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Geometry of Manifolds According to Convex Constraints (Neighborhood & Co - dimenstion)

هندسة الفتحات حسب القيود المحدبة الجوار وشركاه - البعد

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Abstract:

We found that for the co – dimension $k \geq \frac{c\varepsilon^2}{log(1+\varepsilon^{-1})} log\, n$, we can estimate the diameter of the section $E \subset \mathbb{R}^n$ of the convex body $K \subset \mathbb{R}^n$, which guarantee the best clustering of the data up to strong neighborhood $E = B\left(x_0, \alpha_0 2(L_K)^2\left(\frac{c\varepsilon^2}{\log\left(1+\frac{1}{\varepsilon}\right)}\right)\log n\right)$,to decrease the loosing information of geometrical structures of the scattering data in a manifolds.

Key word: Convex body, Concentration of measure, Log – concave measure

ملخص:

وجدنا أنه لأجل البعد $k \geq \frac{c\varepsilon^2}{log(1+\varepsilon^{-1})} log\, n$ يمكننا تقدير قطر القطاع $E \subset \mathbb{R}^n$ للجسم المحدب $K \subset \mathbb{R}^n$ لضمان التقسيم الأفضل للبيانات لأجل أقوى جوار. و ذلك لتقليل فاقد المعلومات للبناءات الهندسية لمتعددات الطيات.

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1. INTRODUCTION

The problem of studying scattering data in an n- dimensional space $\mathbb{R}^n$ plays an important role in many areas of mathematics, Biology, etc. Embedding play a significant role in this area [3,4,6,7,8]. To embed means to send the information data from a space $\mathbb{R}^n$ into some other space $\mathbb{R}^N$ to exhibit their geometrical structure. The technique for that depends on the concept of neighborhood between this pointed data (cluster), and the density of points within that neighborhood. In all modeling of embedding technique, one wants to approximate the geometry of the manifold M from its point cloud ($n - point$).

Theorem: *Let T be a $k - point$ subset of an Euclidean space. Then, for any $\varepsilon > 0$, T is $(1 + \varepsilon) - Lipschitz$ embed into ℓ_2^N with $N \leq \frac{C \log k}{\varepsilon^2}$.*

The graph here is considered as empirical object (net) which constructed from sampled data to construct a geometrical shape from a manifold. For the purposes of embedding the manifold it is necessary to be compact, smooth and isometrically embedded in some Euclidean space ℓ_k^N.

The embedding involves many advantages: one of them is that it induces a measure corresponding to volume from a manifold M. Secondly it induces a vector space at any point on the considering manifold, and also it induces an inner product on the manifold.

As we mentioned above we want to send the information data, so we want to find a linear map from the manifold $\mathbb{R}^n$ to some legally Euclidean space ℓ_k^N for appropriate n such that

$$(1 - \varepsilon) \leq \|Ts\| \leq (1 + \varepsilon) \ldots (1)$$

for all points s in the manifold M .

Theorem: Let X be a metric space. Then there exists a complete metric space $\tilde{X}$and an isometric embedding $f: X \to \tilde{X}$ such that $fX \subset \tilde{X}$ is dense. The space $\tilde{X}$ is unique up to an isometry and it is called the completion of X.□

So the most important parameters to do embedding are; the neighborhood size (width), and the intrinsic dimension.

Theorem: There is a constant $c > 0$ such that for all $n \in \mathbb{N}$ and all $\varepsilon > 0$, every $n - dimensional$ normed space admits a subspace whose Banach – Mazure distance from ℓ_2^k is at most $1 + \varepsilon$ and $k > \frac{c\varepsilon}{\left(\log\left(\frac{1}{\varepsilon}\right)\right)^2} (\log n)$.

The above theorem concerns the importance of the number of point data (dimension of the received space). By the other hand the property of neighborhood also affects the accuracy of embedding. In such way we can consider it as the notion of concentration of measure to cluster the data as groups.

Definition: Let (X, d, μ) be a metric space with metric d and $diam(X) \geq 1$, which is equipped with a Borel probability measure μ. Then the concentration function on X is (isometric constant).

$$\alpha(X;\varepsilon) = 1 - \inf\{\mu(A_\varepsilon) : A\ Borel\ subset\ of\ X\ , \mu(A) \geq \frac{1}{2}\} \dots \dots (2)$$

Where $A_\varepsilon = \{x \in X : d(x, A) \leq \varepsilon\}$ is the $\varepsilon - extension$ of A. And there exists one value L_f such that ;

$$\mu\left(x \in X : \left|f(x) - L_f\right| \geq \varepsilon\right) \leq 2\alpha(X, \varepsilon) \dots \dots (3)$$

Where f is $1 - Lipschitz$ function on X.

Now we had the base theorem for embedding which bounds the dimension of the data space.

Theorem: There is a function $c(\varepsilon) > 0$ such that for all $k \leq c(\varepsilon)E(X^2)$, the space ℓ_2^k $(1 + \varepsilon) - embeds$ into X

Here $E(X^2)$ can be determined as second resolve of the random variable (point of X) which chosen identically and independently, or the variance of the element of X. More specifically for this purposes we had.

Theorem: There is a function $c(\varepsilon) > 0$ such that for all $k \leq c(\varepsilon) \log n$, ℓ_2^k $(1 + \varepsilon) - embeds$ into and normed space of dimension n .

Our paper will concern the suspension of the convex bodies in the technique of embedding as an initial space for the scattering data which belongs to some manifold M. So the first section will concern some definitions and preliminaries about the convex body to obtain its qualification tour purpose. The second section will deal with the notion of concentration of measure and its features to obtain the geometric structure for the convex body. The third section will contain the effect of the log – concave measure in the space of convex geometry. And last we had a few discussions. I apologies that our paper will concern just the theorem as we need to prove our corollaries without proof .

1- Preliminaries and Notations

By a manifold we mean a set M which is locally Euclidean of dimension n. On the other hand for the convex body we deal with an isotropic convex body $K \subset \mathbb{R}^n$, with the norm $\|.\|_K$, and we set $(K, \|.\|_K)$ as Euclidean space. A mathematician Hassler Whitney in 1936 had proved that any $n - dimensional$ differentiable manifold can be embedded in $\mathbb{R}^{2n}$.

1.1 Theorem [Whitney]: Every $n - dimensional$ differentiable manifold can be embedded in $\mathbb{R}^{2n}$.

A set M is called locally Euclidean of dimension n if every point in M has a neighborhood U such that there is a bijective φ from U into an open subset of $\mathbb{R}^n$. The pair (U, φ) is called a chart. A collection of charts are called an atlas on a manifold M which is represents the differentiable structure of a manifold. This differentiable structure is homeomorphic to a polyhedron, in other words, every differentiable manifold is triangulate. The neighborhood of a chart on M constitute a basis for a topology on M. The property of neighborhood can be determines with the notion of metric distance between points, and one of the important advantage of this property appear in the approximation.

1.2 Definition: Let X be a topological space and (Y,d) be a metric space with metric d, and let $\delta: X \to \mathbb{R}_+$ be a continuous function. We say that $g: X \to Y$ is a $\varepsilon - approximation$ of $f: X \to Y$ if

$$d\big(f(x), g(x)\big) < \varepsilon(x) \ldots\ldots (4)$$

So, the property of neighborhood will generate a plausible approximation function to carry the data information into another space. The most appropriate measure for neighborhood is the similarity (dissimilarity) function, which measure the dependency (independency) between the points of the data space. From the other hand, these functions demand a metric to measure the similar (variance) between points in the same group. And that will serve at perfect way for the convex body.

1.3 Definition: Let $x_0, \ldots, x_n$ be affinely independent points in $\mathbb{R}^n$, $m > n$, which means that $\overrightarrow{x_0x_1}, \ldots, \overrightarrow{x_0x_n}$ are linearly independent. We call the set

$$|x_0, \ldots, x_n| = \{X \in \mathbb{R}^m : \overrightarrow{OX} = \lambda_0\overrightarrow{Ox_0} + \cdots + \lambda_n\overrightarrow{Ox_n}; \lambda_0 + \cdots + \lambda_n = 1\}$$

an $n - simplex$. The n is the dimension of the simplex. The union of all simplices belonging to a set K is a polyhedron of K (simplicial complex of dimension n) denoted by

$$|K| = \bigcup_{\sigma \in K} \sigma \subset \mathbb{R}^n \ldots\ldots (5)$$

So, to exhibit the structure of any manifold we need to choose carefully a collection of affinely independent points on it (***net*** or a ***graph***), and these points will generate the faces of the polyhedron. In such a way we can deduce a tangent vectors, tangent spaces and tangent bundles. We will see that the convex set is comfortable for the purposes of embedding.

1.4 Definition: We say that a set K is convex if for any two points $x, y \in K$, the line segment $\alpha x + (1-\alpha)y \in K; \alpha \in [0,1]$, lies in K.

The geometrical features of the convex sets appear in the next definition.

1.5 Definition: Given a closed convex set K, the tangent cone at a point $x \in K$ with respect to K is the set of all directions from x to any other point in K: $T_K(x) = \{\alpha z : z = y - x, y \in K, \alpha \geq 0\}$. And the normal cone at a point $x \in K$ with respect to K is the set of normal vectors to supporting hyper plane of K at x: $\mathcal{M}_K(x) = \{z : \langle z, y - x\rangle \leq 0 \forall y \in K\}$.

1.6 Theorem: A set $K \subset \mathbb{R}^n$ is convex, if and only if all convex combinations of points in K lie in K.□

In other word A is convex iff $(\sum_{i=1}^{n} t_i x_i \subset K; \forall x_i \in K; \sum_{i=1}^{n} t_i = 1)$ and this known as convex hull.□

1.7 Corollary: *A convex set $K \subset \mathbb{R}^n$ is $n - simplex$*

The other advantage of the convex sets is that it has slices (charts) which are locally Euclidean.

1.8 Theorem: For every $\log n \leq k \leq \frac{n}{2}$, and for every convex body K in $\mathbb{R}^n$, there exist a $k - dimesional$ subspace E of $\mathbb{R}^n$ such that

$$d_{E\cap K} \leq C\sqrt{\frac{k}{\log((1+n)k)}} \dots\dots\dots (6)$$

with an absolute constant C.□

1.9 Theorem: For any convex set K and any boundary points $x_0 \in bd(K)$ there exists a supporting hyperplane for K at x_0. □

The other advantage of the convex set is that, the measure which taken for it, is legally relative to Euclidean norm.

1.10 Theorem: Let $K \subset \mathbb{R}^n$ be a convex body. For $1 \leq k \leq n$, the set of all subspaces $E \subset \mathbb{R}^n$ with $codim\ E = k - 1$, such that

$$\sqrt{k/n}\,|x| \leq 2M^*(K)\|x\|_K \quad for\ all\ x \in E \dots\dots\dots (7)$$

has measure larger than $1 - \exp(-\alpha_0 k)$, where $\alpha_0 > 0$ is an absolute constant. k is the $k - convexity$ constant of $(\mathbb{R}^n, K)$.□

The forgoing theorem gives other important features of the convex set, that is; the convex set can be divided into slices which seem to be simplex of the convex body, which determined the geometry of convex body. Secondly, the slices of a convex set can processed with the notion of concentration of measure.

1.11 Corollary: Let $K \subset \mathbb{R}^n$ be a convex body. For $1 \leq k \leq n$, the set of all subspaces $E \subset \mathbb{R}^n$ with $codim\ E = k - 1$, such that Equation (7) above proved . Then,

$$\mu(E) \leq 2\,exp(-\alpha_0 k) \dots\dots (8)$$

where $\alpha_0 > 0$ is an absolute constant

1.12 Theorem: There is a positive number c such that, for any $\varepsilon > 0$ and every natural number n, every symmetric convex body of dimension n has a slice of dimension

$$k \geq \frac{c\varepsilon^2}{\log(1+\varepsilon^{-1})}\log n \dots\dots\dots (9)$$

that is with in distance $(1+\varepsilon)$ of the $k - dimensional$ Euclidean ball.□

1.13 Corollary: Up to Theorem (1.11), every slices $E \subseteq \mathbb{R}^n$ of a convex body of dimension $k \geq \frac{c\varepsilon^2}{log(1+\varepsilon^{-1})} log\, n$ is $(1+\varepsilon) - embeds$ into a normed space of dimension k,and up to Equations (8-9) we have:

$$\mu(E) \leq 2\, exp\left(-\alpha_0 \left(\frac{c\varepsilon^2}{log\left(1+\frac{1}{\varepsilon}\right)}\right) log\, n\right) \ldots \ldots (10)$$

1.14 Corollary: Let $K \subset \mathbb{R}^n$ be a convex body such that D is the ellipsoid of minimal volume containing K, for $\varepsilon \in (0{,}1)$, let $k = [(1-\varepsilon)n]$. Then there exist an orthogonal projection P with $rank\, P = k$ and a subspace $E \subset \mathbb{R}^n$ with $\dim E = k$ such that the Banach – Mazur distance satisfies

$$\max\left(d(PK, \beta_1^k), d(K \cap E, \beta_\infty^k)\right) \leq \left(\frac{C}{\varepsilon^4}\right)\sqrt{n} \ldots \ldots \ldots (11)$$

where C is an absolute constant and β_q^k denotes the unite ball of ℓ_q^k.□

As we deal with the concentration of measure, Dvortzky's theorem state that for any convex body $K \subset \mathbb{R}^n$ and $\varepsilon > 0$, there exist a subspace $E \subset \mathbb{R}^n$ of dimension at least $c\varepsilon^2 \log n$ with $\{(1-\varepsilon)D \subseteq proj_E(K) \subseteq (1+\varepsilon)D\}$, where $D$is some Euclidean ball in the subspace E.

Up to concentration of measure the convex set will generate a graph of points which built the structure of the polyhedron.

1.15 Lemma: Let $K \subset \mathbb{R}^n$ be a compact convex body such that D is the ellipsoid of minimal volume containing K. Let $\varepsilon \in (0{,}1)$. Then there exist $k \geq (1-\varepsilon)n$ and contact points $x_1, x_2, \ldots, x_k$ of K and D such that

$$dis\left(x_j, span\,\{x_i : i \neq j, 1 \leq i \leq k\}\right) \geq \sqrt{\varepsilon} \ldots \ldots \ldots (12)$$

For $j = 1, \ldots, k$.□

2- Convex Bodies and Concentration of measure Phenomenon:

The convex set (body) in high dimension convex geometry is a body K which consists of bulk and outliers. If K is properly scaled, the bulk usually looks like a Euclidean ball. For a proper scaling we had to deal with the notion of concentration of measure. This assumption demands that the convex body had to be isotropic, which means that the random points in K must distributed uniformly in K according to Lebesgue measure with zero mean and identity covariance. For isotropic assumption we had to create a plausible invertible linear transformation, and with this scaling we guarantee that most of volume of the convex body K is located around Euclidean ball.

2.1 Theorem: (Distribution of volume in high – dimensional convex sets)

Let K be isotropic convex body in $\mathbb{R}^n$, and let X be a random vector uniformly distributed in K. Then the following is true:

i- (concentration of volume): For any $t \geq 1$, one has

$$P\{\|X\|_2 > t\sqrt{n}\} \leq \exp(-ct\sqrt{n}) \dots \dots (13)$$

ii- (thin shell): For any $\varepsilon \in (0,1)$, one has

$$P\{\left|\|X\|_2 - \sqrt{n}\right| > \varepsilon\sqrt{n}\} \leq c \exp(-c\varepsilon^2\sqrt{n}) \dots \dots (14)$$

where $\sqrt{n}$ stands for the radius of the unite ball in the Euclidean space

Up to the concept of concentration of measure for the isotropic convex body K, it helps to exhibit the construction of the random section of the convex body K. And that appear in Corollary (1.12) and Lemma (1.14)above.

2.2 Theorem (Dvortzy's Theorem): Let K be an origin – symmetric convex body in $\mathbb{R}^n$ such that the ellipsoid of maximal volume contained in K is the unit Euclidean ball B_2^n. Fix $\varepsilon \in (0,1)$. Let E be random subspace of dimension $d = c\varepsilon^{-2} \log n$ drown from the Grassmanian $G_{n,d}$ according to the Haar measure, then there exist $R \geq 0$ such that with high probability (say, 0.99) we have:

$$(1 - \varepsilon)B(R) \subseteq K \cap E \subseteq (1 + \varepsilon)B(R) \dots \dots (15)$$

The quantity R which appear in Equation (15) refer to as the radius of the ball within the convex body K. To more specific this quantity in high – dimensional space we deal with new criteria which is called the mean width of K denoted by $M(K)$ and it place in the same category as volume and surface area.

2.3 Proposition: The mean width is invariant under translations, orthogonal transformations and taking convex hull.

The mean width of the convex body behaves like the distance function and it's appropriate to deal with the convex geometry area.

2.4 Definition: Let $K \subset \mathbb{R}^n$ be non – empty and convex. The support function $h_K: \mathbb{R}^n \to (-\infty, \infty)$ of K is defined as

$$h_K = \sup_{x \in K}\langle x, u\rangle , u \in \mathbb{R}^n \dots \dots \dots (16)$$

which fail to be as a distance function.□

Another definition for the mean width in the manner of linear function

2.5 Definition : The global mean width of a subset $K \subset \mathbb{R}^n$ is defined as

$$\omega(K) = E \sup_{x,y \in K} \langle g, x - y\rangle \dots \dots \dots (17)$$

where $g_i \sim N(0, I_n)$. The local mean width of a subset $K \subset \mathbb{R}^n$ is a function of scale $t \geq 0$, and is defined as

$$\omega(K) = E \sup_{x,y \in K;\ \|x-y\|_2 \leq t} \langle g, x - y \rangle \ldots \ldots \ldots (18)$$

To get more specific on the mean width, how to compute it and what its advantages, we have,

2.6 Theorem: Let K be a symmetric convex body such that $K \subset B_2^n$. Define

$$M(K) = \int_{S^{n-1}} b_K(\theta) d\sigma(\theta) \ldots \ldots (19)$$

where b_K is the support function of K. Then for all $\varepsilon > 0$, there exists a vector subspace E of $\mathbb{R}^n$ of dimension $k = k(K) = \left[\frac{cn(M(K))^2 \varepsilon^2}{\ln\left(\frac{1}{\varepsilon}\right)}\right]$, such that:

$$(1 - \varepsilon)M(K)P_E B_2^n \subset P_E K \subset (1 + \varepsilon)M(K)P_E B_2^n \ldots \ldots (20)$$

***2.7 Corollary :** The mean width (17) & (18) fail to be as distance in such way that:*

$$\boldsymbol{h_K \propto \omega(K) = M(K) = \int_{S^{n-1}} |sup_{x \in K} \langle x, \theta \rangle| d\sigma(\theta) \ldots (21)} \square$$

***where,** $\boldsymbol{\theta \in S^{n-1}}$*

Another advantage of the concentration of measure is that it generate the neighborhood property, and according to Corollary (1.10) ,Theorem (1.11) and Lemma (1.14) we had the following corollary

***2.8 Corollary:** Let $\boldsymbol{K \subset \mathbb{R}^n}$ be a compact convex body such that $\boldsymbol{B_2^n}$ is the ellipsoid of minimal volume containing $\boldsymbol{K}$. Let $\boldsymbol{\varepsilon \in (0, 1)}$ with $\boldsymbol{k \geq \frac{c\varepsilon^2}{log(1+\varepsilon^{-1})} log\, n}$ and let $\boldsymbol{x_1, \ldots, x_k}$ be a contact points of $\boldsymbol{K}$ and $\boldsymbol{B_2^n}$ up to concentration of volume of $\boldsymbol{K}$ in $\boldsymbol{B_2^n}$, such that:*

$$\boldsymbol{dis\left(x_j, span\{x_i : i \neq j, 1 \leq i \leq k\}\right) \geq \sqrt{R(B_2^n)} \ldots \ldots \ldots (22)}$$

***then, for every** $\boldsymbol{vert(K) = \{x_i\}_{i=1}^k}$ **and** $\boldsymbol{F = conv\{x_i : x_i \in F\}}$**, then each** $\boldsymbol{x_i \in K \cap B_2^n}$ **will be** $\boldsymbol{0 - face}$ **of** $\boldsymbol{K}$**, and every two distinct points** $\boldsymbol{\{(x_i, x_j) : i \neq j\}}$ **will be** $\boldsymbol{1 - face}$**(edge). Where** $\boldsymbol{R(B_2^n)}$ **stand for the radius of the Euclidean ball** $\boldsymbol{B_2^n}$**.** □*

3- Convex bodies and log – concave measure

The level of concentration is determined with respect to the class of linear functional by measuring the size of minimal well – distributed substructures. And these substructures should exhibit a high level of concentration, and at the same time, they should represent the original space in an essential way. All these hypotheses is compatible with the log – concave Borel probability measure μ on $\mathbb{R}^n$.

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3.1 Definition: We say that the measure μ on $\mathbb{R}^n$ is log – concave if: $\mu(\lambda A + (1-\lambda)B \geq \mu(A)^{\lambda}\mu(B)^{1-\lambda}$, for every $A, B \in \mathbb{R}^n$.

The log – concave measure μ is isotropic if : $\int_{\mathbb{R}^n} \langle x, y\rangle^2 \mu d(x) = L_\mu^2$ for every $y \in S^{n-1}$. If we fix $\varepsilon \in (0,1)$ then, $k \geq \frac{c\varepsilon^2}{log(1+\varepsilon^{-1})} log\, n$ points which chosen uniformly and independently from a convex body K with centroid at the origin in $\mathbb{R}^n$ satisfy with probability greater that $1-\delta$

$$A = Co\{x_1, \dots, x_k\} \supseteq c(\delta)\varepsilon K \dots\dots (23)$$

Where, $Co\{x_1, \dots, x_k\}$ stands for the convex hull.

that is, any exponentially ***number*** of random points from a convex body K creates a body which represent K in the distance sense (***neighborhood***).

3.2 Lemma: There exist absolute constants $c_1, c_2 > 0$ such that:

$$c_1 L_\mu \leq \left(\int |\langle x, y\rangle|^p \mu d(x)\right)^{\frac{1}{p}} \leq c_2 C_\alpha \max\left\{1, p^{\frac{1}{\alpha}}\right\} L_\mu \dots\dots (24)$$

for every $p > 0$ and $y \in S^{n-1}$.

3.3 Corollary: Let $K \subseteq \mathbb{R}^n$ be a compact convex body such that B_2^n is the ellipsoid of minimal volume containing K. Let $\varepsilon \in (0,1)$ with $k \geq \frac{c\varepsilon^2}{log(1+\varepsilon^{-1})} log\, n$ and let $x_1, \dots, x_k$ be a contact points of K and B_2^n up to concentration of volume of K in B_2^n, then and with Equation (24) we had:

$$c_1 L_\mu \leq \left(\int |\langle x, y\rangle|^2 \mu d(x)\right)^{\frac{1}{2}} \leq c_2 C_\alpha\, max\left\{1, 2^{\frac{1}{\alpha}}\right\} L_\mu \dots\dots (25)$$

for every $y \in S^{n-1}$.□

Which, in the case of $p = 2$ we can replace c_1, c_2 with $(1-\varepsilon), (1+\varepsilon)$ respectively . Beside of that, for an isotropic convex body also we can replace L_μ with the convex constant L_K

3.4 Corollary: If the log – concave measure is isotropic, then and with Equation (24) we had:

$$(1-\varepsilon) L_K \leq \left(\int |\langle x, y\rangle|^2 \mu d(x)\right)^{\frac{1}{2}} \leq (1+\varepsilon) C_\alpha\, max\left\{1, 2^{\frac{1}{\alpha}}\right\} L_K \dots\dots (26)$$

for every $y \in S^{n-1}$, then :

$$(1-\varepsilon)(L_K)^2 \leq \left(\int |\langle x, y\rangle|^2 \mu d(x)\right) \leq (1+\varepsilon) C_\alpha\, max\left\{1, 2^{\frac{1}{\alpha}}\right\} (L_K)^2 \dots\dots (27)$$

And up to Definition (2.5), Theorem (2.6) and Corollary (2.7) we had

$$\left|\left(\int |\langle x, y\rangle|^2 \mu d(x)\right)\right| \leq C(L_K)^2 \ldots\ldots (28)$$

for every $y \in S^{n-1}$. Where C is a universal constant. □

3.5 Theorem: Every convex body K creats a log – concave measure L_K, and a random set of $\exp \varepsilon n$ points chosen from K creates a body equivalent to K and at the same time, form a $\psi - distribution$ for μ_K

Up to the above theorem we need a plausible distribution function, and as we ex-mention that the supports function of the convex body should be serving as a distance function.

3.6 Definition: A random vector X is called log – concave, if its density has the form

$$f_X(y) = \exp(-u(y)) \ldots\ldots (29)$$

where $u: \mathbb{R}^n \to \mathbb{R} \cup (+\infty)$ is a convex function.

3.7 Theorem: Let $n \geq 1$ be an integer and let X be a random vector in $\mathbb{R}^n$ with an isotropic, log – concave density. Then there exists $\Phi \subset S^{n-1}$ with $\sigma_{n-1}(\Phi) \geq 1 - C\exp(-\sqrt{n})$ such that for all $\theta \in \Phi$, the real – valued random variable $\langle X, \theta\rangle$ has a density $f_\theta: \mathbb{R}^n \to [0, \infty)$ with the following properties:

i- $\int_{-\infty}^{\infty} |f_\theta(t) - \gamma(t)| dt \leq \frac{1}{n^k}$

ii- For all $|t| \leq n^\delta$ we have $\left|\frac{f_\theta(t)}{\gamma(t)} - 1\right| \leq \frac{1}{n^\delta}$

here, $C, k > 0$ are universal constants, and($\left(\gamma(t) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) (t \in \mathbb{R})\right)$ is the standard Gaussian density).□

The quantity $\sqrt{n}$ in the ex-theorem stands for the radius of the maximal neighborhood $\Phi \subset S^{n-1}$ which contains the random vector X.

3.8 Proposition: If X is a normally distributed random variable with mean EX and variance σ^2, then X concentrated around a constant, namely its mean, in the sense that, for every $u \geq 0$

$$P\{|X - EX| \geq u\} \leq \exp\left\{-\frac{u^2}{2\sigma^2}\right\} \ldots\ldots (30)$$

Taking $c = 1$ and $v = \sigma^2$ we have $\mu(A_r) \geq 1 - C\exp\left\{-\frac{r^2}{2v}\right\}$

In the above proposition the quantity r stands for the width of the interval which contains X.

As it mention in the forgoing theorem the property of isotropic stands with zero mean and identity covariance, and if we rethink of the value $(L_K)^2$ as the variance between the points in the same convex body, in such a way it scales the mount of convexity around origin point (Apostolos Giannopoulos and others in their paper [1] mention that ($L_K \gtrsim 1$)). So up to the Gaussian density function we can rewrite Theorem (3.7) with a new assumption, then we have

Corollary: Let $K \subseteq \mathbb{R}^n$ be an isotropic convex body and let $E \subseteq \mathbb{R}^n$ be a section of K with co- dimension $k \geq \frac{c\varepsilon^2}{log(1+\varepsilon^{-1})} log\, n$. Set $n \geq 1$ be an integer and let X be a random vector in $\mathbb{R}^n$ which uniformly distributed according to a log – concave density with mean zero and variance L_K (the convex constant), and up to concentration of Gaussian measure, then for every $X \in E$:

$$P\{|X| \geq r\} \leq exp\left\{-\frac{r^2}{2(L_k)^2}\right\} \dots \dots (31)$$

And with the consideration $X \sim (0, (L_K)^2)$, then the appropriate density function for the convex body has the form:

$$f_X(X) = \frac{1}{L_K\sqrt{2\pi}} exp\left\{-\frac{r^2}{2(L_K)^2}\right\} \dots \dots (32)$$

Such that $\mu(E) \geq 1 - 2\, exp\left\{-\frac{r^2}{2(L_K)^2}\right\}$. And that with Equation (10)we set

$$max_{x_i \in E}|x_0 - x_i| = (x_0 - x_i)^2 = \alpha_0 2(L_K)^2 \left(\frac{c\varepsilon^2}{log\left(1 + \frac{1}{\varepsilon}\right)}\right) log\, n \dots \dots (33)$$

Where, $i = 1, \dots, k$ and $(x_0 - x_i)^2 = r^2$. So we can think of E as a ball in the formula

$$E = B\left(x_0, \alpha_0 2(L_K)^2 \left(\frac{c\varepsilon^2}{log\left(1 + \frac{1}{\varepsilon}\right)}\right) log\, n\right) \dots \dots (34)\square$$

Klartag [68] in his paper find that with concentration up to probability

3.10 Theorem (Klartag): suppose $K \subset \mathbb{R}^n$ is convex and isotropic, and X is distributed uniformly in K. Then $\exists \Phi \subset S^{n-1}$ with $\sigma_{n-1}(\Phi) \geq 1 - \delta_n$, such that for $\theta \in \Phi$,

$$\sup_{A \subset \mathbb{R}} \left| Prop\,\{\langle X, \theta\rangle \in A\} - \frac{1}{L_K\sqrt{2\pi}} \int_A e^{-t^2/2L_K^2} dt \right| \leq \varepsilon_n \dots \dots \dots (35)$$

Here, say, $\delta_n < \exp(-cn^{0.9})$, $\varepsilon_n < Cn^{-\frac{1}{100}}$.□

Many authors was treated to find the appropriate initially point x_0, but

4- **Discussion :** As we has mentioned above the property of neighborhood plays an important role in the space of estimating and that depends originally at the number of elements point in that neighborhood. And for that purposes we had to

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cluster with a strong distribution function which guarantee that, the similar point with in the same group and dissimilar point get out with high accuracy to decrease the loose of information from scattering data which considered as a manifold in $n - dimansion$.

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