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Solving system of nonlinear integral equations by Newton-Kantorovich method

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Abstract. Newton-Kantorovich method is applied to obtain an approximate solution for a system of nonlinear Volterra integral equations which describes a large class of problems in ecology, economics, medicine and other fields. The system of nonlinear integral equations is reduced to find the roots of nonlinear integral operator. This nonlinear integral operator is solved by the Newton-Kantorovich method with initial guess and this procedure is continued by iteration method to find the unknown functions. Finally, numerical examples are provided to show the validity and the efficiency of the method presented.

Keywords: Newton-Kantorovich method, nonlinear operator, Volterra integral equation.
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INTRODUCTION

Finding the solution of nonlinear integral equation is a significant case in engineering, technology, economics, and other fields. In many problems the exact solution is very difficult to obtain, so we use approximate or numerical method to obtain approximate solutions, and one of the well-known approximate method is Newton-Kantorovich method. It attempts to linearize the nonlinear integral equation and finding the approximate solution by processing the convergent sequence. In 1939, Kantorovich [1], published a paper on iterative method for functional equations in a Banach space and derived the convergence theorem for Newton's method. In 1948, he [2], established and proved a semi local convergence theorem for Newton's method in Banach space that is called Newton-Kantorovich method. Many papers are concerned with applying Newton-Kantorovich method to estimate the approximate solution of nonlinear integral equation. In 2009, Amer and Dardery [3], studied the applicability of the Kantorovich method to nonlinear integral equation with shift. In 2010, Jafar and Mahdi [4], developed a method of Newton-Kantorovich and quadrature to solve nonlinear integral equation of the Urysohn form in systematic procedure. In 2012, Ezquerro et al [5], investigated theoretical significance of Newton-Kantorovich method to obtain the existence and uniqueness of solutions of Hammerstein type. In 2013, Babayar and Soltanalizadeh [6], applied Newton-kantorovich to linearize singular nonlinear Volterra integral equation of d dimensions, then found its solution. In 2003, Baikov and Tynda [7], successfully implemented Newton-Kantorovich method to the system of nonlinear Volterra integral equation of the form

$$\begin{cases} x(t) - \int\limits_{y(t)}^{t} h(t,\tau)g(\tau)x(\tau)d\tau = 0, \\ \int\limits_{y(t)}^{t} k(t,\tau)[1-g(\tau)]x(\tau)d\tau = f(t), \end{cases} \tag{1}$$

$0 < t_0 \le t \le T$, $y(t) < t$, where the functions $h(t,\tau), k(t,\tau) \in C_{[0,\infty]\times[t_0,\infty]}$, $f(t), g(t) \in C_{[t_0,\infty]}$ $(0 < g(t) < 1)$, and the unknown function $x(t) \in C_{[0,\infty]}$ $y(t) \in C^1_{[t_0,\infty]}$ $(y(t) < t)$.
In 2010, Eshkuvatov al et [8], solved the system of nonlinear Volterra integral equation of the form

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$$\begin{cases} x(t) - \int\limits_{y(t)}^{t} h(t,\tau)g(\tau)x^n(\tau)d\tau = 0, \\ \int\limits_{y(t)}^{t} k(t,\tau)x^n(\tau)d\tau = f(t), \end{cases} \quad (2)$$

where $n \geq 2, 0 < t_0 \leq t \leq T$, $y(t) < t$, with given functions $H(t,\tau), K(t,\tau) \in C_{[t_0,\infty]\times[t_0,\infty]}$, $f(t) \in C_{[t_0,\infty]}$ and unknown functions $x(t) \in C_{[t_0,\infty]}, y(t) \in C^1_{[t_0,\infty]}$.

In this paper, we consider the system of nonlinear integral equation of the form

$$\begin{cases} x(t) - \int\limits_{y(t)}^{t} H(t,\tau)g(\tau)x^n(\tau)d\tau = g(t), \\ x(t) + \int\limits_{y(t)}^{t} K(t,\tau)x^n(\tau)d\tau = f(t), \end{cases} \quad (3)$$

where $n \geq 2, 0 < t_0 \leq t \leq T$, $y(t) < t$, with given functions $H(t,\tau), K(t,\tau) \in C_{[t_0,\infty]\times[t_0,\infty]}$, $f(t) \in C_{[t_0,\infty]}$ and unknown functions $x(t) \in C_{[t_0,\infty]}, y(t) \in C^1_{[t_0,\infty]}$ .and solve Eq. (3) using Newton-Kantorovich method.

DESCRIPTION OF THE METHOD

The aim of this work is to find the unknown functions $x(t)$ and $y(t)$ approximately satisfying Eq. (3). To do so we rewrite the system (3) in the operator form by introducing

$$\begin{cases} P_1(x(t), y(t)) = x(t) - \int\limits_{y(t)}^{t} H(t,\tau)x^n(\tau)d\tau - g(t) = 0, \\ P_2(x(t), y(t)) = x(t) + \int\limits_{y(t)}^{t} K(t,\tau)x^n(\tau)d\tau - f(t) = 0 \end{cases} \quad (4)$$

Then Eq. (4) can be written as

$$P(X) = \left(P_1(X), P_2(X)\right) = \left(0,0\right), \qquad X = \left(x(t), y(t)\right), \quad (5)$$

To solve (5), we use Newton-Kantorovich method by writing the first iteration in the form

$$P'(X_0)(X - X_0) + P(X_0) = 0, \quad X_0 = (x_0(t), y_0(t)),$$

where $X_0 = \left(x_0(t), y_0(t)\right)$ is the initial guess, and they may be any continuous functions such that, $y(t) < t$, and the derivative $P'(X_0)$ of the nonlinear operator $P(x)$ at the point X_0 is determined by the matrix

$$P'(X_0)=\begin{pmatrix} \left.\frac{\partial P_1}{\partial x}\right|_{(x_0,y_0)} & \left.\frac{\partial P_1}{\partial y}\right|_{(x_0,y_0)} \\ \left.\frac{\partial P_2}{\partial x}\right|_{(x_0,y_0)} & \left.\frac{\partial P_2}{\partial y}\right|_{(x_0,y_0)} \end{pmatrix}$$

Consequently, we have

$$\begin{cases} \left.\frac{\partial P_1}{\partial x}\right|_{(x_0,y_0)}\left(\Delta x(t)\right)+\left.\frac{\partial P_1}{\partial y}\right|_{(x_0,y_0)}\left(\Delta y(t)\right)=-P_1(x_0(t),y_0(t)), \\ \left.\frac{\partial P_2}{\partial x}\right|_{(x_0,y_0)}\left(\Delta x(t)\right)+\left.\frac{\partial P_2}{\partial y}\right|_{(x_0,y_0)}\left(\Delta y(t)\right)=-P_2(x_0(t),y_0(t)), \end{cases} \tag{6}$$

where $\Delta x(t)=x_1(t)-x_0(t)$, $\Delta y(t)=y_1(t)-y_0(t)$ and $\left(x_0(t),y_0(t)\right)$ is the initial given point. To find $x_1(t)$ and $y_1(t)$ we evaluate $P'(X_0)$ by the definition, i.e.

$$\left.\frac{\partial P_1}{\partial x}\right|_{(x_0,y_0)}=x(t)-\int_{y_0(t)}^{t}H(t,\tau)nx^{n-1}{}_0(\tau)x(\tau)d\tau-g(t)=P'_{1x}(x_0,y_0). \tag{7}$$

Likewise

$$\left.\frac{\partial P_2}{\partial x}\right|_{(x_0,y_0)}=x(t)+\int_{y_0(t)}^{t}K(t,\tau)nx^{n-1}{}_0(\tau)x(\tau)d\tau=P'_{2x}(x_0,y_0). \tag{8}$$

Also, by applying L'Hôpital's rule we obtain

$$\left.\frac{\partial P_1}{\partial y}\right|_{(x_0,y_0)}=H(t,y_0(t))x_0^n(y_0(t))y(t)=P'_{1y}(x_0,y_0). \tag{9}$$

In a similar way

$$\left.\frac{\partial P_2}{\partial y}\right|_{(x_0,y_0)}=-K(t,y_0(t))x_0^n(y_0(t))y(t)=P'_{2y}(x_0,y_0). \tag{10}$$

Therefore system (7) reduces to the system of linear Volterra integral equations with respect to $\Delta x_m,\Delta y_m$

$$\left.\begin{aligned}&\Delta x_m(t)-\int_{y_0(t)}^{t} H(t,\tau)nx_0^n(\tau)\Delta x_m(\tau)d\tau+H(t,y_0(t))x_0^n(t)\Delta y_m(t)\\&\quad=\int_{y_{m-1}(t)}^{y} H(t,\tau)x_{m-1}^n(\tau)d\tau-x_0(t)+g(t)\\&\Delta x_m(t)+\int_{y_0(t)}^{t} H(t,\tau)nx_0^n(\tau)\Delta x_m(t)d\tau-K(t,y_0(t))x_0^n(t)\Delta y_m(t)\\&\quad=-\int_{y_{m-1}(t)}^{t} K(t,\tau)x_{m-1}^n(\tau)d\tau-x_{m-1}(t)+f(t)\end{aligned}\right\}. \tag{11}$$

Eq. (11) is a linear Volterra integral equation with respect to. $\Delta x_m(t)$,so we can reduce it to be the following form

$$\Delta x_m(t)-\int_{y_0(t)}^{t} K_1(t,\tau)nx_0^{n-1}(\tau)\Delta x_m(\tau)d\tau=F_{m-1}(t), \tag{12}$$

Where

$$K_1(t,\tau)=\frac{G(t)H(t,\tau)-K(t,\tau)}{1+G(t)}, \tag{13}$$

$$G(t)=\frac{K(t,y_0(t))}{H(t,y_0(t))}., \tag{14}$$

and $F_{m-1}(t)$ has the form

$$F_{m-1}(t)=\int_{y_{m-1}(t)}^{t} K_1(t,\tau)x_{m-1}^n(\tau)d\tau-x_{m-1}(t)+\frac{G(t)}{1+G(t)}g(t)+\frac{1}{1+G(t)}f(t). \tag{15}$$

By continuing this process, we obtain a sequence of approximate solution $(x_m(t),y_m(t))$ where $\Delta x_m(t)=x_m(t)=x_{m-1}(t)$ and $\Delta y_m(t)=y_m(t)=y_{m-1}(t)$, $m=1,2,3,...$

DESCRITIZATION OF THE METHOD

To discretize Eq. (12) we use trapezoidal rule and collocation method. Let the grid (ω) points be chosen as $t_i=t_0+ih,\ i=0,1,2,...,N$,where $h=\frac{T-t_0}{N}$, on the interval $[t_0,T]$.Then from Eq.(12) we have

$$\Delta x_m(t_0)=F_{m-1}(t_0)=-x_{m-1}(t_0)+\frac{G(t_0)}{1+G(t_0)}g(t_0)+\frac{1}{1+G(t_0)}f(t_0), \tag{16}$$

$$\Delta x_m(t_i) - \int_{y_0(t_i)}^{t_i} K_1(t_i,\tau) n x_0^{n-1}(\tau) \Delta x_m(\tau) d\tau = F_{m-1}(t_i), i = 1,2,...,N, \tag{17}$$

On the grid (ω) we set $v_i = y_0(t_i)$, such that $t_{v_i} = \begin{cases} t_{v_i} & t_0 \le y_0(t) < t_{i-1}, \\ t_i & t_{i-1} \le y_0(t_i) < t_i. \end{cases}$

Then the Eq. (17) can be written in the form

$$\begin{aligned} &\Delta x_m(t_i) - n \sum_{j=v_i}^{i-1} \int_{t_j}^{t_{j+1}} K_1(t_i,\tau) x_0^{n-1}(\tau) \Delta x_m(\tau) d\tau \\ &\quad - n \int_{y_0(t_i)}^{t_{v_i}} K_1(t_i,\tau) x_0^{n-1}(\tau) \Delta x_m(\tau) d\tau = F_{m-1}(t_i). \end{aligned} \tag{18}$$

By computing the integrals in Eq. (18) using trapezoidal formula we consider two cases

Case 1: If $v_i \ne i$, then

$$\Delta x_m(t_i) = \frac{F_{m-1}(t_i) + A + B - 0.5n(t_i - t_{i-1}) K_1(t_i, t_{i-1}) x_0^{n-1}(t_i - 1) \Delta x_m(t_{i-1})}{1 - 0.5n(t_i - t_{i-1}) K_1(t_i, t_i) x_0^{n-1}(t_i)} \tag{19}$$

where

$$\begin{aligned} A &= 0.5n(t_{v_i} - y_0(t_i)) \Big[K_1(t_i, t_{v_i}) x_0^{n-1}(t_{v_i}) \Delta x_m(t_{v_i}) \\ &\quad + K_1(t_i, y_0(t_i)) x_0^{n-1}(y_0(t_i) \left(\frac{\Delta x_m(t_{v_i})(t_{v_i} - y_0(t_i)) + \Delta x_m(t_{v_i-1})(y_0(t_i) - t_{v_i-1})}{t_{v_i} - t_{v\ -1}} \right) \Big], \\ B &= 0.5n \sum_{j=v_i}^{i-1} (t_{j+1} - t_j) \Big[K_1(t_i, t_{j+1}) x_0^{n-1}(t_{j+1}) \Delta x_m(t_{j+1}) + K_1(t_i, t_j) x_0^{n-1}(t_j) \Delta x_m(t_j) \Big]. \end{aligned}$$

Case 2: $v_i = i$. In this case $B = 0$,we obtain

$$\begin{aligned} \Delta x_m(t_i) &= \Big(F_{m-1}(t_i) + 0.5n(t_i - t_{i-1}) K_1(t_i, t_i) x_0^{n-1}(t_{i-1}) \Delta x_m(t_{i-1}) \\ &\quad + 0.5n K_1(t_i, y_0(t_i)) x_0^{n-1}(t_{i-1}) \Delta x_m(t_{i-1}) \frac{(t_i - y_0(t_i))(y_0(t_i) - t_{i-1})}{t_i - t_{i-1}} \Big) / \\ &\quad \Big(1 - 0.5n(t_i - t_{i-1}) K_1(t_i, t_i) x_0^{n-1}(t_i) - 0.5(t_i - y_0(t_i)) K_1(t_i, t_i) x_0^{n-1}(t_i) \\ &\quad - 0.5n K_1(t_i, y_0(t_i)) x_0^{n-1}(y_0(t_i)) \frac{(t_i - y_0(t_i))^2}{t_i - t_{i-1}} \Big) \end{aligned} \tag{20}$$

Also, in similar way, $\Delta y_m(t)$ can be computed on the grid (ω).

NUMERICAL RESULTS

Consider the system on nonlinear Volterra integral equation

$$
\begin{cases}
x(t) - \int\limits_{y(t)}^{t} t^2 \tau\, x^3(\tau)\, d\tau = 2t - 8t^2, \\
x(t) + \int\limits_{y(t)}^{t} t\tau x^3(\tau)\, d\tau = 10t,
\end{cases}
\tag{21}
$$

where $t \in [t_0, T] = [10,15]$. .The exact solution of Eq. (21) is $x^*(t) = 2t,\ \ y^*(t) = \sqrt[5]{t^5 - 5}$. The initial function $x_0(t)$ can be any continuous function, also $y_0(t)$ can be any continuous function providing $y_0(t) < t$. In this particular example, we have chosen $x_0(t) = 2t$ and $y_0(t) = 4 + 0.6t$.

TABLE (1). Numerical result for system (21), $N = 20,\ t_0 = 10,\ T = 15,\ h = 0.25$

m	ε_x	ε_y
2	0.0	0.398815541197996
6	0.0	0.082579049719763
10	0.0	0.019117526537547
40	0.0	0.000003351567831

Table 1, shows that the sequence $x_m(t)$ coincides with the exact solution $x^*(t)$ from the first iteration whereas the sequence $y_m(t)$ needs long iteration to agree with the exact solution $y^*(t)$. The notations used are: N is the number of nodes; m is the number of iterations,

Where, $\varepsilon_y = \max\limits_{t\in[t_0,T]} \left| y_m(t) - y^*(t) \right|$ and $\varepsilon_y = \max\limits_{t\in[t_0,T]} \left| y_m(t) - y^*(t) \right|$

CONCLUSION

In the present note, we have developed the Newton-Kantorovich method to solve a system of 2×2 nonlinear Volterra integral equations by reducing it to be nonlinear operator equation, then finding the roots of operator equation by solving linear Volterra integral equation. From the theory of linear Volterra integral equations, we can obtain the unique solution. We have introduced a numerical example in order to show the efficiency of the method used.

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