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General 2 × 2 system of nonlinear integral equations and its approximate solution

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ABSTRACT

In this note, we consider a general 2 × 2 system of nonlinear Volterra type integral equations. The modified Newton method (modified NM) is used to reduce the nonlinear problems into 2 × 2 linear system of algebraic integral equations of Volterra type. The latter equation is solved by discretization method. Nystrom method with Gauss–Legendre quadrature is applied for the kernel integrals and Newton forwarded interpolation formula is used for finding values of unknown functions at the selected node points. Existence and uniqueness solution of the problems are proved and accuracy of the quadrature formula together with convergence of the proposed method are obtained. Finally, numerical examples are provided to show the validity and efficiency of the method presented. Numerical results reveal that the proposed methods is efficient and accurate. Comparisons with other methods for the same problem are also presented.

1. Introduction

The theory of approximation methods and its applications to the solution of nonlinear singular integral equations (Nonlinear SIEs) [1–4], functional equations [5–12] and nonlinear integral equations (NIEs) [13–21] have been developed by many authors. However, system of nonlinear integral equations is not much elaborated [22–27]. In mathematics, many problems of differential equations, integral equations, functional equations and operator equations can be reduced to find the roots of nonlinear operator equation of the form

$$P\left(x\right)=0, \tag{1.1}$$

where P is mainly nonlinear operator and x is a vector function to be determined. The exact solution of (1.1) is available in exceptional cases ([5] and literatures cited there in). Therefore numerical methods are needed to find the approximate solutions. One of the well-known linearization approximation method is modified Newton method [6]. It attempts to linearize the nonlinear equation into linear equation then find the approximate solution by processing the convergent sequence.

$$x_{n+1}=x_n-\left[P'\left(x_0\right)\right]^{-1}P\left(x_n\right),\quad n=0,1,\ldots \tag{1.2}$$

where x_0 is an initial approximation (assume that $\left[P'\left(x_0\right)\right]^{-1}$ exists).

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Each x_n is an approximate solution of (1.1) and the larger the integer n more accurate solution is obtained. Generating sequence x_n in the form of (1.2) is called modified Newton method (modified NM) [7, 525]. If the sequence $\{x_n\}$ converges to a root x^* and $x_0 \in \Omega$ is chosen close enough to x^*, then by the continuity of P', the operator of $P'(x_n)$ and $P'(x_0)$ will only differ by a small amount. This is the justification of modified NM.

Some survey of literatures regarding to the convergence of modified NM are listed as follows. In 1939, Kantorovich [8] proposed iterative method for functional equations in a Banach space and derived the convergence theorem for Newton's method. In 1948, he [9] suggested an extension of Newton's method to functional spaces and established a semilocal convergence result for Newton's method in Banach space, which is called Kantorovich's theorem or, more specifically, the Newton–Kantorovich theorem (NKT). In 1949, Kantarovich [10] has stated the main theorem on the convergence of the Newton process. In it the final conditions of the convergence of the method are given and the convergence rate is established. There are a lot of results published with regards to convergence and error bounded for Newton's method under assumption of the Newton–Kantorovich theorem or under closely related ones. Further developments of the Newton's method can be found in [11,12].

On the solvability of nonlinear Uryson integral equation (IEs) is first carried out by Zabrejko and Majarova [13] in 1978, in Banach space. In 1987, Zabrejko and Nguyen [14] investigated the solvability of the nonlinear algebraic equation and sharp error estimates were obtained by means of the majorant method in the theory of modified NM. In 1991, Appell et al. [15] applied modified NM to a nonlinear integral equation in Banach space to calculate two scalar constants and scalar functions. This is carried out for nonlinear Uruson integral operator in spaces C and L_p, $1 \le p \le 2$. In 2003, Wang [16] has established some results on convergence of Newton's method in Banach spaces under the assumption that derivative of the operators satisfies the radius or center Lipschitz condition with a weak L average. In 2004, Argyros [17] has used Newton–Kantorovich hypothesis as a sufficient condition of the convergence of Newton's method to a solution of an functional equation in connection with the Lipschitz continuity of the Fréchet-derivative and could be able to weakened Newton–Kantorovich hypothesis. In 2010, Saberi and Heidari [18], developed a method of Newton–Kantorovich and quadrature rule to solve nonlinear integral equation of the Urysohn form in a systematic procedure. In 2012, Ezquerro et al. [19] proved the existence and uniqueness solutions of Hammerstein type equation using Newton's method. In 2016, Eshkuvatov et al. [20] have applied modified NM to Volterra-type nonlinear integral equations then the method of Nystrom type Gauss–Legendre quadrature formula (QF) was used to find the approximate solution of a linear Fredholm integral equation. The existence and uniqueness of the approximated method are proved and the convergence rate is established in Banach space. Finally illustrative examples are provided to validate the accuracy of the presented method.

In 1996, Brunnera and Yatsenko [21] considered a system of nonlinear Volterra integral equations (VIE) with unknown delay time

$$\begin{cases} x(t) = \displaystyle\int_{y(t)}^{t} K_1(t,\tau,x(\tau))d\tau, \\ \int_{y(t)}^{t} K_2(t,\tau,x(\tau))d\tau = f(t), \quad t \in [0,T]. \end{cases} \tag{1.3}$$

where $x(\tau)$ and delay time $y(\tau)$ are unknowns, where $\tau \in (-\infty, 0]$. The solution $x(\tau)$ is to agree with a given initial function i.e. $x(\tau) = x_0(\tau), -\infty < \tau \le o$ and the unknown delay time $y(\tau)$ obeys the initial condition $y(0) = y_0 < 0$. They introduced and studied polynomial spline collocation methods for systems of Volterra integral equations (1.3) with unknown lower integral limit arising in mathematical economics. Their discretization leads to the implicit Runge–Kutta type method. The global convergence and local superconvergence properties of this method were proved, and the theory was illustrated by a numerical examples.

In 2003, Boykov and Tynda [22], implemented successfully modified NM to the system of nonlinear Volterra integral equation of the form

$$\begin{cases} x(t) - \displaystyle\int_{y(t)}^{t} h(t,\tau)g(\tau)x(\tau)d\tau = 0, \\ \displaystyle\int_{y(t)}^{t} k(t,\tau)[1-g(\tau)]x(\tau)d\tau = f(t), \end{cases} \tag{1.4}$$

where $0 < t_0 \le t \le T,\ y(t) < t$, with given functions $h(t,\tau),\ k(t,\tau) \in C_{[0,\infty]\times[t_0,\infty]}, f(t),\ g(t) \in C_{[t_0,\infty]}$ $0 < g(t) < 1$, and the unknown function $x(t) \in C_{[0,\infty]},\ y(t) \in C^1_{[t_0,\infty]}$. They also considered n-commodity models described by nonlinear systems of n equations. The uniqueness and existence theorems together with rate of convergence of approximate model were obtained for (1.3).

In 2010, Eshkuvatov et al. [23] solved numerically the system of nonlinear Volterra integral equation of the form

$$\begin{cases} x(t) - \displaystyle\int_{y(t)}^{t} h(t,\tau)g(\tau)x^2(\tau)d\tau = 0, \\ \displaystyle\int_{y(t)}^{t} k(t,\tau)x^2(\tau)d\tau = f(t), \end{cases} \tag{1.5}$$

where $0 < t_0 \le t \le T,\ y(t) < t$, and $f(t) \in C_{[t_0,\infty]},\ H(t,\tau), K(t,\tau) \in C_{[t_0,\infty]\times[t_0,\infty]}$, and the unknown functions $x(t) \in C_{[t_0,\infty]},\ y(t) \in C^1_{[t_0,\infty]}$, and found that Eq. (1.4) has a unique solution and Newton–Kantorovich iteration converges to the exact solution very fast. Numerical examples are provided to show the validity and efficiency of the method presented. In 2015, Hameed et al. [24] developed modified NM to solve the system 2×2 nonlinear Volterra integral equations where the unknown function is of the logarithmic form. A new majorant function was introduced which leads to wider convergence interval. The existence and uniqueness of approximate solution were proved and a numerical example was provided to show the validation of the method. In 2016, Hameed et al. [25] have considered an $n \times n$ system of nonlinear integral equations of Volterra type (nonlinear VIEs) arising from an economic model. By applying the modified NM linear Volterra type integral equations (linear VIEs) is obtained and solved by the Nystrom type Gauss–Legendre quadrature formula (QF). It is found that by increasing the number of collocation points in the sub-grids with fewer iterations, a highly accurate approximate solution was obtained

The aim of the present paper is to investigate the general system of 2×2 nonlinear integral equation of the form

$$\begin{cases} a(t)x(t) - \displaystyle\int_{y(t)}^{t} H(t,\tau)F\left(x(\tau)\right) d\tau = g(t), \\ b(t)x(t) + \displaystyle\int_{y(t)}^{t} K(t,\tau)F\left(x(\tau)\right) d\tau = f(t), \end{cases} \tag{1.6}$$

where $0 < t_0 \le t \le T,\ y(t) < t$, kernels $H(t,\tau),\ K(t,\tau) \in C_{[t_0,T]\times[t_0,T]}$, functions $a(t),\ b(t),\ f(t) \in C_{[t_0,T]}$ and unknown functions $x(t) \in C_{[t_0,T]},\ y(t) \in C^1_{[t_0,T]}$. Here $F\left(x(t)\right)$ is a different type of nonlinear term. In an attempt to solve Eq. (1.6) the modified NM together with Gauss–Legendre quadrature formula and Newton's forward interpolation formula are used.

2. Description of the method

$$\begin{cases} P_1\left(x(t), y(t)\right) = a(t)x(t) - \displaystyle\int_{y(t)}^{t} H(t,\tau)F\left(x(\tau)\right) d\tau - g(t), \\ P_2\left(x(t), y(t)\right) = b(t)x(t) + \displaystyle\int_{y(t)}^{t} K(t,\tau)F\left(x(\tau)\right) d\tau - f(t) \end{cases} \tag{2.1}$$

then rewrite (2.1) in the operator form

$$P(X) = (P_1(X), P_2(X)) = (0,0)\,,\ \ X = (x(t), y(t))\,. \tag{2.2}$$

Apply the first iteration of modified NM

$$P'(X_0)(X - X_0) + P(X_0) = 0, \tag{2.3}$$

to (1.6), where $X_0 = (x_0(t), y_0(t))$ is the initial guess with $y_0(t) < t$. The derivative

$$P'\left(X_0\right)(X) = \lim_{s\to 0} \frac{P\left(X_0 + sX\right) - P(X_0)}{s},$$

is understood as Fréchet derivative of P at $X_0 = (x_0(t), y_0(t))$ and it has the form (details is given in [24])

$$P'(X_0) = \begin{pmatrix} \left.\dfrac{\partial P_1}{\partial x}\right|_{(x_0,y_0)} & \left.\dfrac{\partial P_1}{\partial y}\right|_{(x_0,y_0)} \\ \left.\dfrac{\partial P_2}{\partial x}\right|_{(x_0,y_0)} & \left.\dfrac{\partial P_2}{\partial y}\right|_{(x_0,y_0)} \end{pmatrix} \tag{2.4}$$

where

$$\left.\frac{\partial P_1}{\partial x}\right|_{(x_0,y_0)} = \lim_{s\to 0}\frac{1}{s}\left(P_1\left(x_0 + sx, y_0\right) - P_1(x_0,y_0)\right) = a(t)x(t) - \int_{y_0(t)}^{t} H\left(t,\tau\right) F'\left(x_0(\tau)\right) x(\tau)\, d\tau,$$
$$\left.\frac{\partial P_1}{\partial y}\right|_{(x_0,y_0)} = \lim_{s\to 0}\frac{1}{s}\left(P_1\left(x_0, y_0 + sy\right) - P_1(x_0,y_0)\right) = H\left(t, y_0(t)\right) F\left(x_0(y_0(t))\right) y(t),$$
$$\left.\frac{\partial P_2}{\partial x}\right|_{(x_0,y_0)} = \lim_{s\to 0}\frac{1}{s}\left(P_2\left(x_0 + sx, y_0\right) - P_2(x_0,y_0)\right) = b(t)x(t) + \int_{y_0(t)}^{t} K\left(t,\tau\right) F'\left(x_0(\tau)\right) x\left(\tau\right)\, d\tau,$$
$$\left.\frac{\partial P_2}{\partial y}\right|_{(x_0,y_0)} = \lim_{s\to 0}\frac{1}{s}\left(P_2\left(x_0, y_0 + sy\right) - P_2(x_0,y_0)\right) = -K\left(t, y_0(t)\right) F\left(x_0(y_0(t))\right) y\left(t\right).$$

Substituting (2.4) into (2.3) yields

$$\left.\begin{aligned} &a(t)\Delta x(t)-\int_{y_0(t)}^{t}H(t,\tau)\,F'(x_0(\tau))\,\Delta x(\tau)d\tau\;+H(t,y_0(t))\,F(x_0(y_0(t)))\,\Delta y(t)\\ &\qquad=\int_{y_0(t)}^{y}H(t,\tau)\,F(x_0(\tau))\,d\tau-a(t)x_0(t)+g(t),\\ &b(t)\Delta x(t)+\int_{y_0(t)}^{t}K(t,\tau)\,F'(x_0(\tau))\,\Delta x(\tau)d\tau-K(t,y_0(t))\,F(x_0(y_0(t)))\Delta y(t)\\ &\qquad=-\int_{y_0(t)}^{t}K(t,\tau)\,F(x_0(\tau))\,d\tau-b(t)x_0(t)+f(t). \end{aligned}\right\} \tag{2.5}$$

To solve (2.5), substitution rule is applied and arrive at

$$\left.\begin{aligned} &\Delta x(t)-\frac{1}{c(t)}\int_{y_0(t)}^{t}K_1(t,\tau)F'(x_0(\tau))\Delta x(\tau)d\tau=\psi_0(t),\\ &\Delta y(t)=\frac{1}{d(t)}\left[\int_{y_0(t)}^{t}K(t,\tau)F'(x_0(\tau))\Delta x(\tau)d\tau\right.\\ &\qquad\left.+\int_{y_0(t)}^{t}K(t,\tau)F(x_0(\tau))d\tau+b(t)x_0(t)-f(t)\right], \end{aligned}\right\} \tag{2.6}$$

where $\Delta x(t)=x_1(t)-x_0(t)$ and $\Delta y(t)=y_1(t)-y_0(t)$,

$$\begin{aligned} &c(t)=a(t)+b(t)G(t)\neq 0,\quad d(t)=H(t,y_0(t))F(x_0(y_0(t)))\neq 0,\quad \forall t\in[t_0,T],\\ &K_1(t,\tau)=H(t,\tau)-G(t)K(t,\tau),\quad G(t)=\frac{H(t,y_0(t))}{K(t,y_0(t))},\\ &\psi_0(t)=\frac{1}{c(t)}\left[\int_{y_0(t)}^{t}K_1(t,\tau)F(x_0(\tau))d\tau-c(t)x_0(t)+g(t)+f(t)G(t)\right]. \end{aligned} \tag{2.7}$$

By solving (2.6) in terms of Δx and Δy we obtain $(x_1(t),y_1(t))$. Applying the modified Newton method of the form

$$P'(X_0)(X_m-X_{m-1})+P(X_{m-1})=0, \tag{2.8}$$

to (2.2) we obtain

$$\left.\begin{aligned} &a(t)\Delta x_m(t)-\int_{y_0(t)}^{t}H(t,\tau)F'(x_0(\tau))\Delta x_m(\tau)d\tau+H(t,y_0(t))F(x_0(y_0(t)))\Delta y_m(t)\\ &\qquad=\int_{y_{m-1}(t)}^{y}H(t,\tau)F(x_{m-1}(\tau))\,d\tau-a(t)x_{m-1}(t)+g(t),\\ &b(t)\Delta_m x(t)+\int_{y_0(t)}^{t}K(t,\tau)F'(x_0(\tau))\Delta x_m(\tau)d\tau-K(t,y_0(t))F(x_0(y_0(t)))\Delta y_m(t)\\ &\qquad=-\int_{y_{m-1}(t)}^{t}K(t,\tau)F(x_{m-1}(\tau))d\tau-b(t)x_{m-1}(t)+f(t). \end{aligned}\right\} \tag{2.9}$$

Since (2.9) is a linear Volterra type integral equations, it can easily be solved in terms of Δx_m and Δy_m as follows

$$\left.\begin{aligned} &\Delta x_m(t)-\frac{1}{c(t)}\int_{y_0(t)}^{t}K_1(t,\tau)F'(x_0(\tau))\Delta x_m(\tau)d\tau=\psi_{m-1}(t),\quad m=2,3\cdots\\ &\Delta y_m(t)=\frac{1}{d(t)}\left[\int_{y_0(t)}^{t}K(t,\tau)F'(x_0(\tau))\Delta x_m(\tau)d\tau\right.\\ &\qquad\left.+\int_{y_{m-1}(t)}^{t}K(t,\tau)F(x_{m-1}(\tau))d\tau+b(t)x_m(t)-f(t)\right] \end{aligned}\right\} \tag{2.10}$$

where $\Delta x_m(t)=x_m(t)-x_{m-1}(t)$, $\Delta y_m(t)=y_m(t)-y_{m-1}(t)$ and functions $c(t)$, $d(t)$, $G(t)$ and kernel $K_1(t,\tau)$ are defined in (2.7) and

$$\psi_{m-1}(t)=\frac{1}{c(t)}\left[\int_{y_{m-1}(t)}^{t}K_1(t,\tau)\,F(x_{m-1}(\tau))\,d\tau-c(t)x_{m-1}(t)+g(t)+f(t)G(t)\right]. \tag{2.11}$$

Solving (2.9) with respect to Δx_m and Δy_m, we obtain a sequence of approximate solution $(x_m(t), y_m(t))$.

Remark 1. Assume that $c(t) = b(t) + a(t)G(t) \neq 0, \quad \forall t \in [t_0, T]$ and the kernels $H(t,\tau)$ and $K(t,\tau)$ are continuous with $K(t, y_0(t)) \neq 0, \quad \forall t \in [t_0, T]$ then the first equation of (2.10) has continuous coefficients. Since $0 < t_0 \leq y_0(t) < t$ it follows that the first equation of (2.10) has a unique solution in terms of Δx_m. Once $\Delta x_m(t)$ is calculated from the first equation of (2.10) then approximate solution of (2.6) can be obtained by the method of successive approximations $x_m(t) = \Delta x_m(t) + x_{m-1}(t),\ m = 1, 2, \ldots$. The sequence Δy_m can be uniquely determined from the second equation of (2.10).

3. Quadrature method and its accuracy

3.1. Gauss–Legendre quadrature method

It is known that Legendre polynomials $P_{n+1}(x)$ are orthogonal on $[-1, 1]$ with weights $w(x) = 1$ and its roots $P_{n+1}(x) = 0$ are equally distributed in the interval $[-1, 1]$ centered at $x = 0$. Petras [28] has shown that the efficient numerical calculation of probably the most important quadrature formula is the Gauss–Legendre quadrature formula (QF) on the interval $[-1, 1]$. Extension of Gauss–Legendre QF on any interval $[a, b]$ is shown in [29, pp. 115] and stated that if Gauss–Legendre quadrature formula (QF) is constructed in the form of

$$\int_a^b f(s)ds = \sum_{i=1}^{n+1} w_i f(t_i) + R_n(f), \tag{3.1}$$

where $t_i = \frac{b-a}{2}s_i + \frac{b+a}{2}$ and

$$w_i = \frac{2}{\left(1 - s_i^2\right)\left[P_n'(s_i)\right]^2}, \quad \sum_{i=1}^{n+1} w_i = 2, \tag{3.2}$$

$$P_{n+1}(s_i) = 0,\ i = 1, 2, \ldots, n+1, \tag{3.3}$$

then the error term of Gauss–Legendre QF is

$$R_n(f) = \frac{(b-a)^{2n+3}(n+1)!^4}{(2n+3)\left[(2n+2)!\right]^3} f^{(2n+2)}(\xi), \quad -1 < \xi < 1. \tag{3.4}$$

Theorem 1 (*Kythe and Schaferkotter [29, pp. 113]*). *Gaussian quadrature formula has precision $2n + 1$ only if the points $s_i, \quad i = 1, 2, \ldots, n+1$ are the zeros of orthogonal polynomials $\phi_{n+1}(s)$.*

Eshkuvatov et al. [20] extended Gauss–Legendre QF (3.1) to the kernel integral on the $[y(t_i), t_i],\ i = 1, 2, \ldots, n$ with $t_i = t_0 + ih,\ h = \frac{T - t_0}{n}$ as follows

$$\int_{y(t_i)}^{t_i} K(t_i, \tau)x(\tau)d\tau = \frac{t_i - y(t_i)}{2} \sum_{j=1}^{l} W_j(t_i)x(\tau_j^i) + R_{n+1}(Kx), \tag{3.5}$$

$$W_j(t_i) = K(t_i, \tau_j^i)w_j, \quad \tau_j^i = \frac{t_i - y(t_i)}{2}s_j + \frac{t_i + y(t_i)}{2}, \quad j = 1, 2, \ldots, l,$$

where $\tau_j^i \neq t_i$ with $0 < t_0 \leq y(t) < t \leq T_0$ and l refers to the number of sub partitions of the interval $[y(t_i), t_i] \in [t_0, T]$ and w_j and s_j are the roots of Legendre polynomials and weights defined by (3.2)–(3.3) respectively.

By changing variable interval into fixed interval $[t_0, T]$ and constructing Gauss–Legendre QF in the form

$$Q(t) = \int_{t_0}^{T} K(t,\tau)\,x(\tau)\,d\tau = \frac{T - t_0}{2} \sum_{j=1}^{n+1} W_j(t)\,x\left(\tau_j\right) + R_{n+1}(Kx), \tag{3.6}$$

$$W_j(t) = K\left(t, \tau_j\right) w_j, \quad \tau_j = \frac{T - t_0}{2}s_j + \frac{T + t_0}{2}, \quad j = 1, 2, \ldots, n+1,$$

we prove the following theorem.

Theorem 2. *Let kernel $K(t,\tau)$ and $x(t)$ be in the class of $C^{(2n+2)}[t_0, T]$ then the error term of Gauss–Legendre QF (3.6) has the form*

$$|R_{n+1}(Kx)| \leq E \leq \frac{(T - t_0)^{2n+3}}{(2n+3)} \left[\frac{1 \cdot 2 \cdot 3 \cdots (n+1)}{(n+1)\cdot(n+2)\cdots(2n+2)}\right]^2 \frac{T^{(2n+2)}}{1 \cdot 2 \cdot 3 \cdots (2n+2)}, \tag{3.7}$$

where E denotes the maximum modulus of the error $R_{n+1}(Kx)$ and

$$T^{(q)} = X^{(0)}M_t^{(q)} + b_1X^{(1)}M_t^{(q-1)} + \cdots + b_{q-1}X^{(q-1)}M_t^{(1)} + X^{(q)}M_t^{(0)}, \tag{3.8}$$

$$X^{(0)} = \underset{t_0\le\tau\le T}{ub}|x(\tau)|\,,\;\; X^{(m)} = \underset{t_0\le\tau\le T}{ub}\left|\frac{\partial^m}{\partial\tau^m}x(\tau)\right|,$$

$$M_t^{(0)} = \underset{t_0\le\tau\le t\le T}{ub}|K(t,\tau)|\,,\;\; M_t^{(m)} = \underset{t_0\le\tau\le t\le T}{ub}\left|\frac{\partial^m}{\partial t^m}K(t,\tau)\right|. \tag{3.9}$$

with ub means "upper bound" and $b_i = \dfrac{q!}{i!(q-i)!}$, $i = 1,\ldots,q-1$ *binomial coefficients.*

Proof. Due to product rule of derivative with Binomial coefficients and notations in (3.8)–(3.9) we can easily prove that

$$\begin{aligned}\left|\frac{d^n}{d\tau^n}[K(t,\tau)x(\tau)]\right| &= \left|\frac{\partial^nK(t,\tau)}{\partial\tau^n}x(\tau) + \sum_{i=1}^{n-1}b_n^i\frac{\partial^{n-i}K(t,\tau)}{\partial\tau^{n-i}}\frac{d^ix(\tau)}{d\tau^i} + K(t,\tau)\frac{d^nx(\tau)}{d\tau^i}\right|\\ &\le M_t^{(n)}X^{(0)} + \sum_{i=1}^{n-1}b_n^iM_t^{(n-i)}X^{(i)} + M_t^{(0)}X^{(n)} = T^n.\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\frac{[(n+1)!]^4}{[(2n+2)!]^3} &= \frac{[(n+1)!]^2\cdot[(n+1)!]^2}{[(n+1)!]^2\cdot[(n+1)(n+2)\cdots(2n+2)]^2\cdot[(2n+2)!]}\\ &= \left[\frac{1\cdot2\cdot3\cdots(n+1)}{(n+1)\cdot(n+2)\cdots(2n+2)}\right]^2\frac{1}{1\cdot2\cdots(2n+2)}\end{aligned} \tag{3.11}$$

From (3.4), (3.10), (3.11) and approach in [29, pp.456–457], it follows that

$$\begin{aligned}|R_{n+1}(Kx)| &= \left|\int_{t_0}^T K(t,\tau)\,x(\tau)\,d\tau - \frac{T-t_0}{2}\sum_{j=1}^{n+1}W_j(t)\,x\left(\tau_j\right)\right|\\ &\le \frac{(T-T_0)^{2n+3}(n+1)!^4}{(2n+3)\,[(2n+2)!]^3}\left|\frac{d^n}{d\tau^n}[K(t,\tau)x(\tau)]\right|_{\tau=\xi}\\ &\le \frac{(T-t_0)^{2n+3}}{(2n+3)}\left[\frac{1\cdot2\cdot3\cdots(n+1)}{(n+1)\cdot(n+2)\cdots(2n+2)}\right]^2\frac{T^{(2n+2)}}{1\cdot2\cdot3\cdots(2n+2)}.\quad\square\end{aligned}$$

3.2. The accuracy and stability of quadrature rules

In elementary calculus one learns to evaluate a definite integral

$$I(f) = \int_a^b \rho(x)f(x)dx, \tag{3.12}$$

then the integral in (3.12) is approximated by an n-point quadrature rules, which has the form

$$Q_n(f) = \sum_{i=1}^n \omega_if(x_i), \tag{3.13}$$

where $a \le x_1 < x_2 < x_3 < \cdots < x_n \le b$. The points x_i are called nodes or abscissa, and ω_i are called weights. Quadrature rules can be constructed using polynomial interpolation. In particular, if Lagrange interpolation polynomials are used, then the weights can be represented as

$$\omega_i = \int_a^b \rho(x)\ell_i(x)dx,\quad i = 1,\ldots,n. \tag{3.14}$$

The resulting quadrature rule is called interpolation quadrature formula (QF) [29]. An alternative method for interpolation quadrature rules (QR) is the Gaussian quadrature rule. If Q_n is an interpolatory quadrature rule, and p_{n-1} is the polynomial of degree at most $n-1$ interpolating a sufficiently smooth integrand function f at the knots $x_1, x_2, \ldots, x_n$, then the error bound for the case of $\rho(x) = 1$ [30, pp. 343] can be obtained as follows

$$\begin{aligned}|I(f) - Q_n(f)| &= |I(f) - I(p_{n-1})| = |I(f - p_{n-1})|\\ &\le (b-a)\|f - p_{n-1}\|_\infty \le \frac{1}{4}h^{n+1}\|f^{(n)}\|_\infty,\end{aligned} \tag{3.15}$$

where $h = \max\{x_{i+1} - x_i : i = 1, 2, \ldots, n-1\}$. Thus we can acquire higher accuracy by taking larger n which leads smaller h. In fact, the bound of nth derivative of the function reveals that $Q_n(f) - I(f) \to 0$ as $n \to \infty$, as well as the minimum convergence rate we can expect, provided $f^{(n)}$ remains well behaved.

In addition, Michael [30, pp. 347] states that interpolatory QF constructed in the form of (3.13) can also be concerned with the stability of a quadrature rules. Let $\hat{f}$ be a perturbation to the integrand function f, then we have

$$|Q_n(\hat{f}) - Q_n(f)| = |Q_n(\hat{f} - f)| = \left|\sum_{i=1}^{n} \omega_i(\hat{f}(x_i) - f(x_i))\right|$$
$$\leq \sum_{i=1}^{n} \left(|\omega_i|.\left|\hat{f}(x_i) - f(x_i)\right|\right) \leq \left(\sum_{i=1}^{n} |\omega_i|\right) \|\hat{f} - f\|_{\infty}.$$

If the weights are all nonnegative, then the absolute condition number of the quadrature rule is $b - a$ and thus the quadrature rule is stable. If some of the weights are negative then absolute condition number can be much larger and quadrature rule can be unstable.

Let $w_n(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$ be the polynomials of degree n for the identifying nodes x_k, then Gaussian QF for the integral (3.12) the following theorems are hold.

Theorem 3 (*Israilov [31, pp. 347]*). *Gaussian QF of the form (3.13) to be exact for the polynomials of degree $2n-1$ the following two conditions need to be satisfied*

1. *QF (3.13) must be interpolation*
2. *Polynomials $w_n(x)$ should be orthogonal to any polynomials $Q(x)$ of order less than n with the weights $\rho(x)$ on the interval $[a, b]$ i.e.*

$$\int_a^b \rho(x) w_n(x) Q(x) dx = 0.$$

Theorem 4 (*Israilov [31, pp. 350]*). *If $f(x) \in C^{2n}[a, b]$ then there exists $\xi \in [a, b]$ such that the error term of Gaussian QF for the integral (3.12) with weight function the following equality holds*

$$R_n(f) = I(f) - Q_n(f) = \frac{f^{(2n)}(\xi)}{(2n)!} \int_a^b \rho(x)[w_n(x)]^2 dx. \tag{3.16}$$

For the case $[a, b] = [-1, 1]$ and $\rho(x) = 1$ error term of the Gauss–Legendre QF is

$$R_n(f) = \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\xi) = \frac{2^{2n+1}}{2n+1}\left[\frac{1 \cdot 2 \cdots n}{(n+1)\cdot(n+2)\cdots(2n)}\right]^2 \frac{f^{(2n)}(\xi)}{(2n)!}. \tag{3.17}$$

Error term (3.17) shows that for the bounded derivative of the function leads high accurate and stable QF. Unbounded derivative of the function leads unstable QF.

4. Discretization of the modified NM

For the approximate solution of the linear system (2.5) we introduce a grid point $\Omega_1 = \left\{t_i : \ t_i = t_0 + i\frac{T - t_0}{n},\right.$ $\left. i = 1, 2, \ldots, n\right\}$ where n refers to the number of partitions in $[t_0, T]$. Then from the system (2.10) we obtain

$$\begin{cases} x_m(t_i) - \dfrac{1}{c(t_i)} \displaystyle\int_{y_0(t_i)}^{t_i} K_1(t_i, \tau)\, F'(x_0(\tau))\, x_m(\tau) d\tau \\ \quad = \dfrac{1}{c(t_i)} \left[\displaystyle\int_{y_0(t_i)}^{t_i} K_1(t_i, \tau)\, F'(x_{m-1}(\tau))\, x_{m-1}(\tau) d\tau \right. \\ \quad \left. + \displaystyle\int_{y_{m-1}(t)}^{t} K_1(t_i, \tau)\, F(x_{m-1}(\tau))\, d\tau + g(t_i) + f(t_i)\, G(t_i)\right] \\ \Delta y_m(t_i) = \dfrac{1}{d(t_i)} \left[\displaystyle\int_{y_0(t_i)}^{t_i} K(t_i, \tau)\, F'(x_0(\tau))\, \Delta x_m(\tau) d\tau \right. \\ \quad \left. + \displaystyle\int_{y_{m-1}(t)}^{t} K(t_i, \tau)\, F(x_{m-1}(\tau))\, d\tau + b(t_i)\, x_m(t_i) - f(t_i)\right]. \end{cases} \tag{4.1}$$

Let us introduce a sub grid Ω_2 at each subinterval $[y_0(t_i), t_i]$ and $[y_{m-1}(t_i), t_i]$ of the interval $[t_0, T]$ such that

1. For the interval $[y_0(t_i), t_i]$, assume that $a_i = y_0(t_i) < t_i$ therefore $[a_i, t_i] \subset [t_0, T]$ and we choose Legendre knot points at each open interval (a_i, t_i), i.e.

$$\tau_i^{(k)} = \frac{t_i - a_i}{2} s_k + \frac{t_i + a_i}{2}, \ k = 1, 2, \dots, l, \ i = 1, 2, \dots, n, \ \tau_i^{(k)} \neq t_i \tag{4.2}$$

2. For the interval $[y_{m-1}(t_i), t_i]$, the grid points are chosen as

$$\tau_{i(m-1)}^{(k)} = \frac{t_i - y_{m-1}(t_i)}{2} s_k + \frac{t_i + y_{m-1}(t_i)}{2}, \ k = 1, 2, \dots, l, \ i = 1, 2, \dots, n, \ \tau_{i(m-1)}^{(k)} \neq t_i. \tag{4.3}$$

Applying Gauss–Legendre QF (3.5) for the kernel integrals in (4.1) at the Legendre grid points $\tau_i^{(k)}$ and $\tau_{i(m-1)}^{(k)}$ defined by (4.2) and (4.3), we obtain

$$\begin{aligned} x_m\left(\tau_i^{(k)}\right) &- \frac{1}{c\left(\tau_i^{(k)}\right)} \sum_{j=1}^{l} W_{kji}^{(1)} x_m\left(\tau_j^{(k)}\right) \\ &= \frac{1}{c\left(\tau_i^{(k)}\right)} \left[\sum_{j=1}^{l} W_{kji}^{(1)} x_{m-1}\left(\tau_j^{(k)}\right) + \sum_{j=1}^{l} W_{kji}^{(2)} F\left(x_{m-1}\left(\tau_j^{(k)}\right)\right) \right. \\ &\quad \left. + g\left(\tau_i^{(k)}\right) + f\left(\tau_i^{(k)}\right) G\left(\tau_i^{(k)}\right) \right], \\ &\quad k = 1, 2, \dots, l, \ i = 1, 2, \dots, n, \ m = 1, 2, \dots \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} W_{kji}^{(1)} &= \frac{t_i - a_i}{2} K_1\left(\tau_i^{(k)}, \tau_i^{(j)}\right) F'\left(x_0\left(\tau_i^{(j)}\right)\right) w_j, \\ W_{kji}^{(2)} &= \frac{t_i - y_{m-1}(t_i)}{2} K_1\left(\tau_i^{(k)}, \tau_{i(m-1)}^{(j)}\right) w_j, \quad m = 1, 2, \dots \end{aligned} \tag{4.5}$$

System (4.4) can be written in matrix form as follows

$$DX_m = B_{m-1}, \ m = 1, 2, \dots, \tag{4.6}$$

where

$$D = \begin{bmatrix} 1 - \frac{1}{c(\tau_i^1)} W_{11i}^{(1)} & -\frac{1}{c(\tau_i^1)} W_{12i}^{(1)} & \cdots & -\frac{1}{c(\tau_i^1)} W_{1\ell i}^{(1)} \\ -\frac{1}{c(\tau_i^2)} W_{21i}^{(1)} & 1 - \frac{1}{c(\tau_i^2)} W_{22i}^{(1)} & \cdots & -\frac{1}{c(\tau_i^2)} W_{2\ell i}^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ -\frac{1}{c(\tau_i^\ell)} W_{\ell 1i}^{(1)} & -\frac{1}{c(\tau_i^\ell)} W_{\ell 2i}^{(1)} & \cdots & 1 - \frac{1}{c(\tau_i^\ell)} W_{\ell\ell i}^{(1)} \end{bmatrix}, \quad X_m = \begin{bmatrix} x_m(\tau_i^1) \\ x_m(\tau_i^2) \\ \vdots \\ x_m(\tau_i^\ell) \end{bmatrix}, \ i = 1, 2, \dots, n,$$

and

$$B_{m-1} = \begin{bmatrix} \frac{1}{c(\tau_i^1)} \left[\sum_{j=1}^{\ell} W_{1ji}^{(1)} x_{m-1}(\tau_i^1) + \sum_{j=1}^{\ell} W_{1ji}^{(2)} F\left(x_{m-1}(\tau_i^1)\right) + g(\tau_i^1) + f(\tau_i^1) G(\tau_i^1) \right] \\ \frac{1}{c(\tau_i^2)} \left[\sum_{j=1}^{\ell} W_{2ji}^{(1)} x_{m-1}(\tau_i^2) + \sum_{j=1}^{\ell} W_{2ji}^{(2)} F\left(x_{m-1}(\tau_i^2)\right) + g(\tau_i^2) + f(\tau_i^2) G(\tau_i^2) \right] \\ \vdots \\ \frac{1}{c(\tau_i^\ell)} \left[\sum_{j=1}^{\ell} W_{\ell ji}^{(1)} x_{m-1}(\tau_i^\ell) + \sum_{j=1}^{\ell} W_{\ell ji}^{(2)} F\left(x_{m-1}(\tau_i^\ell)\right) + g(\tau_i^\ell) + f(\tau_i^\ell) G(\tau_i^\ell) \right] \end{bmatrix}.$$

If the determinant $|D| \neq 0$ then (4.6) has a unique solution. Since the values of the unknown functions $x_m\left(\tau_i^{(k)}\right)$ are known at l Legendre grid points in each subinterval $[a_i, t_i]$ for each m iteration, we can find the values of unknown function $x(t_i)$

using Newton forward interpolation formula

$$\begin{aligned} x(t_i) \simeq P_l(t) = x_m\left(\tau_i^{(l)}\right) + x_m\left(\tau_i^{(l)}, \tau_i^{(l-1)}\right)\left(t-\tau_i^{(l)}\right) \\ + x_m\left(\tau_i^{(l)}, \tau_i^{(l-1)}, \tau_i^{(l-2)}\right)\left(t-\tau_i^{(l)}\right)\left(t-\tau_i^{(l-1)}\right) \\ + \cdots + x_m\left(\tau_i^{(l)}, \ldots, \tau_i^{(1)}\right)\left(t-\tau_i^{(l)}\right)\cdots\left(t-\tau_i^{(1)}\right). \end{aligned} \tag{4.7}$$

It is known [31] that the error term of (4.7) is

$$\|x_m(t) - P_l(t)\| \le \frac{M}{(l+1)!}, \tag{4.8}$$

where $M = \max\left\{\left|f^{(l+1)}(\xi)\right|\left|\left(t-\tau_i^{(l)}\right)\cdots\left(t-\tau_i^{(1)}\right)\right|\right\}$.

From (4.8) it follows that by increasing the nodes points l the more accurate solution is obtainable, therefore the Newton forward interpolation method can be used for small m. Since all values of $x_m\left(\tau_i^{(j)}\right)$ are known and due to (4.7) we have values of the unknown function $x(t_i)$, $i = 1, 2, \ldots, n$, then unknown values of $y(t_i)$ are defined by the Gauss–Legendre quadrature

$$\begin{aligned} y_m(t_i) = y_{m-1}(t_i) + \frac{1}{d(t_i)}\Bigg[\sum_{j=1}^{l} W_{ij}^{(1)}\left(x_m\left(\tau_i^{(j)}\right) - x_{m-1}\left(\tau_i^{(i)}\right)\right) \\ + \sum_{j=1}^{l} W_{ij}^{(2)} F\left(x_{m-1}\left(\tau_{i(m-1)}^{(i)}\right)\right) + b(t_i)x_m(t_i) - f(t_i)\Bigg], \end{aligned} \tag{4.9}$$

where W_{ij}^1 and W_{ij}^1 are defined by (4.5).

Hence, we have found all values of $(x_m(t_i), y_m(t_i))$. By the convergence iteration of modified NM [7, pp. 532] and convergence of Gauss–Legendre QF (3.7), we obtain $(x_m(t_i), y_m(t_i)) \xrightarrow[m\to\infty]{} (x(t_i), y(t_i))$.

The steps of application of the proposed method is as follows:

1. Identify operators P_1 and P_2 and write the given equation in the form of (2.1), i.e. $P(X) = (0, 0)$.
2. Reduce the nonlinear problem into linear Volterra type integral equations of the form (2.5) by applying first iteration of modified NM (2.3) i.e. $P'(X_0)(X - X_0) + P(X_0) = 0$.
3. Solve the first equation in (2.6) using Gauss–Legendre quadrature rule (3.5) at the Legendre grid points $\tau_i^{(k)}$ (4.2) for $m = 1$.
4. Use Newton forward interpolation formula (4.7) to find the first value of the iteration $x_1(t_i), i = 1, 2, \ldots, n$. Corresponding value of $y_1(t_i)$ can be defined by the formula (4.9) for $m = 1$.
5. To find the next iteration values of $(x_m(t_i), y_m(t_i)), m = 2, 3, \ldots, i = 1, 3, \ldots, n$ we reduce the given nonlinear problem into linear Volterra integral equation of the form (2.9) by applying modified NM (2.8), i.e. $P'(X_0)(X_m - X_{m-1}) + P(X_{m-1}) = 0$.
6. Repeat steps 3–5 by changing the corresponding formula for any m and obtain the approximate values of $(x_m(t_i), y_m(t_i))$.

5. Convergence analysis

For the convergence of the proposed method we construct new majorant function and based on this the few theorems will be proved with regard to the successive approximations which are characterized by system (2.9). Let us introduce the following classes of functions.

- $C_{[t_0,T]}$ the set of all continuous functions $f(t)$ defined on the interval $[t_0, T]$,
- $C_{[t_0,t]\times[t_0,T]}$ the set of all continuous functions $S(t,\tau)$ defined on the region $[t_0, T] \times [t_0, T]$,
- $\overline{C} = \{X : X = (x(t), y(t)) : x(t), y(t) \in C_{[t_0,T]}\}$,
- $\check{C}_{[t_0,T]} = \{y(t) \in C^1_{[t_0,T]} : y(t) < t\}$.

In addition, define the following norms

$$\|x\| = \max_{t\in[t_0,T]} |x(t)|, \ \|\Delta X\|_{\overline{C}} = \max\{\|\Delta x\|_{C_{[t_0,T]}}, \|\Delta y\|_{C_{[t_0,T]}}\},$$

$$\|X\|_{C^1} = \max\{\|x\|_{C_{[t_0,T]}}, \|x'\|_{C_{[t_0,T]}}\},$$

$$\|\overline{X}\|_{\overline{C}} = \max\{\|\overline{x}\|_{C_{[t_0,T]}}, \|\overline{y}\|_{C_{[t_0,T]}}\},$$

$$\|H(t,\tau)\| = H_1, \ \|H'_\tau(t,\tau)\| = H'_1, \ \|K(t,\tau)\| = H_2, \|K'_\tau(t,\tau)\| = H'_2,$$

$$\min_{t\in[t_0,T]} |y_0(t)| = H_3, \ \|x_0'\| = \max_{t\in[t_0,T]} |x_0'(t)| = H_3', \ \ \|g\| = \max_{t\in[t_0,T]} |g(t)| = H_4, \ \ \|f\| = \max_{t\in[t_0,T]} |f(t)| = H_5$$
$$\|\mathbf{F}(x_0(t))\| = \max_{t\in[t_0,\ T]} |\mathbf{F}(x_0(t))| = c_1, \ \ \left\|\mathbf{F}'(x_0(t))\right\| = \max_{t\in[t_0,\ T]} \left|\mathbf{F}'(x_0(t))\right| = c_2,$$
$$\left\|\mathbf{F}''(x_0(t))\right\| = \max_{t\in[t_0,\ T]} \left|\mathbf{F}''(x_0(t))\right| = c_3, \ \ \ \|G(t)\| = \max_{t\in[t_0,\ T]} |G(t)| = c_4, \ \ \max\left|\frac{1}{c(t)}\right| \le s_0.$$

Let

$$\eta_1 = \max\Big\{ H_1c_3(T-H_3),\ H_1c_2,\ H_1'c_1 + H_1H_3'c_2, \\ H_2c_3(T-H_3),\ H_2c_2,\ H_2'c_1 + H_2H_3'c_2 \Big\}. \tag{5.1}$$

and

$$M = c_2H_1 + c_2c_4H_4. \tag{5.2}$$

Introducing the real valued function

$$\psi(t) = (t-t_0)^2 - (\zeta+\eta)(t-t_0) + \zeta\eta, \tag{5.3}$$

where $\zeta, \eta > 0$ are real coefficients and considering the following equations

$$X = S(X), \tag{5.4}$$
$$t = \phi(t), \tag{5.5}$$

The calculation of coefficients ξ and η of the majorant function (5.3) is due to Theorem 7 where the initial guess $(x_0(t), y_0(t))$ must be in $\Omega_0 = (\|X - X_0\| \le r)$, provided that

$$min\{\xi + t_0, \eta + t_0\} \le r \le max\{\xi + t_0, \eta + t_0\}. \tag{5.6}$$

We define the majorant function as follows.

Definition 1 (*Kantorovich and Akilov [7]*)**.** We say that (5.5) majorizes (5.4) if

$$\|S(X_0) - X_0\| \le \phi(t_0) - t_0, \tag{5.7}$$

$$\|S'(X)\| \le \phi'(t), \quad when \ \ \|X - X_0\| \le t - t_0. \tag{5.8}$$

Theorem 5. *Let the nonlinear operator $P(X) = 0$ in (2.2) is defined in an open set $\Omega = \{X \in C([t_0, T]) : \|X - X_0\| < R\}$ and has continuous second derivative in a closed set $\Omega_0 = \{X \in C([t_0, T]) : \|X - X_0\| \le r\}$ such that $T = t_0 + r \le t_0 + R$. Assume the following conditions are satisfied*

1. $\|\Gamma_0 P(X_0)\| \le \dfrac{\zeta\eta}{\zeta+\eta}$,
2. $\|\Gamma_0 P''(X)\| \le \dfrac{2}{\zeta+\eta}$, *when* $\|X - X_0\| \le t - t_0 \le r$,

then $\psi(t)$ in (5.3) is a majorant function for the nonlinear operator $P(X)$ defined by (2.1).

Proof. Rewrite Eqs. (1.1) and (5.3) in the form

$$t = \phi(t), \quad \phi(t) = t + c_0\psi(t), \tag{5.9}$$
$$X = S(X), \quad S(X) = X - \Gamma_0 P(X), \tag{5.10}$$

where $c_0 = -\dfrac{1}{\psi'(t_0)} = \dfrac{1}{\zeta+\eta}$ and $\Gamma_0 = [P'(X_0)]^{-1}$.

We need to show that Eqs. (5.9) and (5.10) satisfy the majorizing conditions (5.7) and (5.8). Indeed

$$\|S(X_0) - X_0\| = \|-\Gamma_0 P(X_0)\| \le \frac{\zeta\eta}{\zeta+\eta} = \phi(t_0) - t_0. \tag{5.11}$$

Now, to show that $\|X - X_0\| \le t - t_0$, $t \in [t_0, T] = [t_0, t_0 + r]$, we consider a sequence

$$\begin{cases} X_{n+1} = S(X_n),\ n = 0, 1, \ldots \\ t_{n+1} = \phi(t_n),\ n = 0, 1, \ldots \end{cases} \tag{5.12}$$

Due to (5.11)–(5.12), we obtain

$$\|X_1 - X_0\| = \|S(X_0) - X_0\| \le \phi(t_0) - t_0 = t_1 - t_0 \le r,$$

hence $X_1 \in \Omega_0$. Assume that it has already been shown that $X_1, X_2, \ldots, X_n \in \Omega_0$ and that

$$\|X_{k+1} - X_k\| \le t_{k+1} - t_k, \quad k = 0, 1, \ldots, n-1. \tag{5.13}$$

To show that $X_{n+1} \in \Omega_0$ we write X and t for the corresponding points in $[X_{n-1}, X_n]$ and $[t_{n-1}, t_n]$

$$\begin{cases} X = X_{n-1} + \tau(X_n - X_{n-1}), & \tau \in (0, 1) \\ t = t_{n-1} + \tau(t_n - t_{n-1}), & \tau \in (0, 1). \end{cases} \tag{5.14}$$

In view of (5.13)–(5.14), we obtain

$$\begin{aligned} \|X - X_0\| &= \|X_{n-1} + \tau(X_n - X_{n-1}) - X_0\| \\ &= \|\tau(X_n - X_{n-1}) + X_{n-1} - X_{n-2} + X_{n-2} + \cdots + X_1 - X_0\| \\ &\le \tau\|X_n - X_{n-1}\| + \|X_{n-1} - X_{n-2}\| + \cdots + \|X_1 - X_0\| \\ &\le \tau(t_n - t_{n-1}) + t_{n-1} - t_{n-2} + \cdots + t_1 - t_0 \\ &\le \tau(t_n - t_{n-1}) + t_{n-1} - t_0 = t - t_0. \end{aligned}$$

Since $\|X - X_0\| \le t - t_0$, with utilizing the remark in [7, 504]

$$X_{n+1} - X_n = S(X_n) - S(X_{n-1}) = \int_{X_{n-1}}^{X_n} S'(X)dX. \tag{5.15}$$

we have

$$\begin{aligned} \|S'(X)\| &= \|S'(X) - S'(X_0)\| \le \int_{X_0}^{X} \|S''(Y)\|dY = \int_{X_0}^{X} \|\Gamma_0 P''(Y)\|dY \\ &\le \int_{t_0}^{t} c_0\psi''(\tau)d\tau = \int_{t_0}^{t} \frac{2}{\zeta + \eta}d\tau = \frac{2}{\zeta + \eta}(t - t_0) = \phi'(t). \end{aligned} \tag{5.16}$$

Due to Eqs. (5.15) and (5.16) we see that

$$\|X_{n+1} - X_n\| = \left\| \int_{X_{n-1}}^{X_n} S'(X)dX \right\| \le \int_{t_{n-1}}^{t_n} \phi'(t)dt = \phi(t_n) - \phi(t_{n-1}) = t_{n+1} - t_n.$$

Thus, we have proved (5.13) holds for $k = n$. On the other hand

$$\begin{aligned} \|X_{n+1} - X_0\| &\le \|X_{n+1} - X_n\| + \|X_n - X_{n-1}\| + \cdots \|X_1 - X_0\| \\ &\le (t_{n+1} - t_n) + (t_n - t_{n-1}) + \cdots + (t_1 - t_0) = t_{n+1} - t_0 \le T - t_0 = r. \end{aligned}$$

Hence $X_{n+1} \in \Omega_0$ and $\psi(t)$ is a majorant function for $P(X) = 0$. □

Theorem 6 (*Kantorovich and Akilov [7, pp. 529]*). *Let us consider*

$$P(X) = 0, \quad X \in \Omega = (\|X - X_0\| < R),$$

and assume that the operator P has continuous second derivative in a closed ball $\Omega_0 = (\|X - X_0\| \le r)$. Assume that the real valued function

$$\psi(t) = 0. \ \ t \in [t_0, t_0 + t'], \quad t' = t_0 + r,$$

has twice continuously differentiable on the interval $[t_0, t_0 + t']$. Suppose the following conditions are satisfied

1. *there exists a continuous linear operator $\Gamma_0 = [P'(X_0)]^{-1}$,*
2. $c_0 = -\frac{1}{\psi'(t_0)} > 0$,
3. $\|\Gamma_0 P(X_0)\| \le c_0\psi(t_0)$,
4. $\|\Gamma_0 P''(X_0)\| \le c_0\psi''(t_0)$ *if* $\|X - X_0\| \le t - t_0 \le r$,
5. *Eq. (6) has a root* $\bar{t} \in [t_0, t']$, $\ t' = t_0 + r$,
6. $\psi(t') \le 0$.

Then if (6) has a unique root in $[t_0, t_0 + t']$ then (6) has only one solution in Ω_0 as well as modified NM for Eqs. (6) and (6) starting with X_0 and t_0 respectively, converges and yields solution X^* and t^* of these equations, where

$$\|X^* - X_0\| \le \|t^* - t_0\|.$$

Main theorem for the proposed method is as follows:

Theorem 7. *Let the functions* $f(t)$, $g(t) \in C_{[t_0,T]}$, $x_0(t) \in C^1[t_0,T]$, *and the kernels* $H(t,\tau)$, $K(t,\tau) \in C^1_{[t_0,T]\times[t_0,T]}$ *and* $(x_0(t), y_0(t)) \in \Omega_0$. *If*

1. $\Gamma_0 = \left[P'(X_0)\right]^{-1}$ *exists and continuous linear operator with* $\|\Gamma_0\| \le s_0 M e^{M(T-H_3)}$, *where M is defined by* (5.2)
2. $\|\Delta X\| \le \dfrac{\zeta\eta}{\zeta+\eta}$, ζ *and* η *are given in* (5.3)
3. $\|\Gamma_0 P''(X)\| \le \dfrac{2}{\zeta+\eta}$, *with* $\|P''(X)\| \le \eta_1$,
4. *Eq.* (5.3) *has a root* $\bar{t} \in [t_0, t']$, $t' = t_0 + r$ *where* $\min\{\zeta + t_0, \eta + t_0\} < r < \max\{\zeta + t_0, \eta + t_0\}$, *with* $\phi(t') \le t'$.

Then the system (1.6) *has a unique solution* $X^* = (x^*, y^*) \in \Omega_0$ *and the sequence* $X_m(t) = (x_m(t), y_m(t))$, $m \ge 0$ *of successive approximations*

$$\Delta x_m(t) - \frac{1}{c(t)}\int_{y_0(t)}^{t} K_1(t,\tau)\mathbf{F}'(x_0(\tau))\Delta x_m(\tau)d\tau = \psi_{m-1}(t),$$

$$\Delta y_m(t) = \frac{1}{d(t)}\left[\int_{y_0(t)}^{t} K(t,\tau)\mathbf{F}'(x_0(\tau))\Delta x_m(\tau)d\tau + \int_{y_{m-1}(t)}^{t} K(t,\tau)\mathbf{F}(x_{m-1}(\tau))d\tau + b(t)x_m(t) - f(t)\right],$$

converges to the solution X^*. *The rate of convergence is given by*

$$\|X^* - X_m\| \le \left(\frac{2\zeta}{\zeta+\eta}\right)^m \zeta,$$

when $\zeta + t_0$ *is the minimum zero of* (5.3), *i.e* $\zeta < \eta$, *otherwise*

$$\|X^* - X_m\| \le \left(\frac{2\eta}{\zeta+\eta}\right)^m \eta,$$

whenever $\eta + t_0$ *is the minimum zero of* (5.3), *i.e* $\eta < \zeta$.

Proof. First, we need to prove that the first equation of system (2.5) has a unique solution $\Delta x^*(t)$ in terms of resolvent kernel Γ_0, provided that $c(t) = a(t) + b(t)G(t) \ne 0$, $K(t, y_0(t)) \ne 0$, $\forall t \in [t_0, T]$ and $K_1(t,\tau)$ which is defined by (2.6), is a continuous function. Then $\Delta y_m(t)$ can be uniquely determined from the second equation of (2.5). Assume that the integral operator $U : C[t_0,T] \to C[t_0,T]$ is given by

$$Z = U(\Delta x), \qquad Z(t) = \frac{1}{c(t)}\int_{y_0(t)}^{t} K_2(t,\tau)\Delta x(\tau)d\tau, \tag{5.17}$$

where $K_2(t,\tau) = K_1(t,\tau)\mathbf{F}'(x_0(\tau))$, and according to (5.17), the first equation of the system (2.5) can be represented as

$$\Delta x_m - U(\Delta x_m) = \psi_{m-1}(t). \tag{5.18}$$

The solution Δx^* of (5.18) is written in terms of ψ_0 by the formula

$$\Delta x^* = \psi_0 + B(\psi_0), \tag{5.19}$$

where B is an integral operator and can be expanded as a series in powers of U [7, pp. 378],

$$B(\psi_0) = U(\psi_0) + U^2(\psi_0) + \cdots + U^n(\psi_0) + \cdots, \tag{5.20}$$

and it is found that the powers of U are also integral operators. Indeed

$$Z_n = U^n, \qquad Z_n(t) = \frac{1}{c(t)}\int_{y_0(t)}^{t} K_2^{(n)}(t,\tau)\Delta x(\tau)d\tau, \quad (n = 1, 2, \ldots), \tag{5.21}$$

where $K_2^{(n)}$ is the iterated kernel operator with $K_2^{(n)}(t,u) = K_2(t,u)K_2^{(n-1)}(u,\tau)$, $n = 1, 2, \ldots$. Substituting (5.20)–(5.21) into (5.19) we get an expression for the solution of (5.18) in the form of

$$\Delta x^* = \psi_0(t) + \sum_{j=1}^{\infty}\int_{y_0(t)}^{t} K_2^{(j)}(t,\tau)\psi_0(\tau)d\tau. \tag{5.22}$$

Next, we proof that the series in (5.22) is convergent uniformly for all $t \in [t_0, T]$. Since

$$
\begin{aligned}
|K_2(t,\tau)| &= |K_1(t,\tau)\mathbf{F}'(x_0(\tau))| = |K_1(t,\tau)||\mathbf{F}'(x_0(t))| \\
&\le \Big[|H(t,\tau)| + |K(t,\tau)||G(t)|\Big]|\mathbf{F}'(x_0(t))| \\
&\le \big(H_1 + c_4H_2\big)c_2 = M.
\end{aligned}
$$

Then by mathematical induction we obtain

$$
\begin{aligned}
\left|K_2^{(2)}(t,\tau)\right| &\le \int_{y_0(t)}^{t} \left|K_2(t,u)K_2(u,\tau)\right|du \le \frac{M^2(T-H_3)}{(1)!}, \\
\left|K_2^{(3)}(t,\tau)\right| &\le \int_{y_0(t)}^{t} \left|K_2(t,u)K_2^{(2)}(u,\tau)\right|du \le \frac{M^3(T-H_3)^2}{(2)!}, \\
&\vdots \\
\left|K_2^{(n)}(t,\tau)\right| &\le \int_{y_0(t)}^{t} \left|K_2(t,u)K_2^{(n-1)}(u,\tau)\right|du \le \frac{M^n(T-H_3)^{n-1}}{(n-1)!}, \quad (n=1,2,\ldots),
\end{aligned}
$$

then

$$
\|U^n\| = \max_{t\in[t_0,T]} \int_{y_0(t)}^{t} \frac{\left|K_2^{(n)}(t,\tau)\right|}{c(t)}d\tau \le \frac{s_0M^n(T-H_3)^{(n-1)}}{(n-1)!}.
$$

Therefore the nth root test of the sequence implies

$$
\sqrt[n]{\|U^n\|} \le \frac{M(T-H_3)^{1-\frac{1}{n}}\sqrt[n]{s_0}}{\sqrt[n]{(n-1)!}} \to_{n\to\infty} 0.
$$

As a result

$$
\rho = \frac{1}{\lim_{n\to\infty}\sqrt[n]{\|U^n\|}} = \infty
$$

and the first equation of the system (2.5) has no characteristic values. Since the series in (5.22) converges uniformly, solution of (5.19) can be expressed in terms of resolvent kernel of the form

$$
\Delta x^* = \psi_0 + \int_{y_0(t)}^{t} \Gamma_0(t,\tau)\psi_0(\tau)d\tau, \tag{5.23}
$$

where

$$
\Gamma_0(t,\tau) = \sum_{j=1}^{\infty} K_2^{(j)}(t,\tau), \tag{5.24}
$$

is the resolvent kernel which is uniquely determined by $K_2^{(j)}(t,\tau)$, so there is a unique solution $\Delta x^*(t)$ for the first equation of system (2.5). Then there is a unique value $\Delta y(t)$ of the second equation of system (2.5). Since the series in (5.24) is convergent therefore

$$
\|\Gamma_0\| = \|B(\psi_0)\| \le \sum_{j=1}^{\infty}\|U^j\| \le s_0\sum_{j=1}^{\infty} M^j\frac{(T-H_3)^{j-1}}{(j-1)!} \le s_0Me^{M(T-H_3)}.
$$

To evaluate the validity of second condition of (2.5), let us describe the operator equation

$$
P(X) = 0, \tag{5.25}
$$

as in (5.10) and its successive approximations is

$$
X_{n+1} = S(X_n), (n=0,1,2,\ldots). \tag{5.26}
$$

For the initial condition X_0 we have

$$
S(X_0) = X_0 - \Gamma_0P(X_0).
$$

From the first condition of Theorem 2 we have

$$
\|\Delta X\| = \|X_1 - X_0\| = \|S(X_0) - X_0\| = \|\Gamma_0P(X_0)\| \le \phi(t_0) - t_0 = \frac{\zeta\eta}{\zeta+\eta}.
$$

Since $\psi(t)$ is a majorant function of $P(X)=0$, from the second condition of Theorem 2, we have

$$\|\Gamma_0 P''(X)\| \le c_0\psi''(t) = \frac{2}{\zeta+\eta}.$$

Moreover, we need to show that $\|P''(X)\| \le \eta_1$ for all $X\in\Omega_0$ where η_1 is defined in (5.1). It is known that the second derivative $P''(X_0)(X,\overline{X})$ of the nonlinear operator $P(X)$ is expressed by 3-dimensional array $P''(X_0)X\overline{X} = (D_1, D_2)(X,\overline{X})$, which is called bilinear operator, that is

$$\begin{aligned}
P''(X_0)(X,\overline{X}) &= \lim_{s\to0}\frac{1}{s}\Big[P'(x_0+s\overline{X}) - P'(X_0)\Big]\\
&= \Bigg\{\lim_{s\to0}\frac{1}{s}\Bigg[\left(\frac{\partial P_1}{\partial x}(x_0+s\bar{x}, y_0+s\bar{y}) - \frac{\partial P_1}{\partial x}(x_0,y_0)\right)x\\
&\quad + \left(\frac{\partial P_1}{\partial y}(x_0+s\bar{x}, y_0+s\bar{y}) - \frac{\partial P_1}{\partial y}(x_0,y_0)\right)y\Bigg],\\
&\quad \lim_{s\to0}\frac{1}{s}\Bigg[\left(\frac{\partial P_2}{\partial x}(x_0+s\bar{x}, y_0+s\bar{y}) - \frac{\partial P_2}{\partial x}(x_0,y_0)\right)x\\
&\quad + \left(\frac{\partial P_2}{\partial y}(x_0+s\bar{x}, y_0+s\bar{y}) - \frac{\partial P_2}{\partial y}(x_0,y_0)\right)y\Bigg]\Bigg\}.
\end{aligned}$$

$$\begin{aligned}
&= \Bigg\{\lim_{s\to0}\frac{1}{s}\Bigg[\Bigg(\frac{\partial^2 P_1}{\partial x^2}(x_0,y_0)s\bar{x} + \frac{\partial^2 P_1}{\partial y\partial x}(x_0,y_0)s\bar{y} + \frac{1}{2}\Bigg(\frac{\partial^3 P_1}{\partial x^3}(x_0+\theta s\bar{x}, y_0+\delta s\bar{y})s^2\bar{x}^2\\
&\quad + 2\frac{\partial^3 P_1}{\partial x^2\partial y}(x_0+\theta s\bar{x}, y_0+\delta s\bar{y})s^2\bar{x}\,\bar{y} + \frac{\partial^3 P_1}{\partial y^2\partial x}(x_0+\theta s\bar{x}, y_0+\delta s\bar{y})s^2\bar{y}\Bigg)\Bigg)x\\
&\quad + \Bigg(\frac{\partial^2 P_1}{\partial x\partial y}(x_0,y_0)s\bar{x} + \frac{\partial^2 P_1}{\partial y^2}(x_0,y_0)s\bar{y} + \frac{1}{2}\Bigg(\frac{\partial^3 P_1}{\partial x^2\partial y}(x_0+\theta s\bar{x}, y_0+\delta s\bar{y})s^2\bar{x}^2\\
&\quad + 2\frac{\partial^3 P_1}{\partial x\partial y^2}(x_0+\theta s\bar{x}, y_0+\delta s\bar{y})s^2\bar{x}\,\bar{y} + \frac{\partial^3 P_1}{\partial y^3}(x_0+\theta s\bar{x}, \delta s\bar{y})s^2\bar{y}^2\Bigg)\Bigg)y\Bigg],\\
&\quad \lim_{s\to0}\frac{1}{s}\Bigg[\Bigg(\frac{\partial^2 P_2}{\partial x^2}(x_0,y_0)s\bar{x} + \frac{\partial^2 P_2}{\partial y\partial x}(x_0,y_0)s\bar{y} + \frac{1}{2}\Bigg(\frac{\partial^3 P_2}{\partial x^3}(x_0+\theta s\bar{x}, y_0+\delta s\bar{y})s^2\bar{x}^2\\
&\quad + 2\frac{\partial^3 P_2}{\partial x^2\partial y}(x_0+\theta s\bar{x}, y_0+\delta s\bar{y})s^2\bar{x}\,\bar{y} + \frac{\partial^3 P_2}{\partial y^2\partial x}(x_0+\theta s\bar{x}, y_0+\delta s\bar{y})s^2\bar{y}\Bigg)\Bigg)x\\
&\quad + \Bigg(\frac{\partial^2 P_2}{\partial x\partial y}(x_0,y_0)s\bar{x} + \frac{\partial^2 P_2}{\partial y^2}(x_0,y_0)s\bar{y} + \frac{1}{2}\Bigg(\frac{\partial^3 P_2}{\partial x^2\partial y}(x_0+\theta s\bar{x}, y_0+\delta s\bar{y})s^2\bar{x}^2\\
&\quad + 2\frac{\partial^3 P_2}{\partial x\partial y^2}(x_0+\theta s\bar{x}, y_0+\delta s\bar{y})s^2\bar{x}\,\bar{y} + \frac{\partial^3 P_2}{\partial y^3}(x_0+\theta s\bar{x}, \delta s\bar{y})s^2\bar{y}^2\Bigg)\Bigg)y\Bigg]\Bigg\}
\end{aligned}$$

$$\begin{aligned}
&= \Bigg(\frac{\partial^2 P_1}{\partial x^2}(x_0,y_0)\bar{x}x + \frac{\partial^2 P_1}{\partial y\partial x}(x_0,y_0)\bar{y}x + \frac{\partial^2 P_1}{\partial x\partial y}(x_0,y_0)\bar{x}y + \frac{\partial^2 P_1}{\partial y^2}(x_0,y_0)\bar{y}x,\\
&\quad \frac{\partial^2 P_2}{\partial x^2}(x_0,y_0)\bar{x}x + \frac{\partial^2 P_2}{\partial y\partial x}(x_0,y_0)\bar{y}x + \frac{\partial^2 P_2}{\partial x\partial y}(x_0,y_0)\bar{x}y + \frac{\partial^2 P_2}{\partial y^2}(x_0,y_0)\bar{y}x\Bigg),
\end{aligned}$$

where $\theta,\delta\in(0,1)$, so we have

$$P''(X_0)(X,\overline{X}) = \begin{pmatrix} D_1 & D_2\end{pmatrix}\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix},$$

where

$$D_1 = \begin{pmatrix} \frac{\partial^2 P_1}{\partial x^2}\Big|_{(x_0,y_0)} & \frac{\partial^2 P_1}{\partial y \partial x}\Big|_{(x_0,y_0)} \\ \frac{\partial^2 P_1}{\partial x \partial y}\Big|_{(x_0,y_0)} & \frac{\partial^2 P_1}{\partial y^2}\Big|_{(x_0,y_0)} \end{pmatrix},$$

$$D_2 = \begin{pmatrix} \frac{\partial^2 P_2}{\partial x^2}\Big|_{(x_0,y_0)} & \frac{\partial^2 P_2}{\partial y \partial x}\Big|_{(x_0,y_0)} \\ \frac{\partial^2 P_2}{\partial x \partial y}\Big|_{(x_0,y_0)} & \frac{\partial^2 P_2}{\partial y^2}\Big|_{(x_0,y_0)} \end{pmatrix}.$$

Then the norms of every components of D_1 and D_2 have the estimate

$$\left\| \frac{\partial^2 P_1}{\partial x^2} \right\| = \max_{\|X\|\le 1, \|\overline{X}\|\le 1} \left| -\int_{y_0(t)}^{t} H(t,\tau)x(\tau)\overline{x}(\tau)\mathbf{F}''(x_0(\tau))d\tau \right| \le H_1 c_3 (T - H_3),$$

$$\left\| \frac{\partial^2 P_1}{\partial x \partial y} \right\| = \max_{\|X\|\le 1, \|\overline{X}\|\le 1} \left| H(t,y_0(t))x(y_0(t))\mathbf{F}'(x_0(y_0(t)))\overline{y}(t) \right| \le H_1 c_2,$$

$$\left\| \frac{\partial^2 P_1}{\partial y \partial x} \right\| = \max_{\|X\|\le 1, \|\overline{X}\|\le 1} \left| H(t,y_0(t))\overline{x}(y_0(t))\mathbf{F}'(x_0(y_0(t)))y(t) \right| \le H_1 c_2,$$

$$\left\| \frac{\partial^2 P_1}{\partial y^2} \right\| = \max_{\|X\|\le 1, \|\overline{X}\|\le 1} \left| \Big[H'_\tau(t,y_0(t))\mathbf{F}(x_0(y_0(t))) + H(t,y_0(t))\mathbf{F}'(x_0(y_0(t)))x'_0(y_0(t)) \Big] y(t)\overline{y}(t) \right| \le H'_1 c_1 + H_1 H'_3 c_2,$$

$$\left\| \frac{\partial^2 P_2}{\partial x^2} \right\| = \max_{\|X\|\le 1, \|\overline{X}\|\le 1} \left| \int_{y_0(t)}^{t} K(t,\tau)x(\tau)\overline{x}(\tau)\mathbf{F}''(x_0(\tau))d\tau \right| \le H_2 c_3 (T - H_3),$$

$$\left\| \frac{\partial^2 P_2}{\partial x \partial y} \right\| = \max_{\|X\|\le 1, \|\overline{X}\|\le 1} \left| -K(t,y_0(t))x(y_0(t))\mathbf{F}'(x_0(y_0(t)))\overline{y}(t) \right| \le H_2 c_2,$$

$$\left\| \frac{\partial^2 P_2}{\partial y \partial x} \right\| = \max_{\|X\|\le 1, \|\overline{X}\|\le 1} \left| -K(t,y_0(t))\overline{x}(y_0(t))\mathbf{F}'(x_0(y_0(t)))y(t) \right| \le H_2 c_2,$$

$$\left\| \frac{\partial^2 P_2}{\partial y^2} \right\| = \max_{\|X\|\le 1, \|\overline{X}\|\le 1} \left| -\Big[K'_\tau(t,y_0(t))\mathbf{F}'(x_0(y_0(t))) + K(t,y_0(t))\mathbf{F}'(x_0(y_0(t)))x'_0(y_0(t)) \Big] y(t)\overline{y}(t) \right| \le H'_2 c_1 + H_2 H'_3 c_2.$$

Therefore, all the second derivatives exist and bounded,

$$\|P''(X)\| \le \eta_1. \tag{5.27}$$

Let us consider the discriminant of equation $\psi(t) = 0$

$$D = \zeta^2 - 2\zeta\eta + \eta^2 = (\zeta - \eta)^2,$$

and the two roots of $\psi(t) = 0$ are $r_1 = \min\{\zeta + t_0, \eta + t_0\}$ and $r_2 = \max\{\zeta + t_0, \eta + t_0\}$. Therefore, when $r_1 < r < r_2$ implies

$$\psi(r) \le 0, \rightarrow \phi(r) \le r, \tag{5.28}$$

then under the assumption of fourth condition; i.e., $\min\{\zeta + t_0, \eta + t_0\}$ is the unique solution of $\psi(t) = 0$ in $[t_0, t']$ and from Theorem 3 it follows that X^* is the unique solution of operator equation (1.6) and

$$\|X^* - X_0\| \le t^* - t_0,$$

where t^* is a unique solution of $\psi(t) = 0$ in $[t_0, r]$, $r_1 = \min\{\zeta + t_0, \eta + t_0\} < r < r_2 = \max\{\zeta + t_0, \eta + t_0\}$. As for the rate of convergence, let us consider Eq. (5.9). Its successive approximation is

$$t_{m+1} = \phi(t_m), \quad m = 0, 1, 2, \ldots$$

where

$$\phi(t_m) = t_m + \frac{1}{\zeta + \eta}\psi'(t_m).$$

To estimate the difference between t^* and the successive approximation t_m

$$t^* - t_m = \phi(t^*) - \phi(t_{m-1}) = \phi'(\tilde{t_m})(t^* - t_{m-1}), \tag{5.29}$$

where, $\tilde{t_m} \in (t_{m-1}, t^*)$ and

$$\phi'(t) = 1 + c_0\psi'(t) = \frac{2}{\zeta + \eta}(t - t_0), \tag{5.30}$$

therefore, in the case of $\zeta + t_0$ is the minimum root of Eq. (5.3)

$$\phi'(\tilde{t_m}) = \frac{2}{\zeta + \eta}(\tilde{t_m} - t_0) \leq \frac{2}{\zeta + \eta}(t^* - t_0) = \frac{2\zeta}{\zeta + \eta},$$

then

$$t^* - t_m \leq \frac{2\zeta}{\zeta + \eta}(t^* - t_{m-1}),$$
$$t^* - t_{m-1} \leq \frac{2\zeta}{\zeta + \eta}(t^* - t_{m-2}),$$
$$\vdots$$
$$t^* - t_1 \leq \frac{2\zeta}{\zeta + \eta}(t^* - t_0),$$

consequently,

$$t^* - t_m \leq \left(\frac{2\zeta}{\zeta + \eta}\right)^m \zeta.$$

It implies

$$\|X^* - X_m\| \leq (t^* - t_m) = \left(\frac{2\zeta}{\zeta + \eta}\right)^m \zeta.$$

In the same manner, if η the minimum root of Eq. (5.3) we have

$$\|X^* - X_m\| \leq (t^* - t_m) = \left(\frac{2\eta}{\zeta + \eta}\right)^m \eta. \quad \square$$

6. Numerical results and discussion

First, let us refer to the notations used here: n is the number of partitions on $[t_0, T]$, ℓ is the number of sub-partition on $(y_0(t_i), t_i)$ and $(y_{m-1}(t_i), t_i)$, $\quad i = 1, 2, \ldots, n$, where m is the number of iterations, and

$$\epsilon_x = \max_{t\in(0,1]} \left| x_m(t) - x^*(t) \right|,$$
$$\epsilon_y = \max_{t\in(0,1]} \left| y_m(t) - y^*(t) \right|.$$

Example 1. Consider the system of nonlinear equation

$$\begin{aligned} x(t) - \int_{y(t)}^{t} t\tau \sin(x(\tau))d\tau &= t + t\sin\left(\frac{9}{10}t\right) - \frac{9}{10}t^2\cos\left(\frac{9}{10}t\right) - t\sin(t) + t^2\cos(t), \\ x(t) + \int_{y(t)}^{t} t^2\tau^2 \sin(x(\tau))d\tau &= t + \frac{81}{100}t^4\cos\left(\frac{9}{10}t\right) - 2t^2 cos\left(\frac{9}{10}t\right) - \frac{9}{5}t^3\sin\left(\frac{9}{10}t\right) \\ &\quad - t^4 cos(t) + 2t^2 cos(t) + 2t^3\sin(t), \quad t \in (0, 1]. \end{aligned} \tag{6.1}$$

The exact solution is

$$x^*(t) = t,$$
$$y^*(t) = \frac{9}{10}t.$$

The initial guess is chosen as

$$x_0(t) = \frac{t^2}{4},$$
$$y_0(t) = \frac{t}{2}.$$

Table 1
Numerical results for (6.1).

$n = 2$, $\ell = 5$, $h = 0.5$.		
m	ϵ_x	ϵ_y
2	0.00485	0.00167
3	$5.87048E-004$	$1.57273E-004$
4	$7.14562E-005$	$1.48333E-005$
5	$8.69208E-006$	$1.39864E-006$
6	$1.05741E-006$	$1.31881E-007$
11	$2.81707E-011$	$9.81992E-013$

Since $c_2 = 1,\ c_4 = \dfrac{2}{t_0^2},\ H_1 = 1,\ H_3 = 0,\ H_4 = \dfrac{49}{10}$ we have $M = 1 + \dfrac{49}{5t_0^2}$, therefore

$$\|\Gamma_0\| \leq \left(1 + \frac{49}{5t_0^2}\right) \exp\left(1 + \frac{49}{5t_0^2}\right), \quad 0 < t_0 \leq 1.$$

The boundedness of Γ_0 implies continuity and existence. Conditions 2 and 3 of Theorem 7 hold because of the majorant function $\psi(t)$. Since $\psi(t) = 0$ has two roots namely

$$t_1 = t_0 + \zeta, \quad t_2 = t_0 + \eta, \quad t_0 \in (0, 1].$$

We assume that $1 = \zeta < \eta = 4$ and $t_0 > 0$, then $t_0 + 1 < r < t_0 + 4$. Thus conditions of Theorem 4 hold so that the successive approximation $X_{n+1} = S(X_n)$ converges to X^* according to consequence of Theorem 7.

In Example 1, the coefficients of majorant function (5.3) are ($\xi = 1, \eta = 4$) are also chosen according to Theorem 7 provided that $t_0 \in (0, 1]$. When $t_0 = 0.1$, then the majorant function will be

$$\psi(t) = (t - 0.1)^2 - 5(t - 0.1) + 4,$$

which has non-negative real two roots $r_1 = \xi + t_0 = 1 + 0.1 = 1.1$ and $r_2 = \eta + t_0 = 4 + 0.1 = 4.4$, then $1.1 \leq r \leq 4.1$. For the approximate computation, discretization formulas (4.4)–(4.5) and (4.8) are used for the values of $n = 2$, $l = 5$ and $m = \{2, 3, 4, 5, 6, 11\}$. Table 1 shows that eleven iterations are needed for $x_m(t)$ and $y_m(t)$ to be very close to $x^*(t)$ and $y^*(t)$ respectively. The fourth and fifth columns of Table 1 refer to the absolute values of difference between the exact solutions $x^*(t)$ and $y^*(t)$ with the iterated solutions $x_m(t)$ and $y_m(t)$ respectively, and m to be least value such that $|x^*(t) - x_m(t)| \leq \epsilon_x$ and $|y^*(t) - y_m(t)| \leq \epsilon_y$.

Example 2. Consider the system of nonlinear equation

$$\begin{aligned} &x(t) - \int_{y(t)}^{t} t\, \tau \log(|x(\tau)|)d\tau = e^t - \frac{t^2}{3}, \\ &\int_{y(t)}^{t} \tau \log(|x(\tau)|)d\tau = \frac{t}{3}, \quad t \in [10, 15]. \end{aligned} \tag{6.2}$$

The exact solution is

$$x^*(t) = e^t,$$
$$y^*(t) = \sqrt[3]{t^3 - t},$$

and the initial guess is

$$x_0(t) = e^{10}(t - 9),$$
$$y_0(t) = 0.6t + 4.$$

In a similar way of Example 1, it can be shown that all conditions $(1)-(4)$ of Theorem 4 hold and therefore the successive approximation $X_{n+1} = S(X_n)$ converges to X^*.

Since $H(t, \tau) = t\tau$, $K(t, \tau) = \tau$, $G(t) = t$, $a(t) = 1$, $b(t) = 0$ we have $K_1(t, \tau) = 0$, $c(t) = 1$ in (2.5) and it is crucial to note that the first equation of (2.5) has the form

$$x_m(t) = \frac{1}{c(t)}(g(t) + f(t)G(t)) = e^t = x^*(t)$$

which is identical with the exact solution.

Table 2 shows that $x_m(t)$ coincides with the exact $x^*(t)$ from the first iteration due to kernel $K_1(t, \tau) \equiv 0$ in (2.6), whereas only six iterations are needed for $y_m(t)$ to be very close to $y^*(t)$.

Table 2
Numerical results for (6.2).

$n = 2,\ \ell = 5,\ h = 0.5.$		
m	ϵ_x	ϵ_y
1	0.00	0.0029
2	0.00	4.3597E−006
3	0.00	3.1061E−008
4	0.00	1.0140E−009
5	0.00	1.2541E−010
6	0.00	3.9968E−011

Table 3
Numerical results for the system (6.3).

$n = 50,\ h = 0.1,\ \ell = 5,\ t \in [10, 15]$				
m	Boykov and Tynda [22]		Modified Newton method	
	ε_x	ε_y	ε_x	ε_y
1	$7.11E-12$	$6.31E-03$	0	$8.61E-5$
2	$8.14E-12$	$6.26E-06$	0	$6.99E-7$
3	$6.12E-15$	$6.19E-09$	0	$5.69E-09$
4	$3.29E-15$	$6.12E-12$	0	$4.67E-11$
6	$3.00E-15$	$7.80E-14$	0	$4.10E-13$

Example 3 (*Boykov and Tynda [22]*). Consider the system of nonlinear Volterra integral equations

$$\begin{aligned} & x(t) - \int_{y(t)}^{t} t\,(t\tau)x(\tau)d\tau = 0 \\ & \int_{y(t)}^{t} \tau x(\tau)d\tau = 6, \quad t \in [10, 15]. \end{aligned} \tag{6.3}$$

where the exact solution is

$$x^*(t) = 6t,$$
$$y^*(t) = \sqrt[3]{t^3 - 3},$$

and the initial guess is

$$\begin{aligned} & x_0(t) = \frac{t}{2}, \\ & y_0(t) = 0.9t < t. \end{aligned} \tag{6.4}$$

It can be easily shown that $K_1(t, \tau) = 0$ in (2.5) and

$$x_m(t) = \frac{1}{c(t)}(g(t) + f(t)G(t)) = 6t = x^*(t)$$

identical with the exact solution. Approximate solution of $y_m(t)$ can be obtained with the second equation of (2.5). The summary is given in Table 3. Comparisons are also made and is shown in Table 3. In fact $x_m(t)$ coincides with the exact $x^*(t)$ from the first iteration due to kernel $K_1(t, \tau) \equiv 0$ in (2.6) whereas only six iterations are needed for $y_m(t)$ to be very close to $y^*(t)$. In Example 3, Boykov [22] did not show how to choose the initial guess, moreover for the proposed method we have chosen the initial guess as shown in (6.4). The proposed method is comparable with the Boikov's method.

7. Conclusion

In this note, the modified Newton method is presented to solve a general 2×2 system of Volterra type integral equations. New majorant function is introduced and certain conditions are imposed in ensuring the uniqueness of the solution. Moreover by choosing the nonlinear function $F(x(t))$ in different forms we are able to solve many types of nonlinear system of integral equations of Volterra type. We have proposed a new idea by introducing a subgrid collocation points $\tau_i^{(k)}$ and $\tau_{i(m-1)}^{(k)}$ which lie in the intervals $(y_0(t_i), t_i)$ and $(y_{m-1}(t_i), t_i)$ respectively. Gauss–Legendre QF is used for each sub-grid intervals. Numerical examples (Tables 1–3) revealed that the accuracy of the modified NM can be achieved by a few numbers of iterations. It is observed that if kernel $K_1(t, \tau) = 0$ in (2.6) then iterations $x_m(t)$ coincides with the exact solution and iteration $y_m(t)$ is approached to exact solutions for a small number of iteration m.

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