more frequently and hence is likely to be less powerful than the proposed test statistic. Thus from the result of the analysis obtained, the proposed method is probably more efficient than the Bartholomew and Cochran's Q test methods.

IV. Summary And Conclusion:

This paper developed a Ties adjusted non parametric statistical method for the analysis of ordered repeated measures that are related in time, space or condition that takes account of all possible pairwise combinations of treatment levels. A test statistic is developed to determine whether subjects are increasingly performing better or worse over time or space. The proposed method unlike the Cochran's Q test and the Bartholomew's method would enable the researcher to quickly have a bird's eye view of the proportions of subject who are successively improving, experiencing no change or worsening overtime, space or condition to guide the introduction of interventionist measures.

The method is illustrated with some data and shown to be more powerful than Friedman test and shown to be easier to use than the Bartholomew procedure.

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Forecasting the Inflation Rate in Nigeria: Box Jenkins Approach

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Abstract: *This paper aims to forecast the inflation rate in Nigeria using Jenkins approach. The data used for this paper was yearly data collected for a period of 1961-2010. Differencing method were used to obtain stationary process. The empirical study reveals that the most adequate model for the inflation rate is ARIMA (1,1,1). The model developed was used to forecast the year 2011 inflation rate as 16.27%. Based on this result, we recommend effective fiscal policies aimed at monitoring Nigeria's inflationary trend to avoid the consequences in the economy.*

Key words: *ARIMA models, Box-Jenkins, Differencing method, forecasting, Inflation rate.*

I. Introduction

In economics the inflation rate is a measure of inflation, or the rate of increase of a price index such as consumer price index. It is the percentage rate of change in price level over time [1]. The maintenance of price stability is one of the macroeconomic challenges facing the Nigeria government in our economic history [2]. In an inflationary economy, it is difficult for the national currency to act as medium of exchange and a store of value without having an adverse effect on the income distribution [3]. Inflation is characterized by a fall in the value of the country's currency and rise in her exchange rate with other nation's currencies. This is quite obvious in the case of the value of naira(#), which was #1 to $1 (US dollar) in 1980's, average of #100 to $1 in year 2000 , #128 to $1 in 2003 and over #155 to $1 in 2011 [4]. This decline in the value of the Naira coincides with the period of inflationary growth in Nigeria. Increased exchange rate directly affects the prices of imported commodities and an increase in the price of imported goods and services contributes directly to inflation [2].
There are three approaches to measure inflation. They are the Gross National Product(GNP) , the consumer price index(CPI) and the wholesome or producer price index (WPI and PPI). The period to period changes in these two latter approaches (CPI and WPI) are regarded as direct measure of inflation.

Forecasting inflation can be done using time series analysis, relevant literature was scarce. Literature on modeling and forecasting tourism demand in various type of empirical analysis [5]. Some of the researchers apply cross-sectional data, but most of forecasting tourism demand used pure time-series modeling, which was specified based on the famous standard Box-Jenkins method. Many researchers has applied this methodology see for example [6], [5], [7], [4] etc.

Studies which focus on forecasting inflation rate in Nigeria have appeared in various research publications.[4] examined whether monetary aggregates have useful information for forecasting inflation in the case of Nigeria other than that provided by inflation itself using a sample data spanning from 1990 to 1998. The study adopted two approaches, mean absolute percentage Errors (MAPE's) and autoregressive model. The study revealed that the treasury bill rate, domestic debt and M2 (broad money) provide the most important information about price movements.

[8] take a much more simple approach to determining future inflation. In their paper, they acknowledge inflation to be a relatively persistent process, which implies that future inflation rates are greatly affected by past current rates. After determining this, they felt it necessary to include any other economic variables in their analysis. Their paper compares the accuracy of the predictive model. The first are in a simple regression with a sample of 40 quarters. The second is what they refer to as the naive model, which was created by Atkesm and Ohanian in 2001. In it, they simply state that the forecasted inflation for the next year in the four quarter growth rate in CPI in the present year. they concluded both models were capable of performing each other depending on time period.

[9] article makes the simple prediction that even though gas and food prices are rising, which will lead to a higher CPI, inflation stay relatively constant because consumer expectations will remain constant.
[10] provides a similar predictive for both Canada and US inflation rates, however with different methodology .he says both rates should remain fairly steady despite the continued rise in gas and food prices.

[6] use autoregressive integrated moving average(ARIMA) model to predict inflation in Ghana using monthly inflation figures. In building the ARIMA model they use Box-Jenkinis approach, this inflation was found to integrated of order one and follows(6,1,6) order.

Motivated by these research, this paper intends to predict inflation rate in Nigeria through ARIMA approach. Also, the following specific objectives will be pursued: to build an appropriate Autoregressive Integrated Moving Average (ARIMA) model for inflation in Nigeria and forecast the inflation rate in Nigeria.

II. Methodology

The data used for this study consists of annual data on Nigeria- inflation, consumer prices (annual percent) for the period of 1961 to 2010 extracted from the official website of World Bank data base.

In this study we use ARIMA model to forecast one-period ahead of the series by applying Box-Jenkins approach. An ARIMA model is a generalization of an ARMA model. The model is generally referred to as ARIMA (p, d, q) model where p, d and q are integers greater than or equal to zero and refer to the order of the autoregressive, integrated and moving average aspects.

The Box-ARMA model is a combination of the AR(Autoregressive) and MA(Moving Average) models as follows:

$$y_t = \beta_0 + \beta_1 y_{t-1} + \ . \ \ . \ \ . + \beta_p y_{t-p} - \alpha_1 u_{t-1} - \alpha_2 u_{t-2} - \ . \ . \ . - \alpha_q u_{t-q} + u_t \qquad (1)$$

The Box-Jenkins methodology [11], [12] is a five-step process for identifying, selecting, and assessing conditional mean models (for discrete, univariate time series data). The steps are listed below:

1. Establish the stationarity of your time series. If your series is not stationary, successively difference your series to attain stationarity. The sample autocorrelation function (ACF) and partial autocorrelation function (PACF) of stationary series decay exponentially (or cut off completely after a few lags).
2. Identify a (stationary) conditional mean model for your data. The sample ACF and PACF functions can help with this selection. For an autoregressive (AR) process, the sample ACF decays gradually, but the sample PACF cuts off after a few lags. Conversely, for a moving average (MA) process, the sample ACF cuts off after a few lags, but the sample PACF decays gradually. If both the ACF and PACF decay gradually, consider an ARMA model.
3. Specify the model, and estimate the model parameters. When fitting non-stationary models in Econometrics Toolbox, it is not necessary to manually difference your data and fit a stationary model. Instead, use your data on the original scale, and create an arima model object with the desired degree of non-seasonal and seasonal differencing. Fitting an ARIMA model directly is advantageous for forecasting: forecasts are returned on the original scale (not differenced).
4. Conduct goodness-of-fit checks to ensure the model describes your data adequately. Residuals should be uncorrelated, homoscedastic, and normally distributed with constant mean and variance. If the residuals are not normally distributed, you can change your innovation distribution to a Student's *t*.
5. After choosing a model and checking its fit and forecasting ability, you can use the model to forecast or generate Monte Carlo simulations over a future time horizon.

III. Analysis And Result

We obtain the CPI data in Nigeria from 1961 to 2010. The plot of the CPI showed that the time series data was non stationary. But the plot of the differences of the CPI, as shown in fig. 1 showed considerable volatility even though the first differences are stationary. In order to determine whether the data was stationary we conducted an Augmented Dickey Fuller test (fig. 2) with the null hypothesis that the process contains unit roots. The test returned a p-value of approximately 0.00 and the null was rejected at 5% significance level. We continued our analysis with the assumption of stationarity. Since the data was confirmed stationary, it was observed that it was better to use the first difference of the series, so as to build a good model. In other words, the series was an integration of the first order. The autocorrelation function (ACF) and the partial autocorrelation function (PACF) graphs of the first difference confirm the results of the unit root test as discussed earlier (fig. 1).

The next step was to estimate the model, in this regard, a test was carried out on AR(1) and AR(2) as

shown in table 1 with the Akaike Information Criteria (AIC) of 8.029 and 7.96 respectively.

Model	Coefficient	RMSE	AIC	Mean dependent variable	R^2
<u>AR(1)</u> C AR(1)	 17.2499 0.6291	13.14	8.029	16.99	0.398
<u>MA(1)</u> C MA(1)	 16.7989 0.7216	12.67	7.96	16.77	0.43
<u>ARIMA(1,1,1)</u> C AR(1) MA(1)	 17.069 0.3158 0.5653	12.55	7.95	16.99	0.46

Table 1: Regression Results and Diagonistic test for ARIMA models

In carrying out the test for ARIMA process, ARMA (1, 0, 1) was derived and an AIC value of 7.95. However, since the series are integrated of order one as per the result of unit root test our model can be stated as ARIMA (1, 1, 1).

The tentative models that were identified for the set of time series data are AR(1), MA(1) and ARMA(1,1). Considering the tentative models (as shown in table 1) revealed that the best model is ARIMA(1, 1, 1) since it has the smallest root mean square (RMSE) of 12.55, least Akaike Information Criteria (AIC) of 7.95and the highest coefficient of determination of 0.46 which implies that the model is 46% fit.

Diagnostic checking: one simple diagnostic is to obtain the residuals, say, up to lag 25. The Box-Pierce G and Ljung-Box (LB) statistic (Q-statistic) in fig 3 shows that non of the ACF and PACF are statistically significant. In other words, the correlogram of both autocorrelation and partial autocorrelation give the impression that the residuals estimated from ARIMA model (1,1,1) are purely random. Hence, there may not be any need to look for another ARIMA model.

Therefore, the estimator of ARIMA (1,1,1) model is validated, the time series can be described by an ARIMA (1,1,1) process. The inflation rate seasonal adjusted time series and in first-differences (DCPI) is described as:

$$DCPI = 17.069 + 0.3158y_{t-1} + 0.5653u_{t-1}$$

The forecast is done using a statistical package e (E-view), the residual value for the data was also obtained and used to predict the inflation rate for 2011 to be 16.27% using ARIMA (1,1,1).

Date: 09/13/12 Time: 10:23
Sample: 1961- 2010
Included observations: 49

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
. | . |	. | . |	1	0.014	0.014	0.0095	0.922
***| . |	***| . |	2	-0.403	-0.404	8.6646	0.013
.*| . |	.*| . |	3	-0.104	-0.109	9.2527	0.026
.*| . |	**| . |	4	-0.076	-0.289	9.5716	0.048
. | . |	.*| . |	5	0.028	-0.096	9.6172	0.087
. |*. |	. | . |	6	0.186	0.018	11.632	0.071
. | . |	. | . |	7	0.063	0.017	11.864	0.105
.*| . |	.*| . |	8	-0.158	-0.098	13.378	0.099
.*| . |	. | . |	9	-0.081	-0.037	13.788	0.130
. | . |	. | . |	10	0.053	-0.011	13.966	0.175
.*| . |	.*| . |	11	-0.080	-0.174	14.385	0.212
. | . |	. | . |	12	0.020	-0.049	14.413	0.275
. |*. |	.*| . |	13	0.092	-0.069	14.995	0.308
. | . |	. | . |	14	0.050	0.056	15.171	0.367
.*| . |	.*| . |	15	-0.145	-0.181	16.723	0.336
.*| . |	.*| . |	16	-0.134	-0.156	18.080	0.319
. |*. |	. |*. |	17	0.179	0.080	20.582	0.246
. | . |	.*| . |	18	0.062	-0.073	20.894	0.285
. | . |	. | . |	19	0.019	0.060	20.923	0.341
. | . |	. | . |	20	0.069	0.053	21.339	0.377

fig .1 The correlogram of first difference of inflation rate in Nigeria.

Null Hypothesis: D(CPI) has a unit root
Exogenous: Constant
Lag Length: 1 (Automatic - based on SIC, maxlag=10)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-7.181206	0.0000
Test critical values:	1% level	-3.577723	
	5% level	-2.925169	
	10% level	-2.600658	

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Test Equation
Dependent Variable: D(CPI,2)
Method: Least Squares

Date: 09/17/12 Time: 09:07
Sample (adjusted): 1964 2010
Included observations: 47 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
D(CPI(-1))	-1.385779	0.192973	-7.181206	0.0000
D(CPI(-1),2)	0.403840	0.137382	2.939536	0.0052
C	0.392279	2.002205	0.195923	0.8456

R-squared	0.578147	Mean dependent var	0.215745
Adjusted R-squared	0.558971	S.D. dependent var	20.66776
S.E. of regression	13.72546	Akaike info criterion	8.138083
Sum squared resid	8289.079	Schwarz criterion	8.256178
Log likelihood	-188.2450	Hannan-Quinn criter.	8.182523
F-statistic	30.15081	Durbin-Watson stat	2.079621
Prob(F-statistic)	0.000000		

fig. 2 Augmented Dickey fuller test

Date: 09/17/12 Time: 11:06
Sample: 1962 2010
Included observations: 49
Q-statistic probabilities adjusted for 2 ARMA term(s)

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
. | . |	. | . |	1	-0.014	-0.014	0.0098	
. | . |	. | . |	2	-0.001	-0.001	0.0098	
. | . |	. | . |	3	0.052	0.052	0.1550	0.694
. | . |	. | . |	4	0.009	0.011	0.1599	0.923
. | . |	. | . |	5	0.058	0.059	0.3517	0.950
. |*. |	. |*. |	6	0.141	0.141	1.5123	0.824
. | . |	. | . |	7	0.043	0.048	1.6198	0.899
.*| . |	.*| . |	8	-0.084	-0.090	2.0540	0.915
.*| . |	.*| . |	9	-0.093	-0.116	2.5957	0.920
. | . |	. | . |	10	0.058	0.042	2.8078	0.946
.*| . |	.*| . |	11	-0.138	-0.151	4.0678	0.907
. | . |	. | . |	12	0.020	-0.001	4.0946	0.943
. | . |	. | . |	13	-0.022	-0.027	4.1267	0.966
. | . |	. |*. |	14	0.019	0.074	4.1524	0.981
.*| . |	.*| . |	15	-0.097	-0.066	4.8400	0.979
.*| . |	.*| . |	16	-0.128	-0.131	6.0780	0.964
. |*. |	. |*. |	17	0.143	0.168	7.6736	0.936
. | . |	. | . |	18	0.011	0.039	7.6840	0.958
. | . |	. | . |	19	-0.036	-0.043	7.7939	0.971
. |*. |	. | . |	20	0.109	0.073	8.8113	0.964

fig. 3. Correlogram of residual

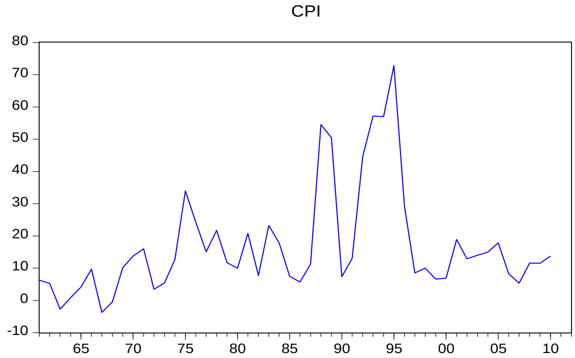

Fig. 4 graph of annual inflation rate during the period of 1961-2010

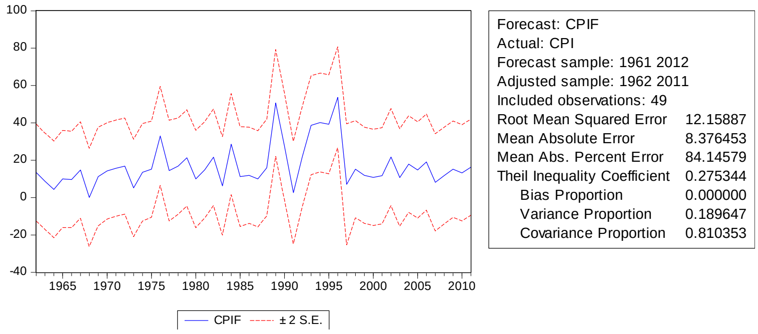

Fig.5 Forecast Graph

IV. Conclusion

This study aimed at predicting inflation rate in Nigeria using ARIMA model. The time series data is not stationary at level. By applying the ADF test for the series of the first order differences we observed that the series becomes stationary, so the initial series of the annual inflation rate is integrated by first order. We then applied the Box-Jenkins procedure on the stationary data series and we identify the corresponding ARIMA (p, q) process. The series correlogram has allowed us to choose appropriate p and q for the data series. Therefore, units root test was conducted and the null of the series integrated of order one was not rejected. We finally, built an ARIMA (1,1,1) model.

The root mean square error (RMSE) which determine the efficiency of the model was estimated at 12.55, this indicate that the model built is efficient. Using an ARIMA (1,1,1) model of annual value series of inflation rate for 2011 is estimated to be 16.27%.

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Triple -Diffusive Convection in a Magnetized Ferrofluid With MFD Viscosity: A Nonlinear Stability Analysis

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Abstract: *A nonlinear stability analysis is performed for a triple- diffusive convection in a magnetized ferrofluid with magnetic field –dependent viscosity (MFD) for stress- free boundaries. The major mathematical emphasis is on how to control the non-linear terms caused by magnetic body force and inertia forces. A suitable generalized energy functional is introduced to perform the nonlinear energy stability analysis. It is found that nonlinear critical stability magnetic thermal Rayleigh number does not coincide with that of linear instability, and thus indicate that the subcritical instabilities are possible. However, it is noted that in case of non-ferrofluid global nonlinear stability Rayleigh number is exactly same as that of linear instability. For lower values of magnetic parameters, this coincidence is immediately lost. The effects of magnetic parameter* M_3 *, solute gradients* S_1 *&* S_2 *and MFD viscosity parameter* δ *, on the subcritical instability region have also been analyzed. The solutes gradients* S_1 *&* S_2 *have stabilizing effect, because both* N_{ce} *,* $N_{c\ell}$ *increases as solute gradients increases. It has also been observed that in the presence of MFD viscosity* (δ)*, both* N_{ce} *,*$N_{c\ell}$ *decrease for lower values of* M_3 *and increase for higher values of* M_3 *.*

Keywords: *nonlinear stability, magnetized ferrofluid, triple- diffusive convection, MFD viscosity , magnetization.*

I. Introduction

Magnetic fluids or ferrofluids are colloidal suspension of fine ferromagnetic mono domain nano particles in non-conducting liquids. The ferromagnetic nanoparticles are coated with a surfactant to prevent their agglomeration. Rosensweig [1985] in his monograph and review article provides a detailed introduction to this subject. Chandrashekher [1981] has given a detailed account of thermal convection problems of Newtonian fluids. The theory of convective instability of ferrofluid begins with Finalyson [1970] and is interestingly continued by Lalas and Carmi [1971], Shliomis [1974], Stile and Kagan [1990], Venkatasubramanian and Kaloni [1994] .In the absence of an applied magnetic field, the particles in the colloidal suspensions are randomly oriented and thus the fluid has no net magnetization. When exposed to a magnetic field, Brownian rotational motions prevent complete alignment of the dipoles with the applied field. As a result when the applied field has a changing direction or magnitude, the magnetization is unable to track the field closely and becomes non-equilibrated. This non-equilibrium state of magnetization leads to the state of asymmetric stress. Rayleigh –Bénard convection in a ferromagnetic fluid layer with internal angular momentum permeated by uniform, vertical magnetic field with free-free, isothermal, spin-vanishing, magnetic boundaries has been considered by Abraham [2002]. She observed that the micropolar ferromagnetic fluid layer heated from below is more stable as compared with the classical Newtonian ferromagnetic fluid. More recently, Suresh [2012] has studied the convection problems in a ferrofluid with internal angular momentum in a porous and non-porous medium.

In the standard Bénard problem, the instability is driven by a density difference caused by a temperature difference between the upper and lower planes bounding the fluid. If the fluid additionally has salt dissolved in it, then there are potentially two destabilizing sources for the density difference, the temperature field and salt field. The solution behavior in the double-diffusive convection problem is more interesting than that of the single component situation in so much as new instability phenomena may occur which is not present in the classical Bénard problem. When temperature and two or more component agencies, or three different salts, are present then the physical and mathematical situation becomes increasingly richer. Very interesting results in triply diffusive convection have been obtained by Pearlstein et al., [1989]. The results of Pearlstein et al., are remarkable. They demonstrate that for triple diffusive convection linear instability can occur in discrete sections of the Rayleigh number domain with the fluid being linearly stable in a region in between the linear instability ones. This is because for certain parameters the neutral curve has a finite isolated oscillatory instability curve lying below the usual unbounded stationary convection one. Straughan and Walker [1987] derive the equations for non-Boussinesq convection in a multi- component fluid and investigate the situation analogous to that of Pearlstein et al., but allowing for a density non linear in the temperature field. Lopez et al., [1990] derive the equivalent problem with fixed boundary conditions and show that the effect of the boundary conditions breaks the perfect symmetry. In reality the density of a fluid is never a linear function of temperature, and so the work of Straughan and Walker applies to the general situation where the equation of state is one of the density quadratic in temperature. This is important, since they find that departure from the linear Boussinesq equation of state changes the perfect symmetry of the heart shaped neutral curve of Pearlstein et al.,. A comprehensive review of the literature concerning convection in porous medium may be found in the book by Nield and Bejan [1998].

There are in general two methods in a stability analysis, the linearlized instability method and energy method. The linear stability method provides sufficient condition for instability, whereas the energy method provides sufficient condition of stability of a basic flow. It is also noticed that the linearized theory alone cannot decide whether a particular flow is stable or not, for this it requires it's response to all physically accepted disturbances. The energy methods on other hand guarantee the exponential decay of arbitrary disturbances at all times and thus can be fully conservative in determining the stable -unstable bounds. The energy method is one of the oldest methods for nonlinear stability and can be traced back to the work of Reynolds (1895) and Orr (1907). The revival of energy method has been acknowledged after the work of Serrin (1959) and Joseph (1965, 1966). Energy methods of nonlinear stability theory are based on the study of time evolution of energy of the perturbation to the basic flow, and leads to variational problem for a critical dimensionless number, below which energy decays to zero. The detailed discussion of literature pertinent to the energy method can be found in Straughan (2001). By introducing the coupling parameters in the energy method and by selecting them optimally, it has been possible to sharpen the stability bound in many physical problems as discussed by Straughan (2004). Nonlinear energy stability analysis for thermal convection with temperature –dependent viscosity in the fluid has been considered by Hill and Carr (2010). A problem of thermal convection in a fluid-saturated porous layer using a global nonlinear stability analysis with a thermal non-equilibrium model has been study by Straughan (2006). He established that the global nonlinear stability boundary obtained using LTNE theory is exactly the same as the linear instability theory by Banu and Rees (2002).Recently a nonlinear stability analysis of magnetized ferrofluid and the same problem in the porous media have been studied by Sunil and Mahajan (2008, 2009).

In this paper, I have studied the nonlinear stability analysis of triple- diffusive convection in a magnetized ferrofluid with MFD viscosity by using generalized energy method. This problem, to the best of my knowledge has not been analyzed yet. It is found that when buoyancy magnetization is absent i.e. in case of non-ferrofluid, there is a coincidence between the nonlinear and linear stability results. For a convection problem in magnetized ferrofluid , the linear critical magnetic thermal Rayleigh number is found higher in values than the nonlinear critical magnetic thermal Rayleigh number, which shows the possibility of the existence of subcritical instability. Finally, the comparison of the results obtained, respectively, by the linear stability analysis and energy method has been discussed in detail.

II. Mathematical formulation of the problem

Here we consider an infinite, horizontal layer of thickness d of an electrically non-conducting incompressible thin –magnetized ferrofluid heated and salted from below having variable viscosity $\mu_1 = \mu(1+\delta.B)$. The temperature T and solute concentrations C^1 and C^2 at the bottom and top surfaces $z=\pm\frac{1}{2}d$ are T_0 and T_1 ; C_0^1 and C_1^1; and C_0^2 and C_1^2 respectively, and a uniform temperature gradient $\beta(=|\frac{dT}{dZ}|)$ and uniform solute gradients *are* β' $(=|\frac{dC^1}{dZ}|)$ and β'' $(=|\frac{dC^2}{dZ}|)$ are maintained. Both the boundaries are taken to be free and perfect conductors of heat. The gravity field **g** = (0,0,-g) and uniform vertical magnetic field intensity **H** = (0,0,H_0) pervade the system. The mathematical equations to discuss the nonlinear stability analysis in triple diffusive convection, for the above model are as follows (Finlayson[1970]) : The continuity equation is

$$\nabla.\boldsymbol{q} = 0 \qquad (1)$$

The momentum equation is

$$\rho_0\left[\frac{\partial}{\partial t} + (\mathbf{q}.\nabla)\right]\mathbf{q} = -\nabla p + \mu_1\nabla^2 q + \rho_0[1-\alpha(T-T_a) + \alpha'(C^1 - C_a^1) + \alpha''(C^2 - C_a^2)] + \mu_0(\mathbf{M}.\nabla)\mathbf{H} \qquad (2)$$

The temperature and solute concentration equations are

$$\left[\frac{\partial}{\partial t} + (\mathbf{q}.\nabla)\right]\mathbf{T} = K_1\nabla^2 T \qquad (3)$$

$$\left[\frac{\partial}{\partial t} + (\mathbf{q}.\nabla)\right]C^1 = K'_1\nabla^2 C^1 \qquad (4)$$

$$\left[\frac{\partial}{\partial t} + (\mathbf{q}.\nabla)\right]C^2 = K''_1\nabla^2 C^2 \qquad (5)$$

Maxwell's equation, simplified for a non-conducting fluid with no displacement currents, become

$\nabla . \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = 0 \quad \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}).$ (6)

We assume that the magnetization is aligned with the magnetic field, but allow a dependence on the magnitude of the magnetic field, temperature and salinity, so that

$\mathbf{M} = \frac{\boldsymbol{H}}{H} M(H, T, C^1, C^2).$ (7)

The magnetic equation of state is linearized about the magnetic field, H_0, an average temperature, T_a, and average concentrations, C_a^1 and C_a^2 to become

$M = M_0 + \chi(H - H_0) - K_2 (T - T_a) + K_3(C^1 - C_a^1) + K_4(C^2 - C_a^2).$ (8)

where magnetic susceptibility, pyromagnetic coefficient and salinity magnetic coefficients are defined by

$\chi \equiv (\frac{\partial M}{\partial H})_{H_0, T_a}$; $K_2 \equiv -(\frac{\partial M}{\partial T})_{H_0, T_a}$; $K_3 \equiv (\frac{\partial M}{\partial C^1})_{H_0, c_a^1}$ and $K_4 \equiv (\frac{\partial M}{\partial C^2})_{H_0, c_a^2}$ respectively. (9)

Here H_0 is the uniform magnetic field of the fluid layer when placed in an external magnetic field $H = H_0^{ext} \hat{k}$, where $\hat{k}$ is a unit vector in the z direction, $H = |\mathbf{H}|$, $M = |\mathbf{M}|$ and $M_0 = M(H_0, T_a, C_a^1, C_a^2)$

The basic state is assumed to be quiescent state and is given by

$\mathbf{q} = \mathbf{q_b} = (0,0,0)$, $\rho = \rho_b(z)$, $p = p_b(z)$, $T = T_b(z) = -\beta z + T_a$, , $C^1 = C^1_b(z) = -\beta' z + C^1_a$

$C^2 = C^2_b(z) = -\beta'' z + C^2_a$, $\beta = (T_0 - T_1)/d$, $\beta' = (C_1^1 - C_0^1)/d$, $\beta'' = (C_1^2 - C_0^2)/d$,

$\mathbf{H_b} = [H_0 - \frac{K_2 \beta z}{1+\chi} + \frac{K_3 \beta' z}{1+\chi} + \frac{K_4 \beta'' z}{1+\chi}] \hat{\boldsymbol{k}}$, $\mathbf{M_b} = [M_0 + \frac{K_2 \beta z}{1+\chi} - \frac{K_3 \beta' z}{1+\chi} - \frac{K_4 \beta'' z}{1+\chi}] \hat{\boldsymbol{k}}$ and $H_0 + M_O = H_0^{ext}$, (10)
where the subscript ' b ' denotes the basic state.

We now examine the stability of the basic state, and assume that the perturbation quantities are small. We write

$\mathbf{q} = \mathbf{q_b} + \mathbf{q}'$, $\rho = \rho_b + \rho'$, $p = p_b(z) + p'$, $T = T_b(z) + \theta$, $C^1 = C^1_b(z) + \gamma$, $C^2 = C^2_b(z) + \gamma'$,
$\mathbf{H} = \mathbf{H_b}(z) + \mathbf{H}'$ and $\mathbf{M} = \mathbf{M_b}(z) + \mathbf{M}'$
(11)

where $\mathbf{q}' = (u, v, w)$, ρ', p', θ, γ, γ', $\mathbf{H}'$, $\mathbf{M}'$ are perturbation in velocity q, pressure p, temperature T, concentrations C^1 and C^2 , magnetic field intensity **H**, and magnetization **M,** respectively. The change in density ρ' , caused mainly by the perturbations θ, γ, and γ' in temperature and concentrations, respectively, is given by

$\rho' = -\rho_0 (\alpha \theta - \alpha' \gamma - \alpha'' \gamma')$
(12)

The non-dimensionless equations for the perturbation are

$$\frac{\partial \mathrm{q}}{\partial t} = -\nabla p + (1 + \delta M_3) \nabla^2 \mathbf{q} + \sqrt{R}(1 + M_1 - M_4) \theta \hat{\boldsymbol{k}} - \frac{\sqrt{S_1}}{L_e}(1 + M_4' - M_1') \gamma \hat{\boldsymbol{k}} - \frac{\sqrt{S_2}}{L_e}(1 + M_4'' - M_1'') \gamma' \hat{\boldsymbol{k}}$$
$$- \sqrt{R}(M_1 - M_4) \phi_{1Z} \hat{\boldsymbol{k}} + \frac{\sqrt{S_1}}{L_e}(M_4' - M_1') \phi_{2Z} \hat{\boldsymbol{k}} + \frac{\sqrt{S_2}}{L_e}(M_4'' - M_1'') \phi_{3Z} \hat{\boldsymbol{k}} - M_1 \theta \phi_{1Z} + \frac{\sqrt{M_4}\sqrt{M_4'}}{\sqrt{L_e}}(\theta \nabla \phi_{2Z}$$
$$+ \gamma \phi_{1Z}) + \frac{\sqrt{M_4}\sqrt{M_4''}}{\sqrt{L_e}}(\theta \nabla \phi_{3Z} + \gamma' \phi_{1Z}) + \frac{\sqrt{M'_4}\sqrt{M_4''}}{\sqrt{L_e}}(\gamma \nabla \phi_{3Z} + \gamma' \phi_{2Z}) + (M_3 - \frac{1}{1+\chi})[M_1 \phi_{1x} \nabla \phi_{1x} - \frac{\sqrt{M_4}\sqrt{M_4'}}{\sqrt{L_e}}$$
$$(\phi_{1x} \nabla \phi_{2x} + \phi_{2x} \nabla \phi_{1x}) - \frac{\sqrt{M_4}\sqrt{M_4''}}{\sqrt{L_e}}(\phi_{1x} \nabla \phi_{3x} + \phi_{3x} \nabla \phi_{1x} - \frac{\sqrt{M'_4}\sqrt{M_4''}}{\sqrt{L_e}}(\phi_{2x} \nabla \phi_{3x} + \phi_{3x} \nabla \phi_{2x}) + \frac{M_1'}{L_e} \phi_{2x} \nabla \phi_{2x}$$
$$+ \frac{M_1''}{L_e} \phi_{3x} \nabla \phi_{3x}] + (M_3 - \frac{1}{1+\chi})[M_1 \phi_{1y} \nabla \phi_{1y} - \frac{\sqrt{M_4}\sqrt{M_4'}}{\sqrt{L_e}}(\phi_{1y} \nabla \phi_{2y} + \phi_{2y} \nabla \phi_{1y} - \frac{\sqrt{M_4}\sqrt{M_4''}}{\sqrt{L_e}}(\phi_{1y} \nabla \phi_{3y} + \phi_{3y}$$
$$\nabla \phi_{1y}) - \frac{\sqrt{M'_4}\sqrt{M_4''}}{\sqrt{L_e}}(\phi_{2y} \nabla \phi_{3y} + \phi_{3y} \nabla \phi_{2y}) + \frac{M_1'}{L_e} \phi_{2y} \nabla \phi_{2y} + \frac{M_1''}{L_e} \phi_{3y} \nabla \phi_{3y}] + (\frac{\chi}{1+\chi})[M_1 \phi_{1z} \nabla \phi_{1z}$$
$$- \frac{\sqrt{M_4}\sqrt{M_4'}}{\sqrt{L_e}}(\phi_{1z} \nabla \phi_{2z} + \phi_{2z} \nabla \phi_{1z}) - \frac{\sqrt{M_4}\sqrt{M_4''}}{\sqrt{L_e}}(\phi_{1z} \nabla \phi_{3z} + \phi_{3z} \nabla \phi_{1z}) - \frac{\sqrt{M'_4}\sqrt{M_4''}}{\sqrt{L_e}}(\phi_{2z} \nabla \phi_{3z} + \phi_{3z} \nabla \phi_{2z})$$

$+\frac{M_1'}{L_e}\phi_{2z}\nabla\phi_{2z}$ - $\mathbf{q}.\nabla\mathbf{q}$ + $\frac{M_1''}{L_e}\phi_{3z}\nabla\phi_{3z}$] - $\frac{M_1'}{L_e}\gamma\nabla\phi_{2z}$ - $\frac{M_1''}{L_e}\gamma'\nabla\phi_{3z}$ + $\delta M_3 (M_\delta\phi_{1x} - M'_\delta\phi_{2x} - M_\delta''\phi_{3x} - M_\delta\phi_{1y}$ $- M_\delta'\phi_{2y} - M_\delta''\phi_{3y})\nabla^2\mathbf{q} + \delta(M_\delta\phi_{1z} - M'_\delta\phi_{2z} - M_\delta''\phi_{3z})\nabla^2\mathbf{q} - \delta M_\delta\,\theta\,\nabla^2\mathbf{q} + \delta M_\delta'\,\gamma\,\nabla^2 q + \delta M_\delta''\,\gamma'\nabla^2\mathbf{q}$ (13)

$\nabla.\mathbf{q} = 0$ (14)

$\frac{\partial\theta}{\partial t} + \mathbf{q}.\nabla\theta = \nabla^2\theta + \sqrt{R}\,w$ (15)

$\frac{\partial\gamma}{\partial t} + \mathbf{q}.\nabla\gamma = \frac{1}{L_e}\nabla^2\gamma + \sqrt{S_1}\,w$ (16)

$\frac{\partial\gamma'}{\partial t} + \mathbf{q}.\nabla\gamma' = \frac{1}{L_e}\nabla^2\gamma' + \sqrt{S_2}\,w$ (17)

$M_3\nabla^2\phi_1 - (M_3 - 1)\phi_{1zz} = \theta_z$ (18)

$M_3\nabla^2\phi_2 - (M_3 - 1)\phi_{2zz} = \gamma_z$ (19)

$M_3\nabla^2\phi_3 - (M_3 - 1)\phi_{3zz} = \gamma'_z$ (20)

Here, the following non dimension quantities and non dimensionless parameters are introduced:

$\gamma^* = \frac{\sqrt{S_1}}{\beta' d}\gamma$, $\gamma'^* = \frac{\sqrt{S_2}}{\beta'' d}\gamma'$, $\phi_1^* = \frac{(1+\chi)\sqrt{R}}{K_1\beta d^2}\phi_1$, $\phi_2^* = \frac{(1+\chi)\sqrt{S_1}}{K_2\beta' d^2}\phi_2$, $\phi_3^* = \frac{(1+\chi)\sqrt{S_2}}{K_3\beta'' d^2}\phi_3$,

$\delta^* = \mu_0 H_0(1+\chi)\delta$, $R = \frac{g\alpha\beta\rho_0 d^4}{\mu K_1}$, $S_1 = \frac{g\alpha'\beta'\rho_0 d^4}{\mu K_1'}$, $S_2 = \frac{g\alpha''\beta''\rho_0 d^4}{\mu K_1''}$, $M_1 = \frac{\mu_0 K_1^2\beta}{(1+\chi)\alpha\rho_0 g}$,

$M_1' = \frac{\mu_0 K_2^2\beta'}{(1+\chi)\alpha'\rho_0 g}$, $M_1'' = \frac{\mu_0 K_3^2\beta''}{(1+\chi)\alpha''\rho_0 g}$, $M_3 = \frac{(1+\frac{M_0}{H_0})}{(1+\chi)}$, $M_4 = \frac{\mu_0 K_1 K_2\beta'}{(1+\chi)\alpha\rho_0 g}$, $M_4' = \frac{\mu_0 K_2 K_3\beta''}{(1+\chi)\alpha'\rho_0 g}$,

$M_4'' = \frac{\mu_0 K_1 K_3\beta}{(1+\chi)\alpha''\rho_0 g}$, $M_5 = \frac{M_4}{M_1} = \frac{M_1'}{M_4'} = \frac{K_2\beta'}{K_1\beta} = \frac{M_1''}{M_4''} = \frac{K_3\beta''}{K_1\beta}$, $M_\delta = \frac{K_1\beta d}{H_0(1+\chi)\sqrt{R}}$, $M_\delta' = \frac{K_2\beta' d}{H_0(1+\chi)\sqrt{S_1}}$

$M_\delta'' = \frac{K_3\beta'' d}{H_0(1+\chi)\sqrt{S_2}}$, $L_e = \frac{K}{K'} = \frac{K}{K''}$,

Where, R is the Rayleigh number, S_1 & S_2 are the solute Rayleigh number, M_1' & M_1'' are the effect of magnetization due to salinity, M_5 represent the ratio of the salinity effect on magnetic field to pyromagnetic coefficient and L_e is the Lewis number.

The functions $\mathbf{q}$, θ, γ, γ', ϕ_1, ϕ_2, ϕ_3 must subject to the boundary conditions and we suppose that $\mathbf{q}$, θ, γ, γ', ϕ_1, ϕ_2, ϕ_3 are periodic in x, y with periods $\frac{2\pi}{a_i}$ for i= 1,2 respectively and the surfaces are stress free so that

$w=0$, $u_z = 0$, $v_z = 0$, $\theta = 0$, $\gamma = 0$, $\gamma' = 0$, $\phi_{1z} = 0$, $\phi_{2z} = 0$, $\phi_{3z} = 0$ at $z = \pm\frac{1}{2}$ (21)

In order to exclude the rigid motions, we assume that the mean values of u and v are zero (Wells and Kloeden [31]) i.e. we require

$\int_v u\,dV = \int_v v\,dV = 0$, (22)

where $V = [0, \frac{2\pi}{a_1}] \times [0, \frac{2\pi}{a_2}] \times [\frac{-1}{2}, \frac{1}{2}]$ is the typical periodicity cell.

Nonlinear analysis

To study the nonlinear stability of triple diffusive convection, we derive an energy equation of the form

$\frac{dE}{dt} = I_0 - D_0 + N_0$ (23)

where

$$E = \frac{1}{2} \parallel \theta \parallel^2 + \frac{\lambda_1}{2} \parallel \boldsymbol{q} \parallel^2 - \frac{\lambda_3}{2} \parallel \gamma \parallel^2 - \frac{\lambda_5}{2} \parallel \gamma' \parallel^2 \quad (24)$$

with coupling parameters λ_i and $\parallel \; \parallel$ denote the norm on L^2 (V) . The terms I_0, D_0, N_0 are as follow:

$$I_0 = \sqrt{R} \{ 1+\lambda_1(1+M_1-M_4)\} \langle w\,\theta\rangle - \sqrt{S_1} \{ \lambda_3 + \frac{\lambda_1}{L_e} (1+M_4' - M_1')\}\langle w\gamma\rangle - \sqrt{S_2} (\lambda_5 + \frac{\lambda_1}{L_e} (M_4'' - M_1''))\}\langle w\gamma'\rangle - \sqrt{R}\,\lambda_1(M_1 - M_4)\langle w\phi_{1Z}\rangle + \frac{\sqrt{S_1}}{L_e}\lambda_1 (M_4' - M_1')\langle w\phi_{2Z}\rangle - \frac{\sqrt{S_2}}{L_e}\lambda_1 (M_4'' - M_1'')\langle w\phi_{3Z}\rangle - \lambda_2 \langle \phi_1\theta_Z\rangle + \lambda_4 \langle \phi_2\gamma_Z\rangle + \lambda_6 \langle \phi_3\gamma_Z'\rangle \quad (25)$$

$$D_0 = \parallel \nabla\theta \parallel^2 + \lambda_1 (1+\delta M_3) \parallel \nabla q \parallel^2 - \frac{\lambda_3}{L_e} \parallel \nabla\gamma \parallel^2 - \frac{\lambda_5}{L_e} \parallel \nabla\gamma' \parallel^2 + \lambda_2 M_3 \parallel \nabla\phi_1 \parallel^2 - \lambda_2(M_3-1) \parallel \nabla\phi_{1z} \parallel^2 - \lambda_4 M_3 \parallel \nabla\phi_2 \parallel^2 (M_3-1) \parallel \nabla\phi_{2z} \parallel^2 - \lambda_6 M_3 \parallel \nabla\phi_3 \parallel^2 - \lambda_6(M_3-1) \parallel \nabla\phi_{3z} \parallel^2 \quad (26)$$

$$N_0 = \lambda_1 M_1\langle \boldsymbol{q}\nabla\theta\phi_{1z}\rangle - \frac{\lambda_1\sqrt{M_4}\sqrt{M_4'}}{\sqrt{L_e}}\langle \mathbf{q}\nabla\theta\phi_{2z}\rangle - \frac{\lambda_1\sqrt{M_4}\sqrt{M_4''}}{\sqrt{L_e}}\langle \mathbf{q}\nabla\theta\phi_{3z}\rangle + \frac{\lambda_1 M_1'}{L_e}\langle \mathbf{q}\nabla\gamma\phi_{2z}\rangle + \frac{\sqrt{M_4'}\sqrt{M_4''}}{\sqrt{L_e}}$$

$$\langle \mathbf{q}\nabla\gamma\phi_{1z}\rangle + \frac{\lambda_1 M_1''}{L_e}\langle \mathbf{q}\nabla\gamma'\phi_{3z}\rangle + \lambda_1(M_3 - \frac{1}{1+\chi})[M_1\langle \phi_{1x}\boldsymbol{q}\nabla\phi_{1x}\rangle - \frac{\sqrt{M_4}\sqrt{M_4'}}{\sqrt{L_e}}\{\langle \phi_{1x}\mathbf{q}\nabla\phi_{2x}\rangle + \langle \phi_{2x}\boldsymbol{q}\nabla\phi_{1x}\rangle\}$$

$$-\frac{\sqrt{M_4}\sqrt{M_4''}}{\sqrt{L_e}}\{\langle \phi_{1x}\boldsymbol{q}\nabla\phi_{3x}\rangle + \langle \phi_{3x}\boldsymbol{q}\nabla\phi_{1x}\rangle\} - \frac{\sqrt{M_4'}\sqrt{M_4''}}{\sqrt{L_e}}\{\langle \phi_{2x}\boldsymbol{q}\nabla\phi_{3x}\rangle + \langle \phi_{3x}\boldsymbol{q}\nabla\phi_{2x}\rangle\} + \frac{M_1'}{L_e}\langle \phi_{2x}\mathbf{q}\nabla\phi_{2x}\rangle$$

$$+\frac{M_1''}{L_e}\langle \phi_{3x}\boldsymbol{q}\nabla\phi_{3x}\rangle] - \lambda_1(M_3 - \frac{1}{1+\chi})[M_1\langle \phi_{1y}\boldsymbol{q}\nabla\phi_{1y}\rangle - \frac{\sqrt{M_4}\sqrt{M_4'}}{\sqrt{L_e}}\{\langle \phi_{1y}\boldsymbol{q}\nabla\phi_{2y}\rangle + \langle \phi_{2y}\boldsymbol{q}\nabla\phi_{1y}\rangle\}$$

$$-\frac{\sqrt{M_4}\sqrt{M_4''}}{\sqrt{L_e}}\{\langle \phi_{1y}\boldsymbol{q}\nabla\phi_{3y}\rangle + \langle \phi_{3y}\boldsymbol{q}\nabla\phi_{1y}\rangle\} - \frac{\sqrt{M_4'}\sqrt{M_4''}}{\sqrt{L_e}}\{\langle \phi_{2y}\boldsymbol{q}\nabla\phi_{3y}\rangle + \langle \phi_{3y}\boldsymbol{q}\nabla\phi_{2y}\rangle\}$$

$$+\frac{M_1'}{L_e}\langle \phi_{2y}\boldsymbol{q}\nabla\phi_{2y}\rangle + \frac{M_1''}{L_e}\langle \phi_{3y}\boldsymbol{q}\nabla\phi_{3y}\rangle] + \lambda_1(\frac{\chi}{1+\chi})[M_1\langle \phi_{1z}\boldsymbol{q}\nabla\phi_{1z}\rangle - \frac{\sqrt{M_4}\sqrt{M_4'}}{\sqrt{L_e}}\{\langle \phi_{1z}\boldsymbol{q}\nabla\phi_{2z}\rangle + \langle \phi_{2z}\boldsymbol{q}\nabla\phi_{1z}\rangle\}$$

$$-\frac{\sqrt{M_4}\sqrt{M_4''}}{\sqrt{L_e}}\{\langle \phi_{1z}\mathbf{q}\nabla\phi_{3z}\rangle + \langle \phi_{3z}\boldsymbol{q}\nabla\phi_{1z}\rangle\} - \frac{\sqrt{M_4'}\sqrt{M_4''}}{\sqrt{L_e}}\{\langle \phi_{2z}\boldsymbol{q}\nabla\phi_{3z}\rangle + \langle \phi_{3z}\boldsymbol{q}\nabla\phi_{2z}\rangle\} + \frac{M_1'}{L_e}\langle \phi_{2z}\boldsymbol{q}\nabla\phi_{2z}\rangle$$

$$+\frac{M_1''}{L_e}\langle \phi_{3z}\boldsymbol{q}\nabla\phi_{3z}\rangle] + \lambda_1\,\delta M_3\{M_\delta\langle \phi_{1x}\mathbf{q}\,\nabla^2\mathbf{q}\rangle - M'_\delta\langle \phi_{2x}\mathbf{q}\,\nabla^2\mathbf{q}\rangle - M_\delta''\langle \phi_{3x}\mathbf{q}\nabla^2\mathbf{q}\rangle - M_\delta\langle \phi_{1y}\mathbf{q}\nabla^2\mathbf{q}\rangle - M_\delta'\langle \phi_{2y}\mathbf{q}\nabla^2\mathbf{q}\rangle - M_\delta''\langle \phi_{3y}\mathbf{q}\,\nabla^2\mathbf{q}\rangle\} + \lambda_1\,\delta\{\langle \phi_{1z}\mathbf{q}\nabla^2\mathbf{q}\rangle - M'_\delta\langle \phi_{2z}\mathbf{q}\,\nabla^2\mathbf{q}\rangle - M_\delta''\langle \phi_{3z}\mathbf{q}\nabla^2\mathbf{q}\rangle\} - \lambda_1\,\delta M_\delta\langle \theta\,\mathbf{q}\nabla^2\mathbf{q}\rangle + \lambda_1\delta M_\delta'\langle \gamma\,\mathbf{q}\nabla^2\mathbf{q}\rangle + \lambda_1\,\delta M_\delta''\langle \gamma'\boldsymbol{q}\nabla^2\mathbf{q}\rangle \quad (27)$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ are positive coupling parameters and $\langle\,\rangle$ is the inner product on L^2 (V).
In equation (24) , it is seen that the energy of the system is consumed due to the solute concentrations (i.e. -ve sign with $\frac{\lambda_3}{2} \parallel \gamma \parallel^2$ & $-\frac{\lambda_5}{2} \parallel \gamma' \parallel^2$). Now, it can be assumed that the energy is consumed, due to solute concentrations is less than the energy produced due to velocity and temperature. Also the energy dissipated by the solute concentrations is less than the energy dissipated by the velocity, temperature and magnetization. These assumptions will ensure that all the terms on the right-hand side of the equations (24) & (26) are always less than the left – hand side of that equation.
Thus from equation (23), we have

$$\frac{dE}{dt} \le -a_0 D_0 + N_0 \quad (28)$$

with $a_0 = 1\text{-}m\ (> 0)$ where

$$m = \max_H \frac{I_0}{D_0} \quad (29)$$

and H is the space of admissible solution.

In order to dominate the nonlinear terms and for studying the nonlinear stability, we now introduce the generalized energy functional as

$$V_g(t) = E(t) + b_0\, E_1(t) \qquad (30)$$

where b_0 is a positive coupling parameter to be chosen and the complementary energy $E_1(t)$ is given by

$$E_1(t) = \frac{1}{2}\,\|\nabla\theta\|^2 + \frac{1}{2}\,\|\nabla q\|^2 + \frac{1}{2}\,\|\nabla\gamma\|^2 + \frac{1}{2}\,\|\nabla\gamma'\|^2 + \frac{1}{2}\,\|\nabla^2\theta\|^2 + \frac{1}{2}\,\|\nabla^2\gamma\|^2 + \frac{1}{2}\,\|\nabla^2\gamma'\|^2 \qquad (31)$$

III. The eigenvalue problem of nonlinear analysis

Now we use the calculus of variation to find the maximum problem at the critical argument $m_1 = 1$ in equation (29). The associated Euler-Lagrange equations, after taking the transformations

$$\hat{\boldsymbol{q}} = \sqrt{\lambda_1}\,\mathbf{q},\ \hat{\phi}_1 = \sqrt{\lambda_2}\,\phi_1,\ \hat{\gamma} = \sqrt{\lambda_3}\gamma,\ \hat{\phi}_2 = \sqrt{\lambda_4}\,\phi_4,\ \hat{\gamma}' = \sqrt{\lambda_5}\,\gamma',\ \hat{\phi}_3 = \sqrt{\lambda_6}\,\phi_3, \qquad (32)$$

and dropping the caps are

$$2(1+\delta M_3)\nabla^2\mathbf{q} + \frac{\sqrt{R}}{\sqrt{\lambda_1}}\{1+\lambda_1(1+M_1-M_4)\}\theta\hat{\boldsymbol{k}} - \frac{\sqrt{S_1}}{\sqrt{\lambda_1}\sqrt{\lambda_3}}\{\lambda_3+\frac{\lambda_1}{L_e}(1+M_4'-M_1')\}\gamma\hat{\boldsymbol{k}} - \frac{\sqrt{S_2}}{\sqrt{\lambda_1}\sqrt{\lambda_5}}(\lambda_5+\frac{\lambda_1}{L_e}(M_4''-M_1''))\}\gamma'\hat{\boldsymbol{k}} - \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_4}}\sqrt{R}\,(M_1-M_4)\,\phi_{1z}\hat{\boldsymbol{k}} + \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_4}}\frac{\sqrt{S_1}}{L_e}(M_4'-M_1')\,\phi_{2z}\hat{\boldsymbol{k}} + \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_6}}\frac{\sqrt{S_2}}{L_e}(M_4''-M_1'')\,\phi_{3z}\hat{\boldsymbol{k}} - 2\nabla p = 0 \qquad (33)$$

$$2\nabla^2\theta + \sqrt{R}\{1+\lambda_1(1+M_1-M_4)\}\frac{1}{\sqrt{\lambda_1}}w + \sqrt{\lambda_2}\phi_{1Z} = 0 \qquad (34)$$

$$\frac{2}{L_e}\nabla^2\gamma + \sqrt{S_1}\{\lambda_3+\frac{\lambda_1}{L_e}(1+M_4'-M_1')\}\frac{1}{\sqrt{\lambda_1}\sqrt{\lambda_3}}w + \frac{\sqrt{\lambda_4}}{\sqrt{\lambda_3}}\phi_{2Z} = 0 \qquad (35)$$

$$\frac{2}{L_e}\nabla^2\gamma' + \sqrt{S_2}(\lambda_5+\frac{\lambda_1}{L_e}(M_4''-M_1''))\}\frac{1}{\sqrt{\lambda_1}\sqrt{\lambda_5}}w + \frac{\sqrt{\lambda_6}}{\sqrt{\lambda_5}}\phi_{3Z} = 0 \qquad (36)$$

$$2M_3\nabla^2\phi_1 - 2(M_3-1)\phi_{1zz} + \sqrt{R}\frac{\sqrt{\lambda_1}}{\sqrt{\lambda_2}}(M_1-M_4)w_z - \sqrt{\lambda_2}\,\theta_z = 0 \qquad (38)$$

$$2M_3\nabla^2\phi_2 - 2(M_3-1)\phi_{2zz} + \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_4}}\frac{\sqrt{S_1}}{L_e}(M_1'-M_4')w_z - \sqrt{\lambda_2}\gamma_z = 0 \qquad (39)$$

$$2M_3\nabla^2\phi_3 - 2(M_3-1)\phi_{3zz} + \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_6}}\frac{\sqrt{S_2}}{L_e}(M_1''-M_4'')w_z - \sqrt{\lambda_2}\gamma'_z = 0 \qquad (40)$$

where $'p'$ is a Lagrange's multiplier, and $'\mathbf{q}'$ is solenoidal.

After taking the third component of the curl curl of equation (33), we have

$$2(1+\delta M_3)\nabla^4 w + \frac{\sqrt{R}}{\sqrt{\lambda_1}}\{1+\lambda_1(1+M_1-M_4)\}\nabla^2{}_1\theta - \frac{\sqrt{S_1}}{\sqrt{\lambda_1}\sqrt{\lambda_3}}\{\lambda_3+\frac{\lambda_1}{L_e}(1+M_4'-M_1')\}\nabla^2{}_1\gamma - \frac{\sqrt{S_2}}{\sqrt{\lambda_1}\sqrt{\lambda_5}}(\lambda_5+\frac{\lambda_1}{L_e}(M_4''-M_1''))\}\nabla^2{}_1\gamma' - \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_4}}\sqrt{R}\,(M_1-M_4)\nabla^2{}_1\phi_{1Z} + \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_4}}\frac{\sqrt{S_1}}{L_e}(M_4'-M_1')\nabla^2{}_1\phi_{2Z} + \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_6}}\frac{\sqrt{S_2}}{L_e}(M_4''-M_1'')\nabla^2{}_1\phi_{3Z} = 0 \qquad (41)$$

Now, we assume a plane tilting form

$$(w, \theta, \gamma, \gamma', \phi_1, \phi_2, \phi_3) = [\,W(z), \Theta(z), \Gamma(z), \Psi(z), \phi_1(z), \phi_2(z), \phi_3(z)\,]\ g(x, y) \qquad (42)$$

Where $\nabla^2{}_1 g + a^2 g = 0$, $'a'$ being the wave number (Chandrasekhar [1981])
The boundary conditions at the free-free surface are

$$W = D^2W = \Theta = \Gamma = \Psi = D\phi_1 = D\phi_2 = D\phi_3 = 0 \quad \text{at } z = \pm\frac{1}{2}, \qquad (43)$$

The exact solution subject to these boundary conditions is written in the form

W = $A_1\cos\pi z$, Θ = $A_2\cos\pi z$, $D\phi_3 = A_3\cos\pi z$,
$\phi_1 = (\frac{A_3}{\pi})\sin\pi z$, $\Gamma = A_4\cos\pi z$, $D\phi_2 = A_5\cos\pi z$,
$\phi_2 = (\frac{A_5}{\pi})\sin\pi z$, $\Psi = A_6\cos\pi z$, $D\phi_3 = A_7\cos\pi z$,
$\phi_3 = (\frac{A_7}{\pi})\sin\pi z$, (44)

Where $A_1, A_2, A_3, A_4, A_5, A_6, A_7$ are constants. Using the plane tiling form and substituting solution (44), we get equations involving coefficients of $A_1, A_2, A_3, A_4, A_5, A_6, A_7$. For the existence of non-trivial solutions, the determinant of the coefficients of $A_1, A_2, A_3, A_4, A_5, A_6, A_7$ must vanish. This determinant on simplification yields the energy thermal Rayleigh number R'_e and then we can performs the optimization

$$R_e = \max_{\lambda_1,\lambda'_2,\lambda_3,\lambda'_4,\lambda_5,\lambda'_6} \min_x R'_e(\lambda_1, \lambda'_2, \lambda_3, \lambda'_4, \lambda_5, \lambda'_6, M_1, M_3, M_5, M'_1, M''_1, \delta, L_e, S_1, S_2) \quad (45)$$

where $R'_e = \frac{R_e}{\pi^4}$, $S_1 = \frac{S}{\pi^4}$ $x = \frac{a^2}{\pi^2}$, $\lambda'_2 = \frac{\lambda_2}{\pi^2}$, $\lambda'_4 = \frac{\lambda_4}{\pi^2}$, $\lambda'_6 = \frac{\lambda_6}{\pi^2}$

To achieve this we need careful selection of $\lambda_1, \lambda'_2, \lambda_3, \lambda'_4, \lambda_5, \lambda'_6$ and are found to be

$$\lambda_1 = \frac{1}{1+M_1(1-M_5)}, \quad \lambda'_2 = \frac{(1+x)M_1(1-M_5)}{1+M_1(1-M_5)}, \quad \lambda_3 = \frac{1+M'_1(\frac{1}{M_5}-1)}{L_e[1+M_1(1-M_5)]},$$

$$\lambda'_4 = \frac{(1+x)M'_1(\frac{1}{M_5}-1)}{L_e^2[1+M_1(1-M_5)]}, \ \lambda_5 = \frac{1+M''_1(\frac{1}{M_5}-1)}{L_e[1+M_1(1-M_5)]}, \ \lambda'_6 = \frac{(1+x)M''_1(\frac{1}{M_5}-1)}{L_e^2[1+M_1(1-M_5)]}, \quad (46)$$

Using Equation (46) in (45), we have

$$R_e = \frac{\{4(1+xM_3)-M_1(1-M_5)/[1+M_1(1-M_5)]\}\left[\{(1+x)^3(1+\delta M_3)\}+xS_1\{1+M_1'\left(\frac{1}{M_5}-1\right)\}+xS_2\{1+M_1''\left(\frac{1}{M_5}-1\right)\}\right]}{x\{4(1+xM_3)[1+M_1(1-M_5)]-2M_1(1-M_5)\}} \quad (47)$$

For M_1 sufficiently large, we obtain the magnetic thermal Rayleigh number

$$N_e = M_1 R_e = \frac{(3+4xM_3)[(1+x)^3(1+\delta M_3)+xS_1\{1+M_1'\left(\frac{1}{M_5}-1\right)\}+xS_2\{1+M_1''\left(\frac{1}{M_5}-1\right)\}]}{x\{(2+4xM_3)(1-M_5)\}} \quad (48)$$

as a function of x, N_e given by equation (48) attains its minimum, when

$$P_5x^5 + P_4x^4 + P_3x^3 + P_2x^2 + P_1x + P_0 = 0 \quad (49)$$

the coefficients $P_0, P_1, \ldots\ldots\ldots P_5$ being quite lengthy, and have not written here. The Newton-Raphson method is used to determine the values of critical wave number in nonlinear stability results by the condition

$$\frac{dN_e}{dx} = 0. \quad (50)$$

With x determined as a solution of Equation (49) , Equation (48) will give the required critical magnetic thermal Rayleigh number N_{ce} . In the absence of solute and MFD viscosity, Equation (48) reduces to

$$N_e = \frac{(3+4xM_3)(1+x)^3}{x(2+4xM_3)} \quad (51)$$

For analyzing the linear instability results, we use the non-dimensional Equations (13)-(20), neglecting the nonlinear terms. We again perform the standard stationary mode analysis and look for the solution of these equations in the form of Equation (42). The boundary conditions in the present case are same i.e. Equation (43). After following the same procedure as stated earlier in the energy stability case, we have

$$R_\ell = \frac{(1+x)^3(1+\delta M_3)(1+xM_3) + xS_1[1+xM_3+xM_1'M_3(\frac{1}{M_5}-1)+xS_2[1+xM_3+xM_1''M_3(\frac{1}{M_5}-1)]}{x[1+xM_3+xM_1M_3(1-M_5)]} \quad (52)$$

We again consider the magnetic thermal Rayleigh number N_e depends on the parameter M_3 . For M_1 sufficiently large, the linear critical magnetic thermal Rayleigh number is

$$N_\ell = \frac{(1+x)^3(1+\delta M_3)(1+xM_3) + xS_1\,[1+xM_3 + xM_1' M_3\,(\frac{1}{M_5}-1)+ xS_2\,[1+xM_3 + xM_1'' M_3\,(\frac{1}{M_5}-1)]}{x^3 M_3\,(1-M_5)} \quad (53)$$

In the absence of the solute and MFD viscosity, Equation (53) reduces to

$$N_\ell = \frac{(1+x)^3(1+xM_3)}{x^2 M_3} \quad (54)$$

which is in good agreement with the previous published (Finlayson[1970])

There are instances in which the two theories coincide. This is true for the classical Bénard problem. In the absence of magnetic parameters (M_1=0, M'_1=0 , $M''_1 = 0$, and $M_3 = 0$), we obtain

$$R_\ell = \frac{(1+x)^3}{x} + S_1 + S_2 = R_e \quad (55)$$

In the absence of solutes (i.e. $S_1 = 0$ & S_2 =0), this further simplifies to

$$R_\ell = \frac{(1+x)^3}{x} = R_e \quad (56)$$

Thus in both the cases the linear instability boundary is equal to linear stability boundary. Here, the energy method leads to the result that arbitrary subcritical instabilities are not possible, which is in good agreement with the previous published work (Joseph[1965'1966]). Thus, for lower values of magnetic parameters, this coincidence is immediately lost.

IV. Results and discussion

The critical magnetic thermal Rayleigh numbers N_{ce} , $N_{c\ell}$ depend upon M_3,M'_1, M''_1, S_1, S_2 and M_5. It has been seen that in the absence of MFD viscosity (δ) , *both* N_{ce} , $N_{c\ell}$ decrease as M_3 increases, there by showing the destabilizing effect of M_3. It has also been observed that in the presence of MFD viscosity (δ) , *both* N_{ce} , $N_{c\ell}$ decrease for lower values of M_3 and increase for higher values of M_3. Thus the MFD viscosity increases *with the increase of* N_{ce} ,$N_{c\ell}$, hence showing the stabilizing effect of MFD viscosity. In the absence of MFD viscosity, the variation in magnetization releases extra energy, which adds up to the thermal energy to destabilize the system. So, in the absence of the MFD viscosity, magnetization always has a destabilizing effect. The presence of MFD viscosity gives rise to a resistive force. It (force) has the tendency to slow down the motion of the fluid in the boundary layer, thus inducing the heat transfer from bottom to top. The decrease in heat transfer is responsible for delaying the onset of convection. Thus, the MFD viscosity promotes stabilization. Since variation in magnetization leads to change in viscosity, hence for large values of coefficient of the MFD viscosity (δ) and large values of magnetization (M_3), the resistive viscous force overcomes the energy released, due to increase in magnetization and thus delays the onset of convection. Hence, increase in magnetization stabilizes the system, and so magnetization plays a dual role depending upon the values of coefficient of the MFD viscosity. Also it is observed that the values of $N_{c\ell}$ are always greater than those of N_{ce} , and this is quite obvious from the fact that linear stability theory gives sufficient conditions for instability, while the energy stability theory gives the sufficient condition for stability. Thus, the difference between the values of N_{ce} *and* $N_{c\ell}$ reveals that there is a band of Rayleigh numbers where subcritical instability may arise. One can note that this band decreases as M_3 increases.

The solutes gradients S_1 & S_2 have stabilizing effect, *because both* N_{ce} , $N_{c\ell}$ increases as solute gradients increases. One can note that the subcritical instability region expands with the increase of solute gradients . Here, in this case heating expands the fluid at the bottom of the layer and this in turn wants to expand, thereby enhance the motion due to thermal convection. On the other side, the heavier salts at the lower part of the layer have exactly the opposite effect and these act to prevent motion through convective overturning. Thus, these two effects are competing against each other. Due to this, the linear theory of instability does not always capture the effect of instability completely and instabilities might arise before the threshold is reached, as we have obtained in this problem.

V. Conclusions

In this paper a nonlinear stability analysis of triple- diffusive convection in a magnetized ferrofluid with magnetic field –dependent viscosity has been investigated. It has been observed that the boundaries of nonlinear stability and linear instability analyses do not intersect. The MFD viscosity and solute gradients always delay the onset of convection. We have derived a nonlinear stability threshold very close to the linear instability one. It has been seen that the magnetic mechanism alone can induce subcritical region of instability. The comparison between the linear and energy stability reveals that for convection problem in ferrofluids, the linear critical magnetic thermal Rayleigh number is higher in values than the nonlinear (energy) critical magnetic thermal Rayleigh number, which shows the possibility of the existence of subcritical instability . It is important to realize that the subcritical instability region decreases as magnetization increases. We also observe that solute gradients cannot induce subcritical region of instability, but in magnetic mechanism, this region expands with the increase of solute gradients. In non-ferrofluids, it is verified that the global stability Rayleigh number is exactly the same as that of linear instability.

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Oscillatory Unsteady Hydrodynamic Viscoelastic Flow in a Porous Channel with Radiative Heat Transfer

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Abstract: *This analysis examines the problem of oscillatory flow of a viscoelastic fluid and heat transfer along a porous oscillating channel with radiative heat transfer. Here we consider the flow through a channel in which the fluid is injected on one boundary of the channel with a constant velocity, while it is sucked off at the other boundary with the same velocity. The two boundaries are considered to be in close contact with two plates parallel to each other. The plates are supposed to be oscillating with a given velocity in their own planes. The analytical expressions for the velocity, the temperature and the wall shear stress have been obtained. The effects of viscoelastic parameter on the velocity profile, shear stress are presented graphically with the combinations of the other flow parameters. It is also observed that the temperature field is not significantly affected by the viscoelastic parameter.*
Keywords: *Viscoelastic fluid, radiative heat transfer, porous wall, oscillating channel.*

I. INTRODUCTION

The problem of hydrodynamic flow in a porous channel with radiative heat transfer received much attention because of its various applications in physiology and in engineering devices such as blood flow in arteries, transpiration cooling of re-entry vehicles and rocket busters, cross-hatching on ablative surfaces. Pulsatile flow of a fluid in a porous channel has been investigated by Wang [1], as well as Bhuyan and Hazarika [2] by considering the periodic pressure gradient. Raptis [3] studied the unsteady free convective flow through a porous medium bounded by an infinite vertical limiting surface with constant suction and time dependent temperature. The effect of Hall current and wall temperature oscillation on convective flow in a rotating fluid through porous medium was studied by Ram [4]. On the other hand, several other researchers (e.g. Makinde and Mhone [5], Prakash and Ogulu [6] as well as Mehmood and Ali [7]) investigated the effects of heat transfer in the flow of fluids. Adhikary and Misra [8] investigated the effects of porosity of the channel wall, magnetic field and radiative heat transfer on unsteady flow of an electrically conducting fluid through a channel. Ghosh [9] investigated the hydrodynamic fluctuating flow of a viscoelastic fluid in a porous channel, where the channels oscillate with a given velocity in their own planes.

The aim of the present work is to investigate the effects of non-Newtonian parameter on the unsteady two dimensional hydrodynamic flow and heat transfer of a viscoelastic fluid. One of the most popular models for non-Newtonian fluids is the model that is called the second-order fluid or fluid of second grade. It is reasonable to use the second-order fluid model to do numerical calculations. The effects of visco-elastic parameter with the combinations of the other flow parameters have been studied thoroughly and presented graphically. The constitutive equation for the incompressible second-order fluid is

$$S = -pI + \mu_1 A_1 + \mu_2 A_2 + \mu_3 (A_1)^2 \tag{1}$$

where S is the stress tensor, p is the hydrostatic pressure, $A_n, n = 1,2$ are the kinematic Rivlin-Ericksen tensors, μ_1, μ_2, μ_3 are the material co-efficients describing the viscosity, visco-elasticity and cross-viscosity respectively, where μ_1 and μ_3 are positive and μ_2 is negative (Coleman and Markovitz [10]). The equation (1) was derived by Coleman and Noll [11] from that of the simple fluids by assuming that the stress is more sensitive to the recent deformation than to the deformation that occurred in the distant past.

II. MATHEMATICAL FORMULATIONS

Consider the channel between two oscillating porous plate $y = 0$ and $y = h,$ the fluid is being injected by one plate with constant velocity V and sucked off by the other plate with the same velocity. Then the continuity equation reduces to $\frac{\partial u^*}{\partial x^*} = 0$ so that u^* is the function of y^* and t^* only.

The momentum equations are given by

$$\rho\left(\frac{\partial u^*}{\partial t^*}+V\frac{\partial u^*}{\partial y^*}\right)=-\frac{\partial p^*}{\partial x^*}+\mu_1\frac{\partial^2 u^*}{\partial y^{*2}}+\mu_2\left(\frac{\partial^3 u^*}{\partial y^{*2}\partial t^*}+V\frac{\partial^3 u^*}{\partial y^{*3}}\right)-\frac{\mu_1 u^*}{k^*}+g\rho\beta(T-T_0) \quad (2)$$

$$0=-\frac{\partial p^*}{\partial y^*}+(2\mu_2+\mu_3)\frac{\partial}{\partial y^*}\left(\frac{\partial u^*}{\partial y^*}\right)^2$$ so that

$0=-\frac{\partial p^*}{\partial y^*}$, assuming $2\mu_2+\mu_3=0$ as $\mu_2<0$ and $\mu_3>0.$ (3)

The heat transfer equation may be put in the form

$$\frac{\partial T}{\partial t^*}+V\frac{\partial T}{\partial y^*}=\frac{k'}{\rho C_p}\frac{\partial^2 T}{\partial y^{*2}}-\frac{1}{\rho C_p}\frac{\partial q}{\partial y^*} \quad (4)$$

where p^* is the pressure, ρ the density of the fluid, k permeability factor, q the radiative heat flux, β the coefficient of volume expansion due to temperature, g the gravitational acceleration, k' the coefficient of thermal conductivity, C_p the specific heat at constant pressure. The last term on the right hand side of equation (4) arises owing to the radiation effect of the heat transfer.
The corresponding boundary conditions of the oscillatory motion are:

$$u^*=U_0e^{i\omega^*t^*}, T=T_w+e^{i\omega^*t^*}(T_w-T_0) \quad \text{at} \quad y^*=h$$
$$u^*=U_0e^{i\omega^*t^*}, T=T_0 \quad \text{at} \quad y^*=0 \quad (5)$$

In these equations, we have taken into account the temperature oscillation on the upper plate $y^*=h$, while the lower plate $y^*=0$ is maintained at the fixed temperature T_0.
The heat flux may be expressed (Cogley et al. [12]) as

$$\frac{\partial q}{\partial y^*}=4\alpha_1^2(T-T_0) \quad (6)$$

where α_1 is the mean radiation and absorption coefficient.
Introduce the following non-dimensional quantities:

$$y=\frac{y^*}{h},\ x=\frac{x^*}{h},\ u=\frac{u^*}{h},\ \mathrm{Re}=\frac{Vh}{k_1},\ k=\frac{k^*}{h^2\rho},\ p=\frac{hp^*}{\rho\upsilon_1V},\ \theta=\frac{T-T_0}{T_w-T_0},$$
$$t=\frac{t^*V}{h},\ Gr=\frac{g\beta(T_w-T_0)h^2}{\upsilon_1V},\ \mathrm{Pr}=\frac{Vh\rho C_p}{k'},\ N^2=\frac{4\alpha^2h^2}{k'},\ \omega=\frac{\omega^*h}{V}. \quad (7)$$

where Re the Reynolds number, Gr the Grashof number, Pr the Prandtl number, N the radiation Parameter, ω the angular frequency.
The governing equations together with the heat equation can be re-written in terms of dimensionless quantities given in (7) as

$$\mathrm{Re}\left(\frac{\partial u}{\partial t}+\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial y^2}-\frac{u}{k}+Gr\theta+\alpha\left(\frac{\partial^3 u}{\partial y^2\partial t}+\frac{\partial^3 u}{\partial y^3}\right) \quad (8)$$

$$0=-\frac{\partial p}{\partial y} \quad (9)$$

and
$$\mathrm{Pr}\left(\frac{\partial\theta}{\partial t}+\frac{\partial\theta}{\partial y}\right)=\frac{\partial^2\theta}{\partial y^2}+N^2\theta \quad (10)$$

while the boundary conditions will assume the form

$$u=U_0e^{i\omega t}, \theta=1+e^{i\omega t} \quad \text{at} \quad y=1$$
$$u=U_0e^{i\omega t}, \theta=0 \quad \text{at} \quad y=0 \quad (11)$$

where $\alpha = \frac{\mu_2 V}{\mu_1 h}$ is the viscoelastic parameter.

III. METHOD OF SOLUTION

From (8) and (9), it follows that $\frac{\partial p}{\partial x}$ is a function of t alone. For the present study, we consider

$$\frac{\partial p}{\partial x} = A + Be^{i\omega t},$$

A and B being undetermined constants. To solve equations (8) and (10) subject to boundary conditions (11), we write the velocity and temperature in temperature in the form:

$$u(y,t) = u_s(y) + u_p(y,t)$$
$$= u_s(y) + u_f(y)e^{i\omega t} \quad (12)$$

and

$$\theta(y,t) = \theta_s(y) + \theta_p(y,t)$$
$$= \theta_s(y) + \theta_f(y)e^{i\omega t} \quad (13)$$

where $u_s(y), u_p(y,t), \theta_s(y), \theta_p(y,t)$ respectively represent the steady and unsteady parts of the velocity and temperature.

Substituting the above expressions in (8) and (10) and comparing the like terms, we have derived the equations that govern the corresponding steady and unsteady flow and heat transfer of the problem under consideration. They are given below:

Steady Case:

$$\alpha \frac{d^3 u_s}{dy^3} + \frac{d^2 u_s}{dy^2} - \mathrm{Re}\frac{du_s}{dy} - \frac{u_s}{k} = A - Gr\theta_s \quad , \quad (14)$$

$$\frac{d^2\theta_s}{dy^2} - \mathrm{Pr}\frac{d\theta_s}{dy} + N^2\theta_s = 0. \quad (15)$$

with the boundary conditions:

$$u_s = 0,\ \theta_s = 1, \quad \text{at} \quad y = 1$$
$$u_s = 0,\ \theta_s = 0. \quad \text{at} \quad y = 0 \quad (16)$$

Unsteady Case:

$$\alpha \frac{d^3 u_f}{dy^3} + (1 + i\alpha\omega)\frac{d^2 u_f}{dy^2} - \mathrm{Re}\frac{du_f}{dy} - (i\omega\mathrm{Re} + \frac{1}{k})u_f = B - Gr\theta_f \quad , \quad (17)$$

$$\frac{d^2\theta_f}{dy^2} - \mathrm{Pr}\frac{d\theta_f}{dy} + \left(N^2 - i\omega\mathrm{Pr}\theta_s\right)\theta_f = 0. \quad (18)$$

with the boundary conditions:

$$u_f = U_0,\ \theta_f = 1, \quad \text{at} \quad y = 1$$
$$u_f = U_0,\ \theta_f = 0. \quad \text{at} \quad y = 0 \quad (19)$$

On solving the equations (15) and (18) along with the boundary conditions (16) and (19) respectively, are found as

$$\theta_s = \frac{1}{e^{m_1} - e^{m_2}}\left(e^{m_1 y} - e^{m_2 y}\right), \quad (20)$$

$$\theta_f = \frac{e^{i\omega t}}{e^{m_3} - e^{m_4}}\left(e^{m_3 y} - e^{m_4 y}\right) \quad (21)$$

where

$$m_1=\frac{\Pr+\sqrt{\Pr^2-4N^2}}{2}\ ,\ m_2=\frac{\Pr-\sqrt{\Pr^2-4N^2}}{2}\ ,$$

$$m_3=\frac{\Pr+\sqrt{\Pr^2-4(N^2-i\omega\Pr)}}{2}\ ,\ m_4=\frac{\Pr-\sqrt{\Pr^2-4(N^2-i\omega\Pr)}}{2}.$$

We note that $|\alpha|<1$ for small shear and so we can assume that

$$u_s(y)=u_{s0}(y)+\alpha u_{s1}(y)+O(\alpha^2)\ ,$$

$$u_f(y)=u_{f0}(y)+\alpha u_{f1}(y)+O(\alpha^2) \qquad (22)$$

Substituting (22) in (14) and (17) together with boundary conditions (16) and (19) up to first order of α and equating the co-efficient of like powers of α, we obtain the following sets of ordinary differential equations and corresponding boundary conditions:

$$\frac{d^2u_{s0}}{dy^2}-\mathrm{Re}\frac{du_{s0}}{dy}-\frac{u_{s0}}{k}=A-Gr\theta_s\ , \qquad (23)$$

$$\frac{d^3u_{s0}}{dy^3}+\frac{d^2u_{s1}}{dy^2}-\mathrm{Re}\frac{du_{s1}}{dy}-\frac{u_s}{k}=0\ . \qquad (24)$$

with

$$u_{s0}=u_{s1}=0, \quad \text{at} \quad y=1$$

$$u_{s0}=u_{s1}=0, \quad \text{at} \quad y=0 \qquad (25)$$

and

$$\frac{d^2u_{f0}}{dy^2}-\mathrm{Re}\frac{du_{f0}}{dy}-(i\omega\mathrm{Re}+\frac{1}{k})u_{f0}=B-Gr\theta_f\ , \qquad (26)$$

$$\alpha\frac{d^3u_{f0}}{dy^3}+\frac{d^2u_{f1}}{dy^2}-\mathrm{Re}\frac{du_{f1}}{dy}-(i\omega\mathrm{Re}+\frac{1}{k})u_{f1}=-i\,\omega\frac{d^2u_{f0}}{dy^2}. \qquad (27)$$

with

$$u_{f0}=U_0,\ u_{f1}=0, \quad \text{at} \quad y=1$$

$$u_{f0}=U_0,\ u_{f1}=0, \quad \text{at} \quad y=0 \qquad (28)$$

The equations (23), (24) and (26), (27) are solved under the boundary conditions (25) and (28) respectively. Substituting these solutions in (22), we get the expressions for u_s and u_f, and thus the expression for u but due brevity the solutions are not presented here.

The non-dimensional wall shear stress at the upper plate is given by

$$\tau_w=\mu_1\left(\frac{\partial u}{\partial y}\right)+\mu_2\left(\frac{\partial^2 u}{\partial y\partial t}+v\frac{\partial^2 u}{\partial y^2}\right) \qquad (29)$$

IV. RESULTS AND CONCLUSIONS

The purpose of this study is to bring out the effects of the visco-elastic parameter α on the governing flow and heat transfer characteristics. We have considered the real parts of the results throughout for numerical validation. The effects of viscoelastic parameter on velocity, shear stress and flow rate of viscoelastic fluid are evaluated numerically and the results are presented in figures 1-4, and 5. The predicted variation of velocity with different values of α and for $\mathrm{N}=2,\Pr=2,\mathrm{k}=1;\ \mathrm{N}=3,\Pr=2,\mathrm{k}=1;\ \mathrm{N}=2,\Pr=5,\mathrm{k}=1;$ $\mathrm{N}=2,\Pr=2,\mathrm{k}=2$ are shown in figures 1-4 respectively.

It is evident from the figures 1-4 that the velocity profile is parabolic in nature and the values of the velocity u decrease with the increasing values of the viscoelastic parameter $|\alpha|(\alpha=0,-0.1,-0.2)$ in comparison with Newtonian fluid. It is also noted from the figures that the behaviours of the velocity profiles remain the same with the increasing values of the viscoelastic parameter $|\alpha|$ when (i) The values of N increase

(Fig.1 and Fig.2) and (ii) Pr increase (Fig.1 and Fig.3) (iii) k increase (Fig.1 and Fig.4). It is also seen that the velocity u increase with the increasing values of the radiative parameter N and the permeability parameter k for both Newtonian and non-Newtonian cases.

The wall shear stresses are calculated from the equation (29). Figure 5 show that the wall shear stress τ_w increases as the values of the viscoelastic parameter $|\alpha|(\alpha = 0, -0.1, -0.2)$ increase in comparison to Newtonian fluid.

It is also observed that the temperature field is not significantly affected by the visco-elastic parameter.

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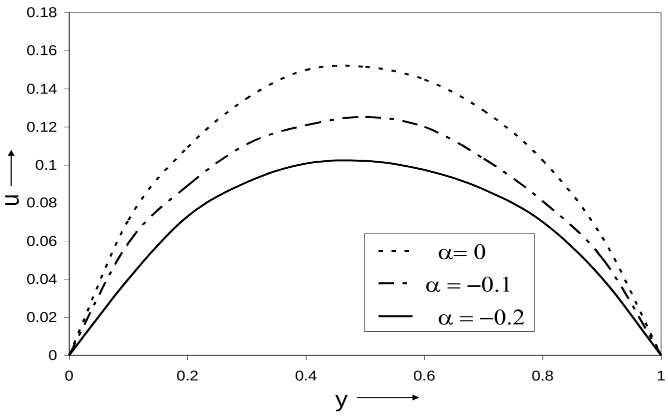

Fig. 1: Variation of u against y when $N = 2, Pr = 2, k = 1, Re = 1, t = \frac{\pi}{2}$.

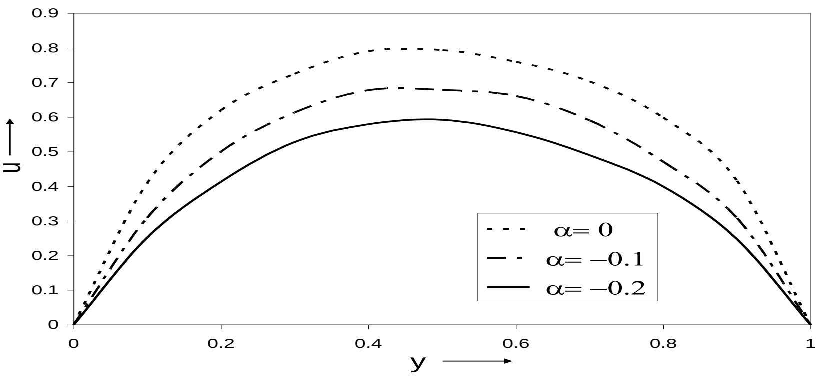

Fig. 2: Variation of u against y when $N = 3, Pr = 2, k = 1, Re = 1, t = \frac{\pi}{2}$.

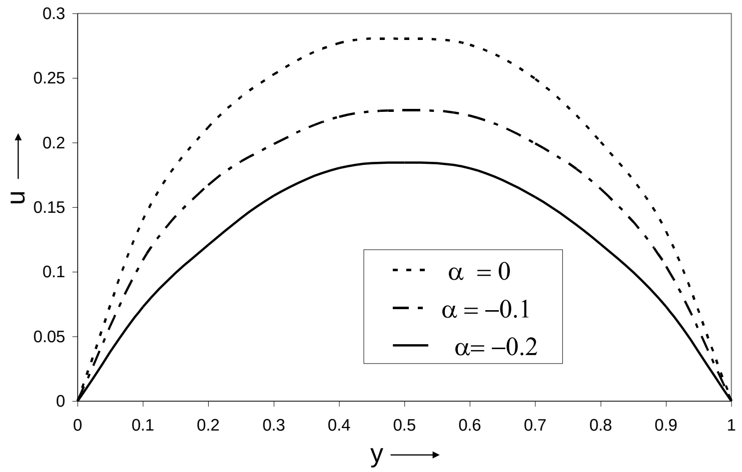

Fig. 3: Variation of u against y when $\mathrm{N} = 2, \mathrm{Pr} = 5, \mathrm{k} = 1, \mathrm{Re} = 1, \mathrm{t} = \frac{\pi}{2}$.

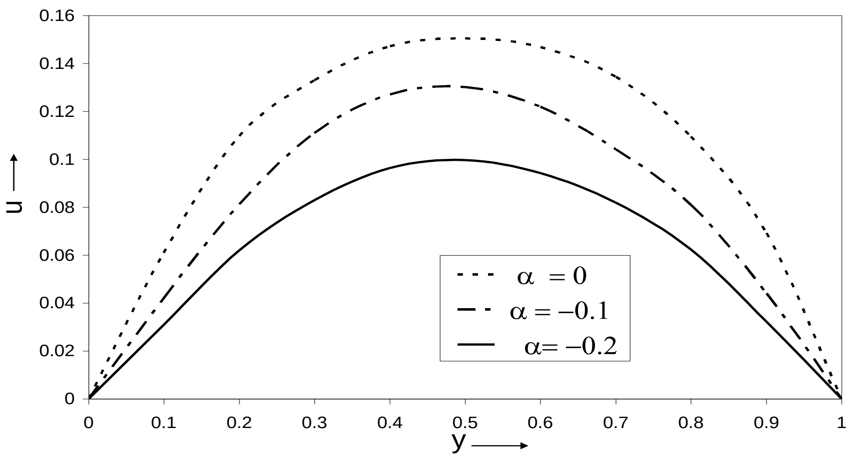

Fig. 4: Variation of u against y when $\mathrm{N} = 2, \mathrm{Pr} = 2, \mathrm{k} = 2, \mathrm{Re} = 1, \mathrm{t} = \frac{\pi}{2}$.

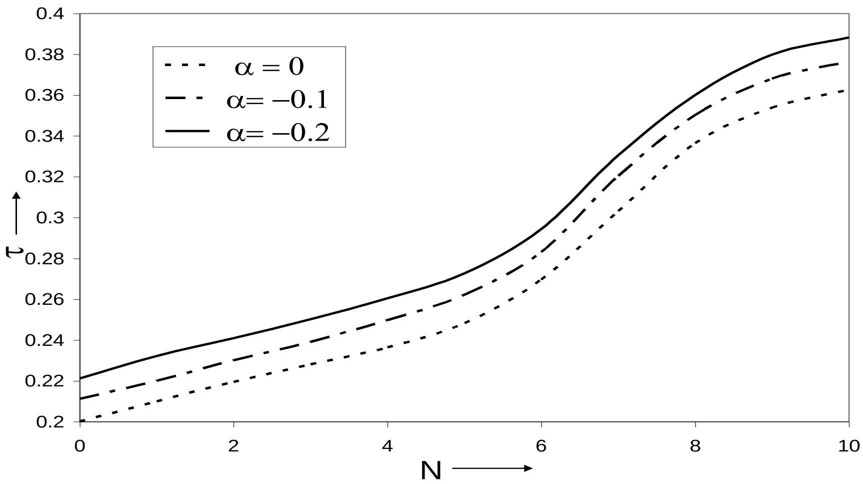

Fig. 5: Wall shear stress versus N when t= π..

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Integral Equations: The Concept Of Integrals

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Abstract: *The use of the double integral is the topic of the paper. The link of the double integral and the single integral is mentioned and the splitting of the double integral in two simple single integrals is involved. Also the equation of the curve formed when the double integral is converted to a single in integral is mentioned.*

The paper deals with the question that if the double integral is solved and it is converted to single integral then what is ment by it, also the equation of the curve formed of that single integral function is discussed and the relation between the surface of that double integral function and the single integral function is discussed.

I. Introduction

This paper deals with the DOUBLE INTEGRALS as what does double integral mean to me. It also consist of some facts and the proofs which provides the relation between the double integral and the single integral.

We have the function as

Y = f(x,y)

Generally, we have to plot or to calculate area under the curve of this region we have to take the double integral of the function

Therefore,

$z = \iint f(x,y)d(x)d(y)$

Solve the double in integral and convert the function to a single integral function generally

We have

$Z=\int g(x)dx$ or $Z =\int g(y)dy$ …eq.(1)

Depending on the term or the parameter (x or y) we keep constant the value of the integral will change as shown in the eq. 1.

The relation between the f(x, y) and g(x) or the relation between the f(x, y) and g(y) is described in this paper. We are known to the single integral function as it can be written and described using the limits as the sum.

We have $\int_a^b f(x)dx = \sum_{r=1}^{n} f(t)$

This concept can be extended further to define the integral of functions of two independent variables as follows.

DOUBLE INTEGRATION : DEFINATION

Let f(x, y) be a continuous function and it is single valued of two variables x, y defined over a region R of area A bounded by a closed simple curve C. let the region be divided into n sub intervals in any manner (eg. By drawing horizontal and the vertical lines) into sub regions R1,…R4 ,…………., Rn of areas A1,…..A4,……An.

Let P(Xr, Yr) be any point inside the rth sub region of area Ar.

We know form the sum

f(x1,y1) &A +…………… f(x7,y7) &A + ………………+f(xn,yn)

ie. $\sum_{r=1}^{n} f(xr\,,yr)\ \&Ar$.

We now increase the number of sub regions such that the area of each sub region becomes smaller and smaller. The limit of the sum (1), when it exits, as tends to infinity and the area of each sub intervals tend to zero is called the double integral of the f(x,y) over the region A and is denoted by

$\iint_A f(x,y)dxdy.$

Thus, $\iint_A f(x,y)dxdy = \lim \sum_{r=1}^{n} f(x,y)\ \&A$

EQUATION OF DOUBLE INTEGRAL :

The double integral as defined above can be evaluated by successive single integration as follows:

If A is a region bounded by the curves y=f(x) , y=g(x).

Then,

$\iint f(x,y)d(x)d(y) = \int_a^b \int_{f(x)}^{g(x)} f(x,y)dy\,dx.$

Where the integration w.r.t. y is performed first by treating x as constant.

Consider the area bounded the two curves y= f(x) and y=g(x) and the ordinates as x=a and x=b.

Now consider a strip parallel to the axis. On this strip y varies from y= f(x) to y=g(x). if the strip is moved to itself so that it will sweep the shown area then x varies from a and b. now it can be shown that

$\iint f(x,y)d(x)d(y) = \int_a^b \int_{f(x)}^{g(x)} f(x,y)dy\,dx.$

II. Concept

1. **CONVERSION TO SINGLE INTEGRAL**:

Generally to find the double integral of any function f(x, y) we have to first convert the double integral to the single integral. So in order to convert to the single integral we have to take solve by using one variable constant as x or y.

1.1 CONVERSION:

When we have the tendency to convert the double integral to single integral .

Let f(x, y) be a function of independent variables x and y. we have to convert in single integral there we take the cases in which we first consider x as constant and y as independent variable.

Therefore

$$\int_a^b \int_{f1(x)}^{f2(x)} f(x,y)\;dx\;dy =$$

Here the function varies from x=a to x=b and y=f1(x) to y=f2(x). so we take the x is independent and y as dependent of x.

1.2 Plotting of curve:

Plot the curve y=f1(x) and y=f2(x) on the x-y plane and find its required region according to specified values of x that is a and b.

We take the integration of the curve according to the limits decided by the y as f1(x) and f2(x).

We got

$$\int_a^b g(x)dx$$

And what actually g(x) is,

we know that limits of y are the extension of the curve f1(x) and f2(x) in the y direction. when the region enclosed between the curves the closed curve must have to be integrated in the x direction for getting the double integral.

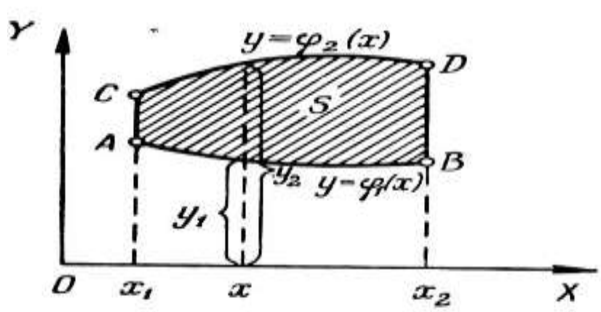
In this case the curve g(x) is actually a line which is parallel to y- axis as the line is made to integrated over x=a to x=b.

1.3 AREA

In the simple integral we have to take the area under the curve of the whole region. Similarly we have the equation of the line here and it has been integrated in the prescribed limits.

Actually the line about which we are talking is the centroidal line which passes from the centroid of the curve and parallel to y-axis.

Here the curve explains the the change in the x as there is a enclosed figure ABCD and the x varies from x1 to x2.



III. Change In Varible:

Consider the second case when the x is dependent on y and y is a dependent variable.

1.2 TAKING X AS VARIBLE:

Similarly we have the same curve as f(x, y) so here we will integrate the x first as it is a dependent variable and then y over the prescribed limits

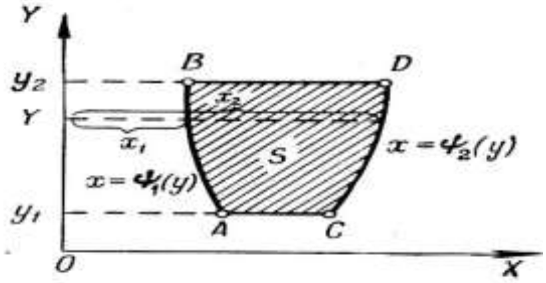
$$\int_a^b \int_{f1(y)}^{f2(y)} f(x,y)\, dy\, dx =$$

So we will first integrate by taking x as variable. So we get

$$\int_a^b g(y)dy$$

Here the curve g(y) is a line which is parallel to x axis which has to be integrated along the y axis limits that is from y=a to y=b.
Actually the line about which we are talking is the centroidal line which passes from the centroid of the curve and parallel to x-axis.

1.3 PLOTTING OF FIGURE:
Here the figure explains the variation of y from y1 to y2 along the figure ABCD as here the x varies from y1 to y2.



Now consider the case of triple integral in this case we have three variables x, y and z.
Then take the case in which we have

$$\int_a^b \int_{f1(x)}^{f2(x)} \int_{g1(x,y)}^{g2(x,y)} f(x,y,z)\ dx\, dy\, dz$$

:
2. TRIPLE INTEGRAL:
The in this case we can have two curves as if we convert the triple integration to a double integral considering either x or y or z as constant variable and the other as the dependent variable.

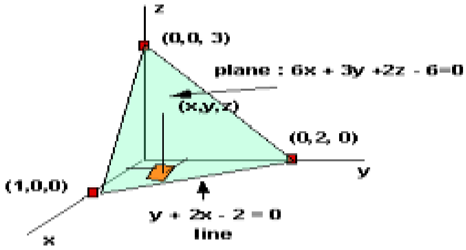
2.1 CONVERSION TO DOUBLE INTEGRALS:
So if we convert in double integral then,

$\int_a^b \int_{f1(x)}^{f2(x)} m(x,y)dx\, dy$ e.q.2

Here the f(x, y, z) is actually a space region or we can say a three dimensional space. Also we can say that the function m(x, y) is the plane which is actually parallel to x-y plane.
2.2 CONVERSION TO SINGLE INTEGRAL:
Also again if again we convert the equation in to single integral then again we can say that it is a equation of a line parallel to axis either x or y depending on the dependent variable.
If we write the equation 2 as
$\int_a^b \int_{f1(y)}^{f2(y)} m(y,z)\, dy\, dz$
Then we say that the dependent variable here is x and the curve m(y, z) is parallel to the y-z plane and also passes from the centroidal plane of that region. Again taking the integral and converting to single integral then the curve is parallel to any axis depending to the variable which is dependent, if the dependent variable is y then the curve formed is parallel to y axis otherwise x-axis.
3.3.CHANGE IN VARIBLES:
Similarly we can define the other condition

$$\int_a^b \int_{f1(x)}^{f2(x)} m(x,z)dx\, dz$$

2.4 Here dependent variable is y and the curve m(x, z) is parallel to the plane x-z and also passes from the centroidal plane of the three dimensional space. again taking the integral then we convert to single integral then the curve is again a line parallel to any axis depending on the dependent variable.



Here the case is taken in which the plane is formed by the three equations and also the plane formed by theequation when there is the case of forming the double integrels then the the plane is parallel to the main x-y-z plane and passes from the centroid of the figure .

3. APPLICATION ON NTH INTEGRAL:

This condition of applying the concept can be used for nth integral as in this case let we have there is nth dimensional plane and if we take the integral of that then the curve which is formed is nothing but the figure which actually consists of the centroidal figure of the main nth degree equation figure.

$$\int_{a}^{b}\int_{f1(x)}^{f2(x)} \dots\dots\dots\dots\dots\dots \int_{b(x,y,z\dots\dots\dots)}^{b2(x,y,z\dots\dots\dots\dots)} f(x,y,z,\dots\dots.)dx\,dy\,dz\dots\dots\dots..$$

Then if we take integral of this case then the figure we get, is the figure which covers the centroidal figure of the main figure and parallel to any of the figure formed by the mixture of the independent variables.

IV. Conclusion

This paper consist of the method through which we can relate the concept of integrals to the mechanics. If we solve the integral then the figure we get, is the figure which covers the centroidal figure of the main figure and parallel to any of the figure formed by the mixture of the independent variables.
This concept is applied in the various fields of mechanics and fluid mechanics.

There can be certain limitations to this concept as it is difficult to analyze the figure regarding the nth order integral when in the future when the nth dimension will be defined this concept can be applied there for finding out the center of gravity etc.

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Homogeneous Dusty Fluid Turbulence In A First Order Reactant For The Case Of Multi Point And Multi Time Prior To The Final Period Of Decay

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Abstract : *Using Deissler's approach, the decay for the concentration of a dilute contaminant undergoing a first-order chemical reaction in dusty fluid homogeneous turbulence at times prior to the ultimate phase for the case of multi-point and multi-time is studied. Here two and three point correlations between fluctuating quantities have been considered and the quadruple correlations are ignored in comparison to the second and third order correlations. Taking Fourier transform the correlation equations are converted to spectral form. Finally, integrating the energy spectrum over all wave numbers we obtained the decay law for the concentration fluctuations in a homogeneous turbulence prior to the final period in presence of dust particle for the case of multi-point and multi-time.*

Keywords: *Deissler's method, Dust particle, First order reactant, Navier-Stock's equation, Turbulent flow.*

I. INTRODUCTION

Chemical kinetics deals with the rates of chemical reactions and with how the rates depend on factors such as concentration and temperature. Such studies are important in providing essential evidence as to the mechanisms of chemical processes. The essential characteristic of turbulent flows is that turbulent fluctuations are random in nature. Chemical reactions occur in the gas phase, in solution in a variety of solvents, at gas-solid and other interfaces, in the liquid state, and in the solid state. It is sometimes convenient to work with amounts of substances instead of with concentrations. Experimental methods, some of them very sophisticated, have been developed for studying the rates of these various types of reaction and even for following very rapid reactions such as explosions. Theoretical treatments also have been worked out for the various types of reaction. Experiments of this kind can be referred to as "bulk" or "bulb" experiments. Chemical reaction as used in chemistry, chemical engineering, physics, fluid mechanics, heat and mass transport. The mathematical models that describe chemical reaction kinetics provide chemists and chemical engineers with tools to better understand and describe chemicals processes such as food decomposition, stratospheric ozone decomposition, and the complex chemistry of biological systems. In recent year; the motion of dusty viscous fluids in a rotating system has developed rapidly. The motion of dusty fluid occurs in the movement of dust –laden air, in problems of fluidization, in the use of dust in a gas cooling system and in the sedimentation problem of tidal rivers. The behavior of dust particles in a turbulent flow depends on the concentrations of the particles and the size of the particles with respect to the scale of turbulent fluid.

Following Deissler's approach [1, 2], the two-point, two-time correlations are obtained by considering the equation for the concentration of a dilute contaminant undergoing a first order chemical reaction. In order to solve the equations for the final period, the triple order correlation terms are neglected in comparison to the second-order ones. Loeffer and Deissler [3] used the theory, developed by Deissler [1, 2] to study the temperature fluctuations in homogeneous turbulence before the final period. In the study of homogeneous fluid turbulence a method is describing theoretically the concentration fluctuations of dilute contaminant a first order reactant prior to the ultimate phage of decay by Kumar and Patel [4]. Kumar and Patel [5] extended their problem [4] for the case of multi-point and multi-time concentration correlation. In [6], Sarker and Kishore studied the decay of MHD turbulence at times before the final period using Chandrasekhar's relation [7].Sarker and Islam [8] discussed the decay of MHD turbulence before the final period for the case of multi-point and multi-time. Aziz *et al.* [9] also extended their previous problem in presence of dust particle. Corrsin [10] obtained on the spectrum of isotropic temperature fluctuations in isotropic turbulence. Azad *et al.* [11] obtained first order reactant in magneto-hydrodynamic turbulence before the final period of decay in presence of dust particles. Azad *et al.* [12] also studied the statistical theory of certain distribution functions in MHD turbulent flow for velocity and concentration undergoing a first order reaction in a rotating system.

In this work, we studied the fluctuation of concentration of a dilute contaminant undergoing a first-order chemical reaction in homogeneous dusty fluid turbulence prior to the final phase of decay for the case of multi-point and multi-time. Here, we have considered two-point and three-point correlation equations and solved these equations after neglecting fourth-order correlation terms. Finally we obtained the decay law of energy fluctuations of concentration of dilute contaminant undergoing a first order chemical reaction for the case of multi-point and multi-time in homogeneous dusty fluid turbulence comes out to the form

$$\left\langle X^2 \right\rangle = \exp(\text{-2RT}_\text{m})\left\{AT_m^{-3/2} + \exp(fQ)BT_m^{-5}\right\}$$

where $\left\langle X^2 \right\rangle$ denotes the concentration fluctuation energy. It is seen that the demolition of the impurity is more rapid than that in the case of pure mixing. This result has been shown in the figure also.

II. BASIC EQUATION

The differential equation governing the concentration of a dilute contaminant undergoing a first-order chemical reaction in dusty fluid homogeneous turbulence could be written as

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_k} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} - Ru_i + f(u_i - v_i) \tag{1}$$

The subscripts can take on the values 1, 2, and 3. Here, $u_i(\hat{x})$ is a random function of position and time at a point p, $u_k(\hat{x},t)$=turbulent velocity, R=constant reaction rate, D =diffusivity, t= time, ε_{mki}=alternating tensor, Ω_m=constant angular velocity components, $f = \frac{kN}{\rho}$, dimension of frequency, N=constant number density of dust particle, $m_s = \frac{4}{3}\pi R_s{}^3 \rho_s$, mass of single spherical dust particle of radius R_s, ρ_s =constant density of the material in dust particle, $p(\hat{x},t) =$ Pressure fluctuation, $\rho =$ Fluid density, $\nu =$ Kinematics viscosity, u_k =turbulent velocity component, v_i = dust particle velocity component, x_k = space-coordinate, and repeated subscript in a term indicates a summation of terms, with the subscripts successively taking on the values 1, 2, 3.

III. TWO-POINT, TWO-TIME CORRELATION AND SPECTRAL EQUATIONS

Under the limitations that (i) the turbulence and the concentration fields are homogeneous (ii) the chemical reaction and the local mass transfer have no effect on the velocity field and (iii) the reaction rate and the diffusivity are constant, differential equation governing the concentration of a dilute contaminant undergoing a first-order chemical reaction we take the Navier-Stokes equations at the point P and the concentration equation at P′ and separated by the vector $\hat{r}$ could be written as

$$\frac{\partial X}{\partial t} + u_k \frac{\partial X}{\partial x_k} = D\frac{\partial^2 X}{\partial x_k \partial x_k} - RX \tag{2}$$

$$\frac{\partial X'}{\partial t'} + u'_k \frac{\partial X'}{\partial x'_k} = D\frac{\partial^2 X'}{\partial x'_k \partial x'_k} - RX' \tag{3}$$

where $X(\hat{x},t)$ is a random function of position and time. The other symbols are as usual.

Multiplying equation (2) by X', equation (3) by X, and averaging, we get

$$\frac{\partial \langle XX' \rangle}{\partial t} + \frac{\partial \langle u_k XX' \rangle}{\partial x_k} = D\frac{\partial^2 \langle XX' \rangle}{\partial x_k \partial x_k} - R\langle XX' \rangle \tag{4}$$

$$\frac{\partial \langle XX' \rangle}{\partial t'} + \frac{\partial \langle u'_k XX' \rangle}{\partial x'_k} = D\frac{\partial^2 \langle XX' \rangle}{\partial x'_k \partial x'_k} - R\langle XX' \rangle \tag{5}$$

where the conditions of continuity and the fact that the quantities at a point at a particular time are independent of the positions at the other points have been utilized.

Using the transformations. $\frac{\partial}{\partial x_k} = -\frac{\partial}{\partial r_k}, \frac{\partial}{\partial x'_k} = \frac{\partial}{\partial r_k}, \frac{\partial}{\partial t} = \left(\frac{\partial}{\partial t}\right)_{\Delta t} - \frac{\partial}{\partial \Delta t}, \frac{\partial}{\partial t'} = \frac{\partial}{\partial \Delta t},$

in to equations (4) and (5), we obtains

$$\frac{\partial \langle XX' \rangle}{\partial t} + \frac{\partial \langle u_k XX' \rangle}{\partial r_k}(-\hat{r},-\Delta t, t+\Delta t) - \frac{\partial \langle u_k XX' \rangle}{\partial r_k}(\hat{r},\Delta t, t) = 2D\frac{\partial^2 \langle XX' \rangle}{\partial r_k \partial r_k} - 2R\langle XX' \rangle \tag{6}$$

$$\frac{\partial \langle XX' \rangle}{\partial \Delta t} + \frac{\partial \langle u_k XX' \rangle}{\partial r_k}(-\hat{r},-\Delta t, t+\Delta t) = D\frac{\partial^2 \langle XX' \rangle}{\partial r_k \partial r_k} - R\langle XX' \rangle \tag{7}$$

In order to reduce Eqs. (6) and (7) to spectral form by using three-dimensional Fourier transform

$$\langle XX'(\hat{r},\Delta t,t)\rangle = \int_{-\infty}^{\infty}\theta(\hat{k},\Delta t,t)\exp(i\hat{k}.\hat{r})d\hat{k} \tag{8}$$

$$\langle XX'(\hat{r},\Delta t,t)\rangle = \int_{-\infty}^{\infty}\phi_k(\hat{k},\Delta t,t)\exp(i\hat{k}.\hat{r})d\hat{k} \,. \tag{9}$$

We get

$$\frac{\partial\theta}{\partial t}+(2Dk^2+2R)\theta = ik_k\phi_k(\hat{k},\Delta t,t)+i(-k_k)\phi_k(-\hat{k}_k,-\Delta t,t+\Delta t)\,, \tag{10}$$

$$\frac{\partial\theta}{\partial\Delta t}+(Dk^2+R)\theta = -ik_k\phi_k(-\hat{k},-\Delta t,t+\Delta t)\cdot \tag{11}$$

IV. Solution for the Ultimate Phase of Decomposing Turbulence

For the ultimate phase of homogeneous turbulence decompose, the third-order correlations can be ignored in comparison to the second-order correlations, with this approximation the solutions of Eqs. (10) and (11) may be obtained as

$$\theta = f_1(\hat{k},\Delta t)\exp[-(2Dk^2+2R)(t-t_0)] \tag{12}$$

$$\theta = f_2(\hat{k},t)\exp[-(Dk^2+R)\Delta t)]. \tag{13}$$

For consistent solution of Eqs (12) and (13) we must have

$$G(k)= f(k)\exp[(-2Dk^2+2R)(t-t_0+\frac{\Delta t}{2})]\,. \tag{14}$$

where $G(k)=2\pi k^2\theta$ is the concentration spectrum function. We evaluate f (k) by Corrsion [10]

i.e. $f(k) = N_o k^2|\pi$ .where N_0 is a constant depend on initial condition. Thus, we obtain

$$G(k)=\frac{N_0k^2}{\pi}\exp[(-2Dk^2+2R)(t-t_0+\frac{\Delta t}{2})] \tag{15}$$

By integrating equation (15) with respect to k, we obtain

$$\langle XX'\rangle(\hat{r},t_m) = \frac{N_oD^{1/2}}{4\sqrt{4\pi}(t_m-t_o)^{3/2}}\ \exp\left\{-\left[\frac{2C(t_m-t_o)+r^2}{8D(t_m-t_o)}\right]\right\} \quad , \quad \text{where } t_m = t+\Delta t/2\,. \tag{16}$$

V. Three-point, Three-time Correlation and Spectral Equations

Under the same assumptions as before, we take the Navier-Stokes equation for dusty fluid homogeneous turbulence at the point P and the concentration equations at P' and P'' as

$$\frac{\partial u_i}{\partial t}+u_k\frac{\partial u_i}{\partial x_k}=-\frac{1}{\rho}\frac{\partial p}{\partial x_k}+\nu\frac{\partial^2u_i}{\partial x_k\partial x_k}-Ru_i+f(u_i-v_i) \tag{17}$$

$$\frac{\partial X'}{\partial t'}+u'_k\frac{\partial X'}{\partial x'_k}=D\frac{\partial^2X'}{\partial x'_k\partial x'_k}-RX' \tag{18}$$

$$\frac{\partial X''}{\partial t''}+u''_k\frac{\partial X''}{\partial x''_k}=D\frac{\partial^2X''}{\partial x''_k\partial x''_k}-RX'' \tag{19}$$

Multiplying equation (17) by $X'X''$, (18) by u_iX'' and (19) by u_iX' and then taking space averages, we obtain.

$$\frac{\partial}{\partial t}\langle u_iX'X''\rangle+\frac{\partial}{\partial x_k}\langle u_iu_kX'X''\rangle=-\frac{1}{\rho}\frac{\partial}{\partial x_k}\langle pX'X''\rangle+\nu\frac{\partial^2}{\partial x_k\partial x_k}\langle u_iX'X''\rangle-R\langle u_iX'X''\rangle+f[\langle u_iX'X''\rangle-\langle v_iX'X''\rangle] \tag{20}$$

$$\frac{\partial}{\partial t'}\langle u_iX'X''\rangle+\frac{\partial}{\partial x'_k}\langle u_iu'_kX'X''\rangle=D\frac{\partial^2}{\partial x'_k\partial x'_k}\langle u_iX'X''\rangle-R\langle u_iX'X''\rangle \tag{21}$$

$$\frac{\partial}{\partial t''}\langle u_iX'X''\rangle+\frac{\partial}{\partial x''_k}\langle u_iu''_kX'X''\rangle=D\frac{\partial^2}{\partial x''_k\partial x''_k}\langle u_iX'X''\rangle-R\langle u_iX'X''\rangle \tag{22}$$

Using the transformations

$$\frac{\partial}{\partial x_k}=-(\frac{\partial}{\partial r_k}+\frac{\partial}{\partial r'_k}),\frac{\partial}{\partial x'_k}=\frac{\partial}{\partial r_k},\frac{\partial}{\partial x''_k}=\frac{\partial}{\partial r'_k},\frac{\partial}{\partial t}t',t''=\left(\frac{\partial}{\partial t}\right)_{\Delta t,\Delta t'}-\frac{\partial}{\partial\Delta t}-\frac{\partial}{\partial\Delta t'},\frac{\partial}{\partial t'}=\frac{\partial}{\partial\Delta t},\frac{\partial}{\partial t''}=\frac{\partial}{\partial\Delta t'}$$

Into equations (20)-(22), we get

$$\frac{\partial}{\partial t}\langle u_i X'X''\rangle - (\frac{\partial}{\partial r_k}+\frac{\partial}{\partial r'_k})\langle u_i u_k X'X''\rangle + \frac{\partial}{\partial r_k}\langle u_i u'_k X'X''\rangle + \frac{\partial}{\partial r'_k}\langle u_i u''_k X'X''\rangle$$

$$= -\frac{1}{\rho}(\frac{\partial}{\partial r_k}+\frac{\partial}{\partial r'_k})\langle pX'X''\rangle + \nu(\frac{\partial}{\partial r_k}+\frac{\partial}{\partial r'_k})^2\langle u_i X'X''\rangle + D(\frac{\partial^2}{\partial r_k \partial r_k}+\frac{\partial^2}{\partial r'_k \partial r'_k})\langle u_i X'X''\rangle$$

$$-(2R)\langle u_i X'X''\rangle + f[\langle u_i X'X''\rangle - \langle v_i X'X''\rangle] \quad (23)$$

$$\frac{\partial}{\partial \Delta t}\langle u_i X'X''\rangle + \frac{\partial}{\partial r_k}\langle u_i u'_k X'X''\rangle = D\frac{\partial^2}{\partial r'_k \partial r'_k}\langle u_i X'X''\rangle - R\langle X'u_i X''\rangle \quad (24)$$

$$\frac{\partial}{\partial \Delta t'}\langle u_i X'X''\rangle + \frac{\partial}{\partial r'_k}\langle u_i u''_k X'X''\rangle = D\frac{\partial^2}{\partial r''_k \partial r''_k}\langle u_i X'X''\rangle - R\langle X'u_i X''\rangle \quad (25)$$

Using the six-dimensional Fourier transform of the type

$$\langle XX'u''_k(\hat{r},\hat{r}',\Delta t,\Delta t',t)\rangle = \int_{-\infty}^{\infty}.\int_{-\infty}^{\infty}\psi_k(\hat{k},\hat{k}',\Delta t,\Delta t',t)\exp(\hat{k}.\hat{r}+\hat{k}'.\hat{r}')d\hat{k}d\hat{k}'$$

and with the fact that, $\langle X'X''v_k\rangle = \int_{-\infty}^{\infty}.\int_{-\infty}^{\infty}\psi'_k(\hat{k},\hat{k}',\Delta t,\Delta t',t)\exp(\hat{k}.\hat{r}+\hat{k}'.\hat{r}')d\hat{k}d\hat{k}'$

and the assumption that the quintuple correlations representing the transfer terms in equations (23)-(25) can be neglected as they decay faster than the lower-order correlation terms. Then the equations (23) - (25) in Fourier space can be written as

$$\frac{\partial \psi_i}{\partial t}(\hat{k},\hat{k}',\Delta t,\Delta t',t) + D[(1+N_S)k^2 + 2N_s kk'\cos\{(1+N_s)k'^2 + (2R-fQ)/D\}]\psi_i(\hat{k},\hat{k}',\Delta t,\Delta t',t) = 0 \quad (26)$$

$$\frac{\partial \psi_i}{\partial \Delta t}(\hat{k},\hat{k}',\Delta t,\Delta t',t) + D[\ (k^2+2R)/D]\psi_i(\hat{k},\hat{k}',\Delta t,\Delta t',t) = 0 \quad (27)$$

$$\frac{\partial \psi_i}{\partial \Delta t'}(\hat{k},\hat{k}',\Delta t,\Delta t',t) + D[\ (k'^2+2R)/D]\psi_i(\hat{k},\hat{k}',\Delta t,\Delta t',t) = 0 \quad (28)$$

where $N_s = \nu / D$, the Schmidt number and $\langle X'X''v_l\rangle = L\psi'_i\langle X'X''v_l\rangle$, *1-L=Q*.

As the pressure force terms are related to higher-order correlations, therefore, these along with the quadruple correlations are also neglected.

Integrating equations (26)-(28) between t_o and t, we obtain

$\psi_i = f_i\exp\{-D[(1+N_s)k^2 + 2N_s kk'\cos\theta + (1+N_s)k'^2 + (2R-fQ)/D](t-t_0)\}$ $\psi_i = g_i\exp\{[-D(k^2+R/D)]\Delta t\}$

$\psi_i = h_i\exp\{[-D(k'^2+R/D)]\Delta t'\}$.

For these relations to be consistent, we have

$$k_i\psi_i = k_i(\psi_i)_0\exp[-D\{(1+N_s)(k^2+k'^2)(t-t_0) + k^2\Delta t + k'^2\Delta t' + 2N_s kk'\cos\theta(t-t_0)$$

$$+(2R/D)[t-t_0+(\Delta t+\Delta t')/2] + [(-fQ)/D](t-t_0)\}] \quad (29)$$

where the subscript 0 refers to the value of ψ_i at $t = t_o$, $\Delta t = \Delta t' = 0$ and θ is the angle between k and k'. The relation between ϕ_i and ψ_i is given by

$$k_i\phi_i(\hat{k},\Delta t,t) = \int_{-\infty}^{\infty}k_i\psi_i(\hat{k},\Delta t,\hat{k}',0,t)dk' \quad (30)$$

Substituting equations (30) and (29) into equation (10), we obtain

$$\frac{\partial G}{\partial t} + (2k^2D + 2R)G = W \quad (31)$$

where $G = 2\pi k^2\theta$ and

$$W = \int_0^{\infty} ik_i(\psi_i)_0 k^2k'^2(2\pi)^2\exp\left\{-D\left[(1+N_S)k^2\left(t-t_o+\frac{\Delta t}{t+N_S}\right)+k'^2(1+N_S)(t-t_o)\right.\right.$$

$$\left.\left.+[(-fQ)/D](t-t_0)+(2R/D)\left(t-t_o+\frac{\Delta t}{2}\right)\right\}\right]\left\{\int_{-1}^{1}\exp[-2N_s Dkk'(t-t_0)\cos\theta]dk'\right.$$

$$+\int_0^{\infty}[i(-k_i)\psi_i(-\hat{k},-\hat{k}')]_0(2\pi)^2k^2k'^2\exp\{-D\left[(1+N_S)k^2\left(t-t_o+\frac{\Delta t}{1+N_S}\right)\right.$$

$$-k'^2(1+N_s)(t-t_0+\Delta t)+[(-fQ)/D](t-t_0)+\frac{2R}{D}(t-t_0+\frac{\Delta t}{2})]\}$$

$$\times\int_{-1}^{1}\exp[-2N_sDkk'(t-t_0+\Delta t)\cos\theta](d\cos\theta)dk' \quad (32)$$

where dk' is written as $2\pi k'^2 d(\cos\theta)dk'$ and the quantity $(\psi_i)_0$ depends on the initial conditions of the turbulence. Now, following Deissler [1, 2], we take

$$(2\pi)^2 i[k_i\psi_i(\hat{k},\hat{k}')_0=-\frac{1}{2}\delta_0(k^2k'^4-k^4k'^2) \quad \text{and}$$

$$(2\pi)^2 i[-k_i\psi_i(-\hat{k},-\hat{k}')_0=\frac{1}{2}\delta_0(k^2k'^4-k^4k'^2) \quad (33)$$

Substituting equation (33) in (32) and completing the integration, we get

$$W=-\frac{\delta_o N_s\pi^{1/2}}{4D^{3/2}(t-t_o)^{3/2}(1+N_s)^{5/2}}\exp\left[-k^2D\frac{1+2N_s}{1+N_s}\left(t-t_o+\frac{1+N_s}{1+2N_s}\Delta t\right)\right.$$

$$\left.+(-fQ)(t-t_0)-2R\left(t-t_o+\frac{\Delta t}{2}\right)\right]\left\{\frac{15k^4}{4N_s^2(t-t_o)^2D^2}\frac{N_s}{1+N_s}+\left[5\left(\frac{N_s}{1+N_s}\right)^2-\frac{3}{2}\right]\right.$$

$$\left.\times\frac{k^6}{N_sD(t-t_o)}+\left[\left(\frac{N_s}{1+N_s}\right)^3-\frac{N_s}{1+N_s}\right]k^8\right\}-\frac{\delta o N_s\sqrt{\pi}}{4D^3/2(t-t_o+\Delta t)^{3/2}(t+N_S)^{5/2}}$$

$$\times\exp\left[-k^2D\frac{1+2N_s}{1+N_s}\left(t-t_o+\frac{N_s}{+1+N_s}\Delta t\right)+(-fQ)(t-t_0)-2R\left(t-t_o-\frac{\Delta t}{2}\right)\right]$$

$$\times\left\{\frac{15k^4}{4D^2N_s^2(t-t_o+\Delta t)^2}\left(\frac{N_s}{1+N_s}\right)+\left[5\left(\frac{N_s}{1+N_s}\right)2-\frac{3}{2}\right]\frac{k^6}{N_sD(t-t_o+\Delta t)}\right.$$

$$\left.+\left[\left(\frac{N_s}{1+N_s}\right)^3-\frac{N_s}{1+N_s}\right]k^8\right\}. \quad (34)$$

This represents the transfer function arising due to the consideration of concentration at three- point and three- time. When Δt = 0 and *R* = 0, the expression for reduces to the case of pure mixing .It may also be noted that (for Δt = 0)

$$\int_0^\infty W.dk=0 \quad (35)$$

This means that the conditions of continuity and homogeneity are satisfied. Physically, it was to be expected as *W* is a measure of the energy transfer and the total energy transferred to all wave numbers must be zero. With the help of equations (31) and (34), one can get

$$G=\frac{N_0k^2}{\pi}\exp\left\{-2(k^2D+2R)\left(t-t_o+\frac{\Delta t}{2}\right)\right\}+\frac{\sqrt{\pi}N_s}{D^{3/2}(1+N_s)^{7/2}}\frac{\delta_o}{4}$$

$$\times\exp\left[-D\frac{1+2N_s}{1+N_s}k^2\left(t-t_o+\frac{(1+N_s)\Delta t}{1+2N_s}\right)+(-fQ)(t-t_0)-2R\left(t-t_o+\frac{\Delta t}{2}\right)\right]$$

$$\times\left[\frac{3k^4}{2N_sD^2(t-t_o)^{5/2}}+\frac{(7N_s-6)k^6}{3D(1+N_s)(t-t_o)^{3/2}}-\frac{4(3N_s^2-2N_s+3)k^8}{3(1+N_s)^2(t-t_o)^{1/2}}\right.$$

$$\left.+\frac{8D^{1/2}(3N_s^2-2N_s+3)k^9}{3(1+N_s)^{5/2}}F\left(k\sqrt{\frac{(t-t_o)D}{1+N_s}}\right)\right]+\frac{\sqrt{\pi}N_s}{D^{3/2}(1+N_s)^{7/2}}\frac{\delta_o}{4}$$

$$\times\exp\left[-D\left(\frac{1+2N_s}{1+N_s}\right)k^2\left(t-t_o+\frac{N_s}{1+2N_s}\Delta t\right)+(-fQ)(t-t_0)-2R\left(t-t_o+\frac{\Delta t}{2}\right)\right]$$

$$\times\left[\frac{3k^4}{2D^2N_s(t-t_o+\Delta t)^{5/2}}+\frac{(7N_s-6)k^6}{3D(1+N_s)(t-t_o+\Delta t)^{3/2}}-\frac{4}{3}\frac{(3N_s^2-2N_s+3)k^8}{(1+N_s)^2(t-t_o+\Delta t)^{1/2}}\right.$$

$$\left.+\frac{8D^{1/2}(3N_s^2-2N_s+3)k^9}{(1+N_s)^{5/2}}F\left(k\sqrt{\frac{(t-t_o+\Delta t)D}{1+N_s}}\right)\right] \tag{36}$$

where, $F(\omega)=\exp(-\omega^2)\int_0^{\omega}\exp(x^2)dx$, $\omega = k\sqrt{\frac{(t-t_o)D}{1+N_s}}$ or $k\sqrt{\frac{(t-t_o+\Delta t)D}{1+N_s}}$

As in the previous section, by integrating equation (36) with respect to k, we obtain

$$\left\langle \frac{XX'}{2}(\Delta t,t_m)\right\rangle=\int_0^{\infty}Gdk=\frac{N_o}{8D^{3/2}\sqrt{2\pi}\left(T+\frac{\Delta t}{2}\right)^{3/2}}\exp\left[-2R\left(T+\frac{\Delta t}{2}\right)\right]$$

$$+\frac{\pi}{D^6(1+N_s)(1+N_s)^{5/2}}\exp[(fQ)]\exp[-2R(T+\frac{\Delta t}{2})]$$

$$\times\frac{\delta_0}{4}\left[\frac{9}{16T^{5/2}\left(T+\frac{1+N_s}{1+2N_s}\Delta T\right)^{5/2}}+\frac{9}{16(T+\Delta T)^{5/2}\left(T+\frac{1+N_s}{1+2N_s}\Delta T\right)^{5/2}}\right.$$

$$+\frac{5N_s(7N_s-6)}{16(1+2N_s)T^{3/2}\left(T+\frac{1+N_s}{1+2N_s}\Delta T\right)^{7/2}}+\frac{5N_s(7N_s-6)}{16(1+2N_s)(T+\Delta T)^{3/2}\left(T+\frac{N_s}{1+2N_s}\Delta T\right)^{7/2}}$$

$$+\frac{35N_s(3N_s^2-2N_s+3)}{8(1+2N_s)T^{1/2}\left(T+\frac{1+N_s}{1+2N_s}\Delta T\right)^{9/2}}+\frac{35N_s(3N_s^2-2N_s+3)}{8(1+2N_s)(T+\Delta T)^{1/2}\left(T+\frac{N_s}{1+2N_s}\Delta T\right)^{9/2}}$$

$$\left.+\frac{8N_s(3N_s^2-2N_s+3)(1+2N_s)^{5/2}}{3.2^{23/2}(1+N_s)^{11/2}}\sum_{n=0}^{\infty}\frac{1.3.5\ldots(2n+9)}{n!(2n+1)2^{2n}(1+N_s)^n}\times\frac{T^{(2n+1)/2}}{\left(T+\frac{\Delta T}{2}\right)^{(2n+11)/2}}+\frac{(T+\Delta T)^{(2n+1)/2}}{\left(T+\frac{\Delta T}{2}\right)^{(2n+1)/2}}\right] \tag{37}$$

where, $T = t - t_o$. For $T_m = T+\Delta T/2$, equation (37) becomes

$$\left\langle \frac{XX'}{2}(t_m)\right\rangle=\exp(-2RT_m)\frac{N_0}{8\sqrt{2\pi}D^{3/2}T^{3/2}{}_m}+\frac{\delta_o\pi}{4D^6(1+N_s)(1+2N_s)}\exp[fQ]\frac{9}{16}\frac{1}{\left(T_m-\frac{\Delta T}{2}\right)^{5/2}\left(T_m+\frac{\Delta T}{1+2N_s}\right)^{5/2}}$$

$$+\frac{9}{16(T_m+\frac{\Delta T}{2})^{5/2}(T_m-\frac{\Delta T}{2})^{5/2}}+\frac{5N_s(7N_s-6)}{16(1+2N_s)}\frac{1}{(T_m+\frac{\Delta T}{2})^{7/2}(T_m-\frac{\Delta T}{2})^{3/2}}+\frac{5N_s(7N_s-6)}{16(1-2N_s)}\frac{1}{\left(T_m+\frac{\Delta T}{2}\right)^{3/2}\left(T_m-\frac{\Delta T}{1+2N_s}\right)^{7/2}}+\ldots \tag{38}$$

If $\Delta t=0$, then equation (38) reduces to the form

$$\left\langle \frac{X^2}{2}\right\rangle=\exp(-2RT_m)\left\{\frac{N_o}{8\sqrt{2\pi}D^{3/2}}\frac{1}{T_m^{3/2}}+\frac{\delta_o\pi}{2D^6(1+N_s)(1+2N_s)}\exp[fQ]\left[\frac{9}{16T_m^5}+\frac{5}{16}\frac{N_s(7N_s-6)}{(1+2N_s)T_m^5}\right]\right\}$$

Therefore, $\left\langle X^2\right\rangle=\exp(-2RT_m)\left[\frac{N_o}{4\sqrt{2\pi}D^{3/2}}\frac{3}{T_m^{3/2}}+\exp[fQ]\frac{\delta_o\alpha}{D^6}\frac{1}{T_m^5}\right]$ (39)

where, $\alpha=\frac{\pi}{(1+N_s)(1+2N_s)}\left[\frac{9}{16}+\frac{5}{16}\frac{N_s(7N_s-6)}{(1+2N_s)}+\ldots\right].$

Thus, the decay law for the concentration energy fluctuation of dusty fluid homogeneous turbulence in a first order reactant for multi-point and multi-time prior to the ultimate phase may be written as

$$\left\langle X^2\right\rangle=\exp(-2RT_m)\left\{AT_m^{-3/2}+\exp[fQ]BT_m^{-5}\right\} \tag{40}$$

where $A = \dfrac{3N_0}{4\sqrt{2\pi}D^{3/2}}$, $B = \dfrac{\delta_0 \alpha}{D^6}$

In equation (40) we obtained the concentration fluctuation energy of dusty fluid homogeneous turbulence. In the absence of dust particles the equation (40) becomes

$$\left\langle X^2 \right\rangle = \exp(\text{-}2\mathrm{RT_m})\left\{AT_m^{-3/2} + BT_m^{-5}\right\} \tag{41}$$

Which was obtained earlier by Kumar and Patel [5].For large times, the last term of equation (41) becomes negligible and the decay law for the ultimate period becomes $\exp(\text{-}2\mathrm{RT_m})(AT_m^{-3/2})$ which in the case of pure-mixing is similar to the law obtained by Corrsin [12].

In Figs. 1-4 we observe that the variation of chemical reaction in presence of dust particle i.e. for *exp (fQ)* =.75, .50, .25, 0 causes significant changes in the concentration fluctuation decay of energy of homogeneous turbulence. In the presence of dust particles the energy decay of the fluid particles more rapidly which indicated in the Figs. 3-1 respectively. In Fig. 4, we observe that in the absence of dust particles energy decay more slowly than with the present of dust particles. It is noted that *y*1,*y*2,*y*3,*y*4,*y*5,*y*6 and *y*7 are solution curves of equation (40) but in the absence of dust particles *y*1,*y*2,*y*3,*y*4,*y*5,*y*6 , *y*7 are represented by equation (41) at the different values of *R* and dust particles and plotted are shown from Figs. 1 -3 and Fig. 4 respectively.

VI. Figures

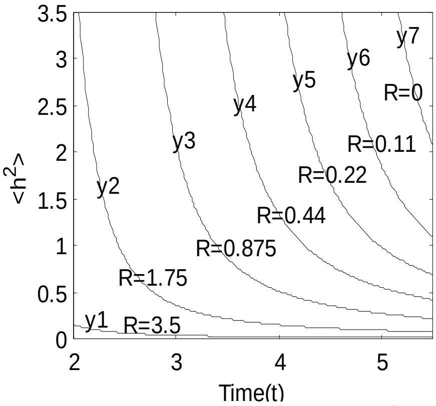

Fig. 1.Energy decay curves for exp (*Qf*) = 0.75.

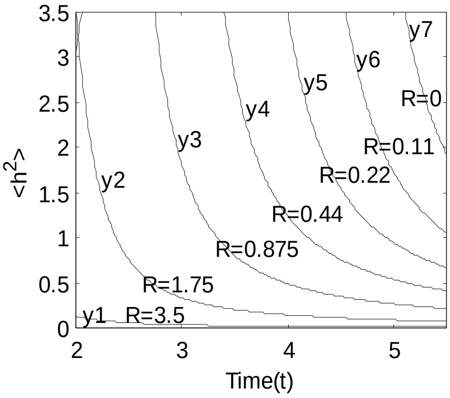

Fig. 2.Energy decay curves for exp (*Qf*) = 0.50.

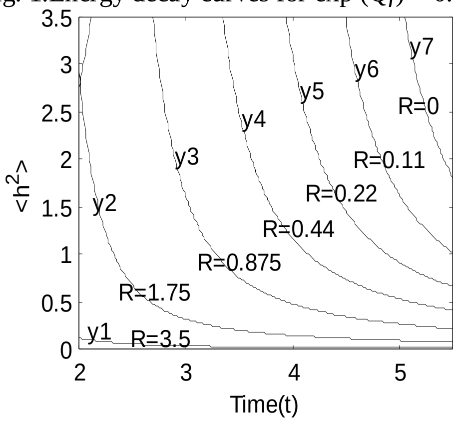

Fig. 3.Energy decay curves for exp (*Qf*) = 0.25.

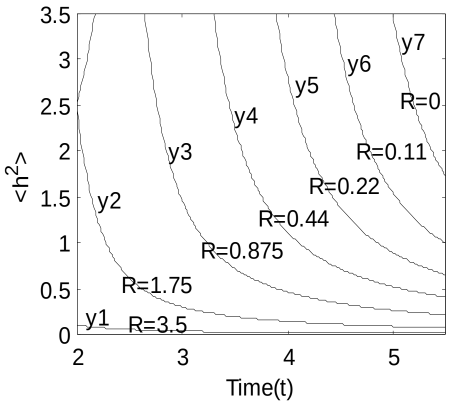

Fig. 4.Energy decay curves for exp (*Qf*) = 0.

VII. CONCLUSION

In the case of pure mixing, the concentration fluctuation decays with time in a natural manner. This study shows that if the concentration selected is the chemical reactant of the first order, then the effect is that the decomposition of the concentration fluctuation in homogeneous turbulence in the presence of dust particle for the case of multi-point and multi-time is much more rapid and the faster rate of decomposition is governed by $\exp(\text{-}2\mathrm{RT_m})$ .The decomposition of the concentration fluctuation in homogeneous turbulence is more slowly due to the absence of dust particles than any other type of chemical reactant as stated above. In a normal way, it takes a lot of time to get rid of a pollutant in the fluid. From the above figures and discussion, we conclude that in the absence of dust particles energy decay of the fluid particles more slowly but in the presence of dust particles the decomposition of the concentration fluctuation for the case of multi-point and multi-time in homogeneous turbulence are decreases due to the increases of the first order chemical reaction and maximum at the point where the chemical reaction is zero.

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Contraction Type Mapping on 2-Metric Spaces

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Abstract: *The superimposition of infinite number of intervals $[a_1, b_1]$, $[a_2, b_2]$, $[a_3, b_3]$,......., $[a_n, b_n]$ follows two laws of randomness if*

(a) $a_i \neq a_j$; i,j= 1, 2,......, n,
(b) $b_i \neq b_j$; i,j= 1, 2,......, n,
(c) $max(a_i) \leq min(b_i)$; i= 1, 2,..., n, where $n \rightarrow \infty$

Keywords: *Superimposition of sets, Probability distribution function, Glivenko – Cantelli Lemma.*

I. Introduction

Construction of normal fuzzy number has been discussed in ([1], [2]) based on the randomness – fuzziness consistency principle deduced by Baruah ([3], [4], [5]). Based on this aforesaid principle by including two more conditions which are not mentioned by Baruah, we have shown that if we superimpose infinite number of intervals $[a_1, b_1]$, $[a_2, b_2]$, $[a_3, b_3]$,......., $[a_n, b_n]$, then the values $a_{(1)}, a_{(2)}, \ldots, a_{(n)}$ follows an uniform probability distribution function and the values $b_{(1)}, b_{(2)}, \ldots, b_{(n)}$ follows an another complementary uniform probability distribution function where $a_{(1)}, a_{(2)}, \ldots, a_{(n)}$ and $b_{(1)}, b_{(2)}, \ldots, b_{(n)}$ are arranged in increasing order of magnitude of $a_1, a_2, a_3, \ldots, a_n$ and $b_1, b_2, b_3, \ldots, b_n$ respectively. If $\alpha = \min(a_i)$, $\beta = \max(a_i)$, $\mu = \min(b_i)$, $\gamma = \max(b_i)$, by satisfying the condition $a_i \neq a_j$, $b_i \neq b_j$ and $\max(a_i) < \min(b_i)$; i,j= 1, 2,......, n, then we can define the function $\psi(x)$ as

$$\begin{aligned}\psi(x) &= \psi_1(x) && \text{if } \alpha \leq x \leq \beta,\\ &= 1-\psi_2(x) && \text{if } \mu \leq x \leq \gamma,\\ &= 1 && \text{if } \beta \leq x \leq \mu,\\ &= 0 && \text{otherwise}.\end{aligned}$$

Where $\Psi_1(x)$ being a continuous distribution function in the interval $[\alpha, \beta]$, and $(1- \Psi_2(x))$ being a continuous distribution function in the interval $[\mu, \gamma]$, with $\Psi_1(\alpha) = \Psi_2(\gamma) = 0$ and $\Psi_1(\beta) = \Psi_2(\mu) = 1$.

II. The Operation Of Set Superimposition

The operation of set superimposition of two real intervals $[a_1, b_1]$ and $[a_2, b_2]$ as

$$[a_1,b_1](S)[a_2,b_2] = [a_{(1)},a_{(2)}] \cup [a_{(2)},b_{(1)}]^{(2)} \cup [b_{(1)},b_{(2)}]$$

Where $a_{(1)} = \min(a_1,a_2), a_{(2)} = \max(a_1,a_2), b_{(1)} = \min(b_1,b_2)$ and $b_{(2)} = \max(b_1,b_2)$.Here we have assumed without any loss of generality that $a_1 \neq a_2$, $b_1 \neq b_2$ and $[a_1,b_1] \cap [a_2,b_2]$ is not void or in other words that $\max(a_i) < \min(b_i), i = 1,2$

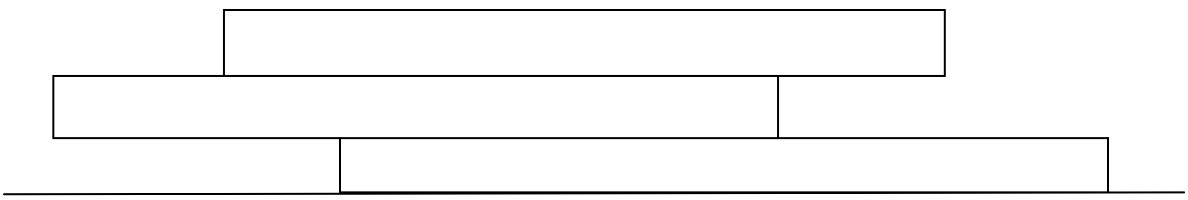

Figure1: Superimposition of $[x_1, y_1]^{\left(\frac{1}{3}\right)}, [x_2, y_2]^{\left(\frac{1}{3}\right)}$ and $[x_3, y_3]^{\left(\frac{1}{3}\right)}$

Figure2: Cumulative and complementary cumulative distribution functions

For the three intervals $[x_1, y_1]^{(1/3)}, [x_2, y_2]^{(1/3)} \, and \, [x_3, y_3]^{(1/3)}$ all with elements with a constant probability equal to 1/3 everywhere, we shall have

$$[x_1, y_1]^{(1/3)} (S) [x_2 y_2]^{(1/3)} (S) [x_3, y_3]^{(1/3)} = [x_{(1)}, x_{(2)}]^{(1/3)} \cup [x_{(2)}, x_{(3)}]^{(2/3)} \cup [x_{(3)}, y_{(1)}]^{(1)} \cup [y_{(1)}, y_{(2)}]^{(2/3)} \cup [y_{(2)}, y_{(3)}]^{(1/3)}$$

where, for example $[y_{(1)}, y_{(2)}]^{(2/3)}$ represents the interval $[y_{(1)}, y_{(2)}]$ with probability 2/3 for all elements in the entire interval, $x_{(1)}, x_{(2)}, x_{(3)}$ being values of x_1, x_2, x_3 arranged in increasing order of magnitude, and similarly $y_{(1)}, y_{(2)}, y_{(3)}$ being values of y_1, y_2, y_3 arranged in increasing order of magnitude again. We here presumed that $[x_1, y_1] \cap [x_2, y_2] \cap [x_3, y_3]$ is not void and $x_1 \neq x_2 \neq x_3$ and $y_1 \neq y_2 \neq y_3$.

It can be seen that for n intervals $[a_1, b_1]^{1/n}$, $[a_2, b_2]^{1/n}$, ..., , $[a_n, b_n]^{1/n}$ all with probability equal to 1/n everywhere, we shall have

$$[a_1, b_1]^{\frac{1}{n}} (S) [a_2, b_2]^{\frac{1}{n}} (S) .... [a_n, b_n]^{\frac{1}{n}} = [a_{(1)}, a_{(2)}]^{\frac{1}{n}} \cup [a_{(2)}, a_{(3)}]^{\frac{2}{n}} \cup .... \cup [a_{(n-1)}, a_{(n)}]^{\frac{n-1}{n}}$$

$$\cup [a_{(n)}, b_{(1)}]^{(1)} \cup [b_{(1)}, b_{(2)}]^{\frac{n-1}{n}} \cup .... \cup [b_{(n-2)}, b_{(n-1)}]^{\frac{2}{n}} \cup [b_{(n-1)}, b_{(n)}]^{\frac{1}{n}},$$

Where, for example, $[b_{(1)}, b_{(2)}]^{\frac{n-1}{n}}$ represents the interval $[b_{(1)}, b_{(2)}]$ with probability $\frac{n-1}{n}$ in the entire interval, $a_{(1)}, a_{(2)}, ..., a_{(n)}$ being values of $a_1, a_2, ..., a_n$ arranged in increasing order of magnitude, and $b_{(1)}, b_{(2)}, ..., b_{(n)}$ being values of $b_1, b_2, ..., b_n$ arranged in increasing order of magnitude. Thus for the intervals $[a_1, b_1]^{\frac{1}{n}}, [a_2, b_2]^{\frac{1}{n}}, ... [a_n, b_n]^{\frac{1}{n}}$, all with uniform probability $\frac{1}{n}$, the probabilities of the superimposed intervals are $\frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1, \frac{n-1}{n}, ..., \frac{2}{n}$ and $\frac{1}{n}$. These probabilities considered in two halves as

$$\left(0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\right)$$

and

$$\left(1, \frac{n-1}{n}, ..., \frac{2}{n}, \frac{1}{n}, 0\right)$$

would suggest that they can define an empirical distribution and a complementary empirical distribution on $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ respectively. In other words, for realizations of the values of $a_{(1)}, a_{(2)}, ..., a_{(n)}$ in increasing order and of $b_{(1)}, b_{(2)}, ..., b_{(n)}$ again in increasing order, we can see that if we define

$$\psi_1(x) = \begin{cases} 0 & if \; x < a_{(1)} \\ \frac{r-1}{n} & if \; a_{(r-1)} \leq x \leq a_{(r)}, r = 2, 3, ..., n \\ 1 & if \; x \geq a_{(n)}, \end{cases}$$

$$\psi_2(x)=\begin{cases}1 & if\ x<b_{(1)}\\ 1-\dfrac{r-1}{n} & if\ b_{(r-1)}\le x\le b_{(r)}, r=2,3,...,n\\ 0 & if\ x\ge b_{(n)},\end{cases}$$

Then the Glivenko – Cantelli Lemma on Order Statistics assures that

$$\psi_1(x)\to\prod{}_1[\alpha,x], \qquad \alpha\le x\le\beta,$$
$$\psi_2(x)\to 1-\prod{}_2[\mu,x], \quad \mu\le x\le\gamma,$$

where $\prod_1[\alpha,x], \alpha\le x\le\beta$ and $\psi_2(x), \mu\le x\le\gamma$ are two probability distributions. Here in this case we have considered that max(a_i) □ min(b_i), but for large number of observation when max(a_i) = min(b_i) that is β = μ then $a_{(n)} = b_{(1)}$ and we can write

$$\psi_1(x)\to\prod{}_1[\alpha,x], \qquad \alpha\le x\le\beta,$$
$$\psi_2(x)\to 1-\prod{}_2[\beta,x], \quad \beta\le x\le\gamma,$$

where $\prod_1[\alpha,x], \alpha\le x\le\beta$ and $\psi_2(x), \beta\le x\le\gamma$ are two probability distributions.

III. CONCLUSION

The superimposition of an infinite number of intervals [a_1, b_1], [a_2, b_2], [a_3, b_3],......., [a_n, b_n] by satisfying the conditions $a_i \neq a_j$, $b_i \neq b_j$ and max(a_i) ≤ min(b_i); i,j= 1, 2,......, n, follows two laws of randomness, one of which is $a_{(1)}$, $a_{(2)}$, ... , $a_{(n)}$ follows an uniform probability distribution function and the other one is $b_{(1)}$, $b_{(2)}$,.... , $b_{(n)}$ follows an another complementary uniform probability distribution function where $a_{(1)}$, $a_{(2)}$, ... , $a_{(n)}$ and $b_{(1)}$, $b_{(2)}$, ... , $b_{(n)}$ are arranged in increasing order of magnitude of a_1, a_2, a_3, ..., a_n and b_1, b_2, b_3, ..., b_n respectively. If α = min(a_i), β = max(a_i), μ = min(b_i), γ = max(b_i) and max(a_i) □ min(b_i), then we can define the function $\psi(x)$ as

$$\begin{aligned}\psi(x)&=\psi_1(x) && if\ \alpha\le x\le\beta,\\ &=1-\psi_2(x) && if\ \mu\le x\le\gamma,\\ &=1 && if\ \beta\le x\le\mu,\\ &=0 && otherwise.\end{aligned}$$

Where Ψ_1(x) being a continuous distribution function in the interval [α, β], and (1- Ψ_2(x)) being a continuous distribution function in the interval [μ, γ], with Ψ_1 (α) = Ψ_2 (γ) = 0 and Ψ_1 (β) = Ψ_2 (μ) = 1.

Again if max(a_i) = min(b_i), then β = μ and we can define the function $\psi(x)$ as

$$\begin{aligned}\psi(x)&=\psi_1(x) && if\ \alpha\le x\le\beta,\\ &=1-\psi_2(x) && if\ \beta\le x\le\gamma,\\ &=0 && otherwise.\end{aligned}$$

Where Ψ_1(x) being a continuous distribution function in the interval [α, β], and (1- Ψ_2(x)) being a continuous distribution function in the interval [β, γ], with Ψ_1 (α) = Ψ_2 (γ) = 0 and Ψ_1 (β) = Ψ_2 (β) = 1.

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