Fibrewise Separation axioms in Fibrewise Topological Group

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Abstract: In this paper we will introduce and study the notion of fibrewise separation axioms in fibrewise topological group and show that fibrewise T_2 space $\Rightarrow$ fibrewise T_1 space $\Rightarrow$ fibrewise T_0 space.

1. Introduction

The fibrewise viewpoint is standard in the theory of fibre bundles, however, it has been recognized relatively recently that the same viewpoint is also of as important in other areas such as general topology. A fibrewise topological space over B is just a topological space X together with a continuous function $p: X \rightarrow B$ called projection. Most of the results obtained so far in this field can be found in James [4] (1984) and James [5] (1989). Our aim in this paper is to study the fibrewise separation axioms in fibrewise topological group. We study many properties and obtained some new results. Also we investigate some important theorems and properties of fibrewise separation axioms in fibrewise topological group*s* , especially for the fibre G_{e_B} over the identity element e_B of B .

2. Preliminaries

Throughout this section we give the basic concepts and notations which we shall use in this paper:

2.1. Fibrewise topological space [5]

Definition 2.1.1: Let B be any set. Then a fibrewise set over B consists of a set X together with a function $p: X \rightarrow B$, called the projection, where B is called a base set.

For each $b \in B$, the fibre over b is the subset $X_b = p^{-1}(b)$ of X. Also for each subset W of B, we regard $X_W = p^{-1}(W)$ is a fibrewise set over W with the projection determined by p.

Proposition 2.1.2: Let X be a fibrewise set over B, with projection p. Then Y is fibrewise set over B with projection $p\alpha$ for each set Y and function $\alpha : Y \rightarrow X$.

In particular X' is a fibrewise set over B with projection $P/_{X'}$ for each subset X' of X. Also X is fibrewise set over B' with projection βp for each superset B' of B and function $\beta : B \rightarrow B'$.

Definition 2.1.3: If X and Y are fibrewise sets over B, with projections p and q respectively, a function $\varphi : X \rightarrow Y$ is said to be fibrewise function if $q\varphi = p$, in other words $\varphi(X_b) \subseteq Y_b$ for each $b \in B$.

Definition 2.1.4: Let $\{ X_r\}$ be an index family of fibrewise sets over B. Then the fibrewise product $\prod_B X_r$ is defined, as a fibrewise set over B, and comes equipped with the family of fibrewise projections $\pi_r : \prod_B X_r \rightarrow X_r$. Specifically the fibrewise product is defined as the subset of the ordinary product $\prod X_r$, in which the fibres are the products of the corresponding fibres of the factors X_r.

Definition 2.1.5: Let B be a topological space. Then a fibrewise topology on a fibrewise set X over B is any topology on X for which the projection p is continuous.

A fibrewise topological space over the space B is defined to be a fibrewise set over B with fibrewise topology.

The coarsest fibrewise topology on a fibrewise set X over B is the topology induced by p, in which the open sets of X are precisely the inverse images of the open sets of the B, this is called the fibrewise indiscrete topology, and the discrete topology on a fibrewise set X over B is called fibrewise discrete.

Definition 2.1.6 : The fibrewise topological space X over B is called fibrewise closed (fibrewise open) if the projection p is closed (open).

Definition 2.1.7: Let X be a fibrewise topological space over B. If $x \in X_b$, where $b \in B$,then the family Γ of neighborhoods of $x \in X$ is fibrewise basic if for each neighborhood U of x, there exists a neighborhood W of b in B such that $X_W \cap V \subseteq U$, fo some member V of Γ.

Definition 2.1.8: Let X be a fibrewise topological space over B. Then:

i. X is fibrewise $T_0(T_1)$ if whenever $x, y \in X_b$, where $b \in B$, and $x \neq y$, either there exist a neighborhood of x which does not contain y, or there exists a neighborhood of y which does not contain x (there exists a neighborhood of x which does not contain y and there exists a neighborhood of y which does not contain x).

ii. X is fibrewise Hausdorff(T_2) if whenever $x, y \in X_b$, where $b \in B$, and $x \neq y$, there exists disjoint neighborhoods $V , U$of x, y , respectively, in X.

iii. X is fibrewise R_0 if for each point $x \in X_b$, where $b \in B$, and each neighborhood Vof x in X, there exists a neighborhood W of b in B such that $X_W \cap \overline{\{x\}} \subset V$.

iv. X is fibrewise functionally Hausdorff if whenever $x, y \in X_b$, where $b \in B$, and $x \neq y$, there exists a neighborhood W of b in B and a continuous function $\alpha : X_W \to I$ such that $\alpha(x) = 0$ and $\alpha(y) = 1$.

v. X is fibrewise regular if for each point $x \in X_b$, where $b \in B$, and each neighborhood Vof x in X, there exists neighborhood W of b in B and a neighborhood U of x in X_W such that $X_W \cap \overline{U} \subseteq V$. A fibrewise regular and fibrewise T_0 is called fibrewise T_3.

vi. X is fibrewise completely regular if for each $x \in X_b$, where $b \in B$, and for each neighborhood Vof x in X, there exists neighborhood W of b in B and a continuous function $\alpha : X_W \to I$ such that $\alpha(x) = 1$ and $\alpha(x) = 0$ for all x away fromV.

vii. X is fibrewise normal if for each point b of B and for each pair H, K of disjoint closed sets of X, there exists a neighborhood W of b in B and a pair U , V of disjoint neighborhoods of $X_W \cap H$, $X_W \cap K$ in X_W .

viii. X is fibrewise functionally normal if for each point b of B and for each pair H, K of disjoint closed sets of X, there exists a neighborhood W of b in B and a continuous function $\alpha : X_W \to I$ such that $\alpha = 0$ throughout H_W and $\alpha = 1$ throughout K_W.

2.2. Topological group, Fibrewise Group and fibrewise topological group

Definition 2.2.1[3]: A topological group G is a group which is also a topological space on G such that the maps $g \to g^{-1}$ and $(g, h) \to gh$ are continuous.

Theorem 2.2.2[3]: A group G endowed with any topology, is a topological group if and only if, the mapping $(g, h) \to gh^{-1}$ is continuous.

Theorem 2.2.3[3]: Let a be a fixed element of a topological group G, then $r_a : g \to ga$ and $I_a : g \to ag$ of G onto G are homeomorphisms of G.

Corollary 2.2.4[3]: Let F be a closed set, E be an open set, A be any subset of a topological group G and $a \in G$. Then aF, Fa, F^{-1} are closed sets, aE, Ea, E^{-1}, AE, EA are all open sets.

Proposition2.2.5[3]: For each neighborhood U of the identity e in a topological group G there exists a symmetric neighborhood V of e such that $VV \subset U$.

Corollary 2.2.6[3]: Let U be any neighborhood of the identity e in a topological group G. Then there is a neighborhood V of e such that $\overline{V} \subset U$. And this is true at each $g \in G$.

Theorem 2.2.7[3]: Let G be a topological group, let e denoted the identity in G, and let F be a closed subset of G such that $e \notin F$. Then there is a continuous function $f: G \to [0,1]$ such that $f(e) = 0$ and $f(x) = 1$ for every $x \in F$.

Definition 2.2.8[10]: Let B be a group. A fibrewise group over B is a fibrewise set G with any binary operation makes G a group such that the projection $p : G \to B$ is homomorphism.

Definition 2.2.9[10]: Let G be a fibrewise group over B. Then any subgroup H of G is a fibrewise group over B with projection $p_{/H}: H \to B$, we call this group a fibrewise subgroup of G over B.

Definition2.2.10[10]: Let G and K be two fibrewise groups over B. Then any homomorphism $\varphi : G \to K$ is called a fibrewise homomorphism if φ is a fibrewise map.

Definition2.2.11[10]: A bijective fibrewise homomorphism is called a fibrewise isomorphism.

Theorem2.2.12[10]: Let G be a fibrewise group over B with projection p and H be a fibrewise normal subgroup of G. Then G/H is fibrewise group over B, with projection $q : G/H \to B$ such that $q\pi = p$.

Theorem2.2.13[10]: let $\varphi : G \to K$ be a fibrewise function, where G and K are fibrewise groups over B, with p, q respectively. Then:

1. If q is injective then φ is a fibrewise homomorphism, and consequently:

i.) $\varphi(e_G) = e_K$, where e_G, e_K denotes the identities of G, K respective.
ii) $\varphi(\ker(P)) = e_K$.
iii) If H is fibrewise subgroup of G, then $\varphi(H)$ is fibrewise subgroup of K.
iv) If H' is fibrewise subgroup of K , then $\varphi^{-1}(H')$ is fibrewise subgroup of G.
v) If H is fibrewise normal subgroup of G, then $\varphi(H)$ is fibrewise normal subgroup of K.

2. If p is bijective and q is injective then if G is abelian then K is abelian.
3. If q is bijective and p is surjective then if G is cyclic then K is cyclic.
4. If p, q are bijective then φ is fibrewise isomorphism.

Definition 2.2.14[11]: A fibrewise topological group G is a fibrewise group endowed with fibrewise topology such that the mapping $g \to g^{-1}$ of G onto G and $(g, h) \to gh$ of $G \times G$ onto G are fibrewise continuous maps.

Proposition 2.2.15[11]: Let G be a fibrewise topological group over B. Then G_{B^*} is fibrewise topological group over B^* for each subgroup B^* of B.

Proposition2.2.16[11]: Let G be a fibrewise topological group over B with projection p and H be a fibrewise normal subgroup of G. Then the quotient group G/H is a fibrewise topological group with projection $q: G/H \to B$ such that $q\pi = p$.

3. Fibrewise Separation Axioms

In fibrewise topology X, if X is fibrewise T_2 then X is fibrewise T_1, but the converse does not hold in general, however If G is a fibrewise topological group we will prove in this section, that the converse is true.

Theorem 3.1: Let G be a fibrewise topological group over B. G is fibrewise Hausdorff (fibrewise T_1, fibrewise T_0) if and only if G_{e_B} is Hausdorff (fibrewise T_1, fibrewise T_0).

Proof:

Frist, if G is fibrewise Hausdorff then from the definition the fibre G_{e_B} is Hausdorff

Second, let $b \in B$ and $x, y \in G_b$: $x \neq y \Rightarrow xy^{-1} \neq yy^{-1} = e_G$ and $x, y \in G_b \Rightarrow p(x) = p(y) = b \Rightarrow p(x)(p(y))^{-1} = e_B \Rightarrow p(x)p(y^{-1}) = e_B \Rightarrow p(xy^{-1}) = e_B \Rightarrow xy^{-1} \in G_{e_B}$ but $e_G \in G_{e_B}$ and $xy^{-1} \neq e_G$. Since G_{e_B} is Hausdorff then there exist open sets U, V such that $xy^{-1} \in U$, $e_G \in V$ and $U \cap V = \emptyset$. now $xy^{-1} \in U \Rightarrow x \in Uy$ and $y \in Vy$, where Uy, Vy open sets and to show that $Uy \cap Vy = \emptyset$, suppose $Uy \cap Vy \neq \emptyset$, this is implies there exist an element $a \in (Uy \cap Vy) \Rightarrow \exists r_1 \in U, r_2 \in V$ such that $a = r_1 y = r_2 y \Rightarrow r_1 = r_2$, then $r_1 \in U \cap V$ but $U \cap V = \emptyset$ and this is a contradiction, hence $Uy \cap Vy = \emptyset$,, thus G is fibrewise Hausdorff. Similarly, we can prove the case of fibrewise T_1 and fibrewise T_0.

The following results prove the converse : "If a fibrewise topological group is fibrewise T_0 then it is fibrewise T_1" (If a fibrewise topological group is fibrewise T_0 then it is fibrewise T_1)

Proposition 3.3: Let G be a fibrewise topological group over B. If G is fibrewise T_0 then G is fibrewise T_1.

Proof:

Let G be a fibrewise T_0 and for $b \in B$ let $x, y \in G_b$, $x \neq y$ then $xy^{-1} \neq e_G$ and $xy^{-1} \in G_{e_B}$ but $e_G \in G_{e_B}$ since G is fibrewise T_0, then there exist open set U of G contains $e_G$and does not contain xy^{-1} from Proposition 2.2.5 $\Rightarrow$ exist open symmetric neighborhood V of e_G such that $VV \subseteq U$, then Vx is open and contains x but does not contain y, and Vy is open and contain y but does not contain x. Where if $y \in Vx$ then exist $v_1 \in V$ such that $y = v_1 x \Rightarrow xy^{-1} = {v_1}^{-1} \in V^{-1} = V \subseteq U$. This is contradiction.

And, if $x \in Vy$ then exist $v_2 \in V$ such that $x = v_2 y \Rightarrow xy^{-1} = v_2 \in V \subseteq U$. This is contradiction. Then G is fibrewise T_1.

Proposition 3.4 : Let G be a fibrewise topological group over B. If G is fibrewise T_1 then G is fibrewise T_2.

Proof:

Let G be a fibrewise T_1 and any $b \in B$ let $x, y \in G_b : x \neq y$ then $xy^{-1} \neq e_G$ and $xy^{-1} \in G_{e_B}$ but $e_G \in G_{e_B}$ since G is fibrewise T_1, then there exist open set U of G contains e_G and does not contain xy^{-1} from Proposition 2.2.5 $\Rightarrow$ exist open symmetric neighborhood V of e_G such that $VV \subseteq U$, then Vx , Vy are open sets contains x and y respectively and $Vx \cap Vy = \emptyset$. Where if $Vx \cap Vy \neq \emptyset$ then there exist an element $r \in (Vx \cap Vy)$ and there exist two elements v_1, v_2 in V such that $r = v_1 x = v_2 y$ hence $xy^{-1} = {v_1}^{-1} v_2 \in V^{-1}V = VV \subseteq U$. This is a contradiction, then G is fibrewise T_2.

Proposition 3.5: Let G and K be fibrewise topological groups over B. Let $\varphi: G \to K$ be a continuous fibrewise homomorphism and let the kernel($\varphi) = \{ e_G\}$. Then if K is fibrewise T_0(fibrewise T_1, fibrewise T_2) then G is so.

Proof:

Let K be a fibrewise T_o and any $b \in B$ let $x, y \in G_b : x \neq y$ then $xy^{-1} \neq e_G$ and $\varphi(xy^{-1}) \neq e_k$ this is implies $\varphi(x)\varphi(y^{-1}) \neq e_k$, hence $\varphi(x) \neq \varphi(y)$, since K is fibrewise T_0 and $\varphi(x), \varphi(y) \in K_b$ then there exist a neighborhood V of $\varphi(x)$ in K which does not contain $\varphi(y)$ or vice versa, then $\varphi^{-1}(V)$ is neighborhood of x in G which does not contain y. The proof is similar for the cases if fibrewise T_1 and fibrewise T_2.

Proposition 3.6 : Let G be a fibrewise Hausdorff over B. Then $G_{B'}$ is fibrewise Hausdorff over B' for each subgroup B' of B.

Proof:

Let B' be any subgroup of B and any $b' \in B'$ let $, y \in G_{b'} : x \neq y$ since $b' \in B' \subseteq B$ and G is fibrewise Hausdorff then there exist disjoint neighborhoods U, V of x, y in G. let $U' = U \cap G_{b'}$,
$V' = V \cap G_{b'}$, then U', V' are disjoint neighborhoods of x, y in $G_{B'}$ this is implies $G_{B'}$ is fibrewise Hausdorff.

Theorem 3.7 : Any fibrewise topological group is fibrewise regular.

Proof:

Let G be a fibrewise topological group over B and any b∈B let $x \in G_b$ and W be neighborhood of b, then from Corollary2.2.6 any neighborhood U of x there exist neighborhood V of x such that $\overline{V} \subseteq U$, then $G_W \cap \overline{V} \subseteq \overline{V} \subseteq U$. Hence G is fibrewise regular.

Theorem 3.8 : Any fibrewise topological group is fibrewiseR_o.

Proof:

Let G be a fibrewise topological group over B and any b∈B let $x \in G_b$, then any neighborhood U of x in G there exist neighborhood V of x in G such that $\overline{V} \subseteq U$ from Corollary2.2.6. Hence any neighborhood W of b in B is $G_W \cap \overline{\{x\}} \subseteq G_W \cap \overline{V} \subseteq \overline{V} \subseteq U$. This implies G is fibrewiseR_o.

Corollary 3.9: If G is fibrewise T_2 then G is fibrewise T_3.

Proof:

Let G be fibrewise T_2, then G is fibrewiseT_0 and from Theorem 3.8 G is fibrewiseR_o. Hence G is fibrewise T_3.

Theorem 3.10 : If G is a fibrewise topological group over B, which is fibrewise T_1, then G is fibrewise completely regular.

Proof:

Let G be a fibrewise T_1 and any b∈B let $x \in G_b$ and F be a closed set of G such that $x \notin F$. Then $x^{-1}F$ is closed set of G not containing e_G and from Theorem2.2.7 there is a continuous function $f: G \to I$ such that $f(e_G) = 0$ and $f(y) = 1$ for $y \in x^{-1}F$. Now, the function $\alpha(g) = f(x^{-1}g)$, $g \in G$ is continuous from G to I, then any neighborhood W of b, the restricted $\alpha_{G_W}: G_W \to I$ is continuous and $\alpha_{G_W}(x) = f(e_G) = 0$ and$\alpha_{G_W}(x') = f(x^{-1}x') = 1$, for $x' \in F \cap G_W \subseteq F$. Hence G is fibrewise completely regular.

Proposition 3.11 : A closed fibrewise subgroup of fibrewise normal space is fibrewise normal.

Proof:

Let G be a fibrewise normal space and let H be a closed fibrewise subgroup of G. Let E, F be disjoint closed sets of H and b∈B, then E, F are disjoint closed sets of G. Since G is fibrewise normal then there exists a neighborhood W of b in B and two disjoint neighborhoods U, V of

$G_W \cap E$, $G_W \cap F$ in G_W. Let $U' = U \cap H_W$, $V' = V \cap H_W$ where $H_W = G_W \cap H$. Then U', V' are disjoint neighborhoods of $H_W \cap E$, $H_W \cap F$ in H_W. Hence H is fibrewise normal space.

Proposition 3.12: Let G be a fibrewise topological group over B. If G is fibrewise Hausdorff then G is fibrewise functionally Hausdorff.

Proof:

Let G be a Hausdorff and any b∈B let $x, y \in G_b$ $x \neq y$ then $xy^{-1} \neq e_G$ and $xy^{-1} \in G_{e_B}$ but $e_G \in G_{e_B}$ since G is fibrewise Hausdorff then there exist two disjoint open sets U, V such that $xy^{-1}U$, $e_G \in V$ then V^c is closed and does not contain e_G from Theorem2.2.7 there exist a continuous function $f: G \to I$ such that $f(e_G) = 0, f(g) = 1$ for $g \in V^c$ and $xy^{-1} \in V^c \Rightarrow f(xy^{-1}) = 1$. And $\alpha(h) = f(hy^{-1})$ is continuous function from G to I and $\alpha(y) = f(e_G) = 0$, $\alpha(x) = f(xy^{-1}) = 1$ and any nbd W of b the restricted function $\alpha_{G_W}: G_W \to I$ is continuous and $\alpha_{G_W}(y) = 0, \alpha_{G_W}(x) = 1$. Hence G is fibrewise functionally Hausdorff.

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