Stability Study of Stationary Solutions of the Viscous Burgers Equation

Mohammad Sabawi
College of Computer sciences and Mathematics
University of Mosul

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الملخص

تمت دراسة استقرارية الحلول اللازمنية لمعادلة Burgers اللزجة باستخدام تحليل الأستقرارية من النمط Fourier للحلول اللازمنية $u_1 = D$ إذ D ثابت و $u_1 = u_1(x)\,, 0 \le x \le 1$ في حالتين: الأولى في حالة كون السعة الموجية A ثابتة والثانية في حالة كون السعة الموجية A متغيرة وان النتائج التي تم الحصول عليها في حالة السعة الثابتة هي: الحل اللازمني $u_1 = D$ هو دوماً مستقر في حين أن الحل $u_1 = u_1(x)$ مستقر على نحو مشروط. وان النتائج التي تم الحصول عليها في حالة السعة المتغيرة هما الحلان اللازمنيان $u_1 = D$ و $u_1 = u_1(x)\,, 0 \le x \le 1$ وهما مستقران على نحو مشروط.

ABSTRACT

Stability study of stationary solutions of the viscous Burgers equation using Fourier mode stability analysis for the stationary solutions $u_1 = D$, where D is constant and $u_1 = u_1(x)\,, 0 \le x \le 1$, in two cases is analyzed. Firstly when the wave amplitude A is constant and secondly when the wave amplitude A is variable. In the case of constant amplitude, the results found to be: The solution $u_1 = D$ is always stable while the solution $u_1 = u_1(x)$ is conditionally stable. In the case of variable amplitude, it has been found that the solutions $u_1 = D$ and $u_1 = u_1(x)\,, 0 \le x \le 1$ are conditionally stable.

1. Introduction

Consider a system of any nature whatsoever that exists in a state S. We say that S is stable, in one sense or another, if small perturbations or changes in the system do not drastically affect the state S. For example, the solar system currently exists in a time–dependent state in which the planets move about the sun in an orderly fashion. It is known that if a small additional celestial body is introduced into the system, then the original state is not disturbed to any significant degree. We say that the original state is stable to small perturbations. Similar questions of stability arise in every physical problem [19]. The notorious Burgers equation was the subject of interest study in different fields such as analytical solutions, numerical solutions, mathematical modeling, fluid mechanics, stability and bifurcation. Roy and Baker [27] presented and derived the numerical results using a nonlinear subgrid embedded (SGM) finite element basis for $1D$, $2D$ and $3D$ verification/benchmark linear and nonlinear convection–diffusion problems such as Burgers equation in steady state.

Burns et al [8] considered the numerical stationary solutions for a viscous Burgers equation on the interval $(0,1)$ with Neumann boundary conditions. Roy and Fleming [28] developed a nonlinear subgrid embedded (SGM) finite element basis for generating multidimensional solutions for convection–dominated computational fluid dynamics (CFD) applications and they applied them to a stationary Burgers equation. Balogh and Krstic [4] considered the viscous Burgers equation under recently proposed nonlinear boundary conditions and they showed that it guarantees global asymptotic stabilization and semi global exponential stabilization in H^1 sense. Balogh et al [5] studied the stationary solutions of a one–parameter family of boundary control problems for a forced viscous Burgers equation. They assumed that the forcing term possesses a special symmetry. Allen et al [2] studied numerically the equilibrium solutions of Burgers equation. Moller [23] studied and conducted some numerical experiments on the $1\boldsymbol{D}$ viscous Burgers equation in linear and nonlinear cases with the same stationary solution.

Di Francesco and Markowich [11] studied the large time behavior for the viscous Burgers equation with initial data in $L^1(R)$. They reduced the rescaled Burgers equation to the linear Fokker–Planck equation and then employed well known results concerning the decay in relative entropy and in Wassertstein metric towards stationary solutions for the Fokker–Planck equation. Holm and Staley [14] studied the exchange of stability in the dynamics of solitary wave solutions under changes in the nonlinear balance in a $1+1$ evolutionary PDE related both to shallow water waves and turbulence such as Burgers equation.

Bakhtin [3] considered the existence and uniqueness of stationary solutions for $3D$ Navier–Stokes system in the Fourier space with regular forcing given by a stationary in time stochastic process satisfying a smallness condition. The method of constructing stationary solutions is actually applicable for the Burgers equation. Kowalczyk et al [17] studied in details the linear stability analysis of homogeneous solutions to some aggregation models such as in viscid Burgers like equations. Konicek et al [16] derived a new approximate solution of the inhomogeneous Burgers equation for real fluid in stationary state regime using Prandtl's technique and verified the validity of the approximate solution by comparison with the numerical one. Roy [26] examined the numerical solutions to $1D$ Burgers equation in unsteady and steady states.

In this paper, the stability of stationary solutions of viscous Burgers equation using Fourier mode stability analysis is investigated.

2. The Mathematical Model

One of the famous nonlinear diffusion equations is the generalized Burgers–Huxley (gBH) equation [30]:

$$u_t + \alpha\, u^d u_x - \in u_{xx} = bu\left(1-u^d\right)\left(u^d - a\right) \qquad (1)$$

where α , b , d , $\in$ *and* a are constant parameters

$\alpha \geq 0$, $b \geq 0$, $d > 0$, $\in > 0$, $-1 \leq a < 1$

where $\in$ is the diffusion coefficient and in fluid flow problems it represents the viscosity and is the reciprocal of the Reynolds number.

Equation (1) is an extended form of the famous Huxley, Newell–Whitehead (NW) and Burgers equations [12]. When $\alpha = 0$, equation (1) is reduced to the generalized Huxley or generalized Fitzhugh–Nagumo (gFN) equation.

$$u_t - \in u_{xx} = b\, u\left(1-u^d\right)\left(u^d - a\right) \qquad (2)$$

Huxley equation is a particular case of Eq.(1) and (2) when $\alpha = 0$, $d = 1$ and $d = 1$, respectively [22].

$$u_t - \in u_{xx} = b\, u(1-u)(u-a) \qquad (3)$$

which describes nerve pulse propagation in nerve fibers and wall motion in liquid crystals. The parameter a arises in genetics and other fields, the case with $0 < a < 1$ is what the geneticists refer to as the heterozygote inferiority case [15]. Manaa and Moheemmeed studied the stability [20] and the numerical solution [21] of this case.

The standard real Newell–Whitehead (rNW) equation is a special case of Eq. (2) and (3) when $a = -1, d = 1$ and $d = 1$, respectively.

$$u_t - \in u_{xx} = b\,u\left(1-u^2\right) \qquad (4)$$

Newell and Whitehead examined this equation in 1969 [24]. When $b=0$, Eq. (1) is reduced to the generalized Burgers equation

$$u_t + a\,u^d u_x - \in u_{xx} = 0 \qquad (5)$$

The well known viscous Burgers equation is a special case of Eq.(1) and (5) when $b=0$, $d=1$ and $d=1$,respectively [18].

$$u_t + a\,u u_x - \in u_{xx} = 0 \qquad (6)$$

Burgers equation provides remarkable system that has been studied for some time by Bateman in 1915 and was extensively developed by Burgers in 1940 and 1948 as a simplified fluid flow model which, nonetheless, exhibits some of the important aspects of turbulence. It was later derived by Lighthill in 1956 as a second–order approximation to the one–dimensional unsteady Navier–Stokes equation [5]. The Burgers equation can be seen as a reduction of the Navier– Stokes equation to the case of a single space dimension. In this equation, a controls the nonlinearity and $\in$ stands for viscosity. It is perhaps the simplest nonlinear differential second order equations, and it has been considered to describe different physical problems such as sound waves in viscous media, the far field of wave propagation in nonlinear dissipative systems, shock waves, magnetohydrodynamic waves in media with finite electrical conductivity, nonlinear heat diffusion and viscous effects in gas dynamics [6]. The study of the viscous Burgers equation is naturally related to that of the in viscid Burgers equation [11]:

$$u_t + a u u_x = 0 \qquad (7)$$

The heat equation corresponds to the linearized Burgers equation

$$u_t - \in u_{xx} = 0 \qquad (8)$$

It is known that nonlinear diffusion equations (3) and (6) play important roles in nonlinear physics. They are of special significance for studying nonlinear phenomena. If we take $d=1$ and $a \neq 0$, $b \neq 0$, equation (1) becomes the following Burgers–Huxley (BH) equation:

$$u_t + a u u_x - \in u_{xx} = b u(1-u)(u-a) \qquad (9)$$

Equation (9) shows a prototype model [30] for describing the interaction between reaction mechanisms, convection effects and diffusion transport. Also, Burgers equation is a particular case of following convection–reaction–diffusion equation.

$$u_t + au^d u_x - \in u_{xx} = bu\left(1 - u^d\right) \tag{10}$$

The equation (10) is the generalized Burgers–Fisher (gBF) equation, the generalized Burgers and Burgers equations correspond to the cases $b = 0$ and $b = 0$, $d = 1$, respectively. When $a = 0$, equation (10) is reduced to the generalized Fisher equation

$$u_t - \in u_{xx} = bu\left(1 - u^d\right) \tag{11}$$

when $d = 1$, We have Fisher–Kolmogorov–Petrovskii–Piskunov (Fisher–KPP) or Fisher equation [13]:

$$u_t - \in u_{xx} = bu\left(1 - u\right) \tag{12}$$

The case $d = 1$ in Eq.(10) is the Burgers–Fisher equation [30]:

$$u_t + auu_x - \in u_{xx} = bu(1 - u) \tag{13}$$

There is another Burgers type equation named the generalized Burgers–Korteweg–de Vries equation [31]:

$$u_t + au^d u_x - \in u^t u_{xx} + gu_{xxx} = 0 \tag{14}$$

where a, d *and* t are positive real numbers. It reduces to the generalized Burgers and Burgers equations for $t = 0$, $g = 0$ and $t = 0$, $g = 0$, $d = 1$, respectively. It also reduces to the generalized Korteweg–de Vries (gKdV) and standard Korteweg–de Vries (KdV) equations for $\hat{I} = 0$ and $\hat{I} = 0$, $d = 1$, respectively.

$$u_t + a\, u^d u_x + gu_{xxx} = 0 \tag{15}$$

$$u_t + a\, u u_x + gu_{xxx} = 0 \tag{16}$$

The Burgers–Korteweg–de Vries or Burgers–KdV equation [18] is special case of Eq. (14) when $d = 1$ and $t = 0$.

$$u_t + au u_x - \in u u_{xx} + gu_{xxx} = 0 \tag{17}$$

which reduces to the Burgers and KdV equations when $g = 0$ and $\in = 0$, respectively.

3. Introduction to the Burgers Type Equations

Burgers type equations are famous nonlinear equations which, appear in different scientific fields and play significant role in the study of the nonlinear evolution equations in applied mathematics. Satsuma–Burgers–Huxley (SBH) equation [9], [10] considers another type of the Burgers type equation with reaction term:

$$u_t - (1-3\in) u\, u_x - \in u_{xx} = (1-\in)\left(-u^3 + bu + d\right) \qquad (18)$$

where $b, d \neq 0$. Burgers equation corresponds to the case $\in = 1$. When $\in = 1/3$, SBH equation reduces to Fitzhugh–Nagumo–Kolmogorov–Petrovskii–Piskunov (FN–KPP) equation, which arises in population dynamics and other fields

$$u_t - \left(\frac{1}{3}\right) u_{xx} = \left(\frac{2}{3}\right)\left(-u^3 + bu + d\right) \qquad (19)$$

The case $\in = 0$, corresponds to the first order equation

$$u_t - u\, u_x = \left(-u^3 + bu + d\right) \qquad (20)$$

One of the important models related to both shallow water waves and to turbulence is the b–equation [14]:

$$\underbrace{m_t}_{\textbf{Evolution}} + \underbrace{u\, m_x}_{\textbf{Convection}} + \underbrace{bm u_x}_{\textbf{Stretching}} - \underbrace{\in m_{xx}}_{\textbf{Viscosity}} = 0 \qquad (21)$$

with $m = u - s^2 u_{xx}$, $b = 0, \pm 1, \pm 2, \pm 3, \ldots$. The equation (21) contains a family of equations. For $b = 0$, $s = 0$, equation (21) is reduced to Burgers equation. The case $b = 2$ restricts (21) to the Cammassa–Holm (CH) equation

$$m_t + um_x + 2mu_x - \in m_{xx} = 0 \qquad (22)$$

The case $b = 3$ is the Degasperis–Procesi (DP) equation

$$m_t + um_x + 3mu_x - \in m_{xx} = 0 \qquad (23)$$

Let us consider the generalized Burgers equation (5), this equation is named generalized since it contains the quantity $\boldsymbol{u^d}$ in the convection term $a\, u^d u_x$. We can get another generalized Burgers equations by changing the properties of the nonlinear term $a\, u^d u_x$. The generalizations of Burgers and

Burgers–Huxley equations, for which only relaxation of the assumption of weak nonlinearity is made. This means that no change in the original equations is made to introduce other effects, like including a new term to describe dispersion for instance, but just changing the nonlinear properties of the original system, for the generalized Burgers equation, for example, the consideration of the dynamics of diffusion in media where nonlinearity is not just restricted to the simplest case. If we replace the nonlinear term $a\, u^d u_x$ in (5), we get another generalized Burgers equation [6]:

$$u_t + g(u)u_x - \in u_{xx} = 0 \tag{24}$$

where $\boldsymbol{g}(\boldsymbol{u})$ is a smooth function of u. The Burgers equation (6) is obtained with the linear function $\boldsymbol{g}(\boldsymbol{u}) = a\, \boldsymbol{u}$. Like the Burgers equation (6), the generalized Burgers equation (24) also combines nonlinearity and diffusion, but now nonlinearity is controlled by $\boldsymbol{g}(\boldsymbol{u})$ and may vary according to the model one considers, note that the Burgers equation is defined with the simplest nontrivial function $\boldsymbol{g} = \boldsymbol{g}(\boldsymbol{u})$. If we take $\boldsymbol{g}(\boldsymbol{u}) = 3a\, \boldsymbol{u}^2$ in (24), we get:

$$u_t + 3a\, u^2 u_x - \in u_{xx} \tag{25}$$

This equation is named the modified Burgers equation, since it contains nothing but the change $u \to 3u^2$ in its nonlinear term. Equation (24) can be written in the form:

$$u_t + f_x - \in u_{xx} = 0 \tag{26}$$

and for $f = f(u)$, we get:

$$u_t + \frac{df}{du} u_x - \in u_{xx} = 0 \tag{27}$$

This form is interesting since it allows a natural extension to systems where two or more configurations interact with each other. The equation (27) can be extended to the system of two coupled Burgers type equations

$$\left.\begin{array}{l} u_t + f_x - \in u_{xx} = 0 \\ v_t + g_x - \overline{\in}\, v_{xx} = 0 \end{array}\right\} \tag{28}$$

where $u = u(x,t)$ *and* $v = v(x,t)$ are the two interacting configurations. For $f = f(u,v)$ *and* $g = g(u,v)$, we can write (28) as:

$$\left.\begin{aligned} u_t + f_u u_x + f_v v_x - \in u_{xx} = 0 \\ v_t + g_u u_x + g_v v_x - \overline{\in}\, v_{xx} = 0 \end{aligned}\right\} \qquad (29)$$

The generalized Burgers equation (24) can be further extended to the following form

$$u_t + \frac{df}{du} u_x - \in u_{xx} = bh(u) \quad or \quad u_t + f_x - \in u_{xx} = bf_u \qquad (30)$$

where $f = f(u)$ *and* $h = h(u)$ are smooth functions. Equation (30) represents another generalized Burgers–Huxley or generalized Burgers–Fisher equations, which differ from the equations (1) and (10) by changing the nonlinear term. If we take $f(u) = h(u)$ this is very interesting since we can relate the equation (30) to relativistic $1+1$ dimensional systems of scalar fields, and so we can get different equations and solutions given in terms of different functions $f = f(u)$. If we take $f(u) = h(u) = u(a - u^2)$ in (30), we have:

$$u_t + (a - 3u^2)u_x - \in u_{xx} = b\,u(a - u^2) \qquad (31)$$

which is named the modified Burgers–Huxley (mBH) equation. Equation (30) can be further generalized to the case where several configurations interact with each other. In the case of two configurations $u(x,t)$ and $v(x,t)$, equation (30) is extended to the following system of pair of coupled Burgers–Huxley equations [6]:

$$\left.\begin{aligned} u_t + f_u u_x + f_v v_x - \in u_{xx} = b\ f(u,v) \\ v_t + \bar{f}_v v_x + \bar{f}_u u_x - \overline{\in}\, v_{xx} = \bar{b}\ \bar{f}(u,v) \end{aligned}\right\} \qquad (32)$$

If KdV and Burgers–Huxley equations are added, we get the generalized KdV–Burgers–Huxley (gKdVBH) equation [7]:

$$u_t + f_x + g_{xx} + g\,u_{xxx} = h(u) \qquad (33)$$

where f, g *and* h are smooth functions in *u*. It contains several interesting particular cases. For $h(u) = 0$, it corresponds to the generalized KdV–Burgers (gKdVB) equation:

$$u_t + f_x + g_{xx} + g\,u_{xxx} = 0 \quad or \quad u_t + f_u u_x + g_{uu} u_{xx} + g\,u_{xxx} = 0 \qquad (34)$$

For $f = f(u,v)$ *and* $g = g(u,v)$, we get the standard (KdVB) equation (17). The (KdV) and Burgers equations were first added [7] to describe properties of waves in liquid–filled elastic tubes.

For $f(u) = a\,u^3$, $g(u) = -\in u$ and $h(u) = 0$, it represents the modified KdVB (mKdVB) equation:

$$u_t + 3a\,u^2 u_x - \in u_{xx} + g\,u_{xxx} = 0 \qquad (35)$$

For $\boldsymbol{g}$ trivial, i.e. $\boldsymbol{g} = 0$, we get the generalized KdV–Huxley (gKdVH) equation since it is similar to the generalized BH equation but with diffusion term present in the BH case changed by the dispersion term present in the KdV case. The equation (34) can also be extended to a system of coupled gKdVB equations in the form [7]:

$$\left.\begin{aligned} &u_t + f_x + g_{xx} + g\,u_{xxx} = 0 \\ &v_t + \bar{f}_x + \bar{g}_{xx} + \bar{g}v_{xxx} = 0 \end{aligned}\right\} \qquad (36)$$

Here $f = f(u,v)$ *and* $g = g(u,v)$ are odd in u and even in v, and $\bar{f} = \bar{f}(u,v)$ *and* $\bar{g} = \bar{g}(u,v)$ are even in u and odd in v**,** in order to preserve the symmetries in the (u,v) space of the original equations. These smooth functions allow us to write the above equations in the form:

$$\left.\begin{aligned} &u_t + f_u u_x + f_v v_x + g_u u_{xx} + g_v v_{xx} + g_{uu}(u_x)^2 \\ &+ 2g_{uv} u_x v_x + g_{vv}(v_x)^2 + g\,u_{xxx} = 0 \\ &v_t + \bar{f}_u u_x + \bar{f}_v v_x + \bar{g}_u u_{xx} + \bar{g}_v v_{xx} + \bar{g}_{uu}(u_x)^2 \\ &+ 2\bar{g}_{uv} u_x v_x + \bar{g}_{vv}(v_x)^2 + \bar{g}\,u_{xxx} = 0 \end{aligned}\right\} \qquad (37)$$

The nonlinear differential equations in the generic form [25]:

$$u_t + P(u)u_x - \in u_{xx} + g\,u_{xxx} = A(u) \qquad (38)$$

with polynomial functions defined as:

$$P(u) = \sum_{i=0}^{Np} p_i u_i \qquad (39)$$

$$A(u) = \sum_{i=0}^{Np} a_i u_i \qquad (40)$$

The general form of equation (38) allows the identification of several interesting cases. For instance, the gKdVBH equation is recovered from Eq.(38) for

$$\boldsymbol{\frac{df}{du} = p(u)\ ,\ \ g(u) = g_o - \hat{\text{I}}\ u\ ,\ \ and\ \ h(u) = A(u)}\ .$$

Furthermore, the standard KdVB equation corresponds to identifying $\boldsymbol{P(u)} = a\ \boldsymbol{u}$, ***and*** $\boldsymbol{A(u)} = 0$, and the modified KdVB equation [25] requires $\boldsymbol{P(u)} = 3a\ \boldsymbol{u}^2$, ***and*** $\boldsymbol{A(u)} = 0$, with the particular case $\in = 0$ accounting respectively for the standard and modified KdV equations:

$$u_t + 3au^2u_x + gu_{xxx} = 0 \qquad (41)$$

Equation (41) represents the mKdV equation. On the other hand, the BH equation represents the situation in which $\boldsymbol{P(u)} = a\ \boldsymbol{u}$, $g = 0$, ***and*** $\boldsymbol{A(u)} = \boldsymbol{h(u)}$.

$$u_t + a\ uu_x - \in u_{xx} = h(u) \qquad (42)$$

with the case $h(u) = 0$ corresponding to the standard Burgers equation , which has an important connection with the deterministic Kardar–Parisi–Zhang (KPZ) equation in one spatial dimension, known to provide the evolution of the profile of a growing interface or a domain wall of general nature. Eq.(42) sometimes is named the inhomogeneous Burgers equation or Burgers equation with reaction term [29] but when $h(u) = 0$ it is named the homogeneous Burgers equation.

4. The Non-dimensional Transformations

For non-dimensional form, we introduce the following non-dimensional quantities:

$$x' = \frac{x}{L} \quad , \quad t' = \frac{a\ t}{L} \quad , \quad 0 \le x \le L$$

By substituting these dimensionless quantities in (6), we get:

$$u_{t'} + u\,u_{x'} - \frac{1}{\text{Re}} u_{x'x'} = 0 \qquad , \qquad 0 \le x' \le 1$$

Here $aL/\in$ represents the Reynolds number if we set $\text{Re} = aL/\in$ and omit the primes in the equation in above, we get:

$$u_t + u\,u_x - \frac{1}{\text{Re}} u_{xx} = 0 \qquad (43)$$

$$\boldsymbol{u}(0,\boldsymbol{t}) = \boldsymbol{a} \ , \ \boldsymbol{u}(1,\boldsymbol{t}) = -\boldsymbol{a} \ , \ 0 \le \boldsymbol{x} \le 1 \quad , \ \boldsymbol{a} > 0$$

The equation (43) with the boundary conditions represents the non-dimensional Burgers equation in x and t.

5. Fourier Mode Stability Analysis

Let the solution of equation (43) has the following form [19]:

$$u(x,t)=u_1(x)+u_2(x,t) \tag{44}$$

where $u_1(x)$ is the steady state solution and $u_2(x,t)$ is the disturbance or perturbation .

Substitute (44) in (43), with its boundary condition, we have:

$$\frac{\partial u_2}{\partial t}+(u_1+u_2)\left(\frac{du_1}{dx}+\frac{\partial u_2}{\partial x}\right)-\frac{1}{\text{Re}}\frac{d^2u_1}{dx^2}-\frac{1}{\text{Re}}\frac{\partial^2 u_2}{\partial x^2}=0 \tag{45}$$

$$\Rightarrow \quad \frac{\partial u_2}{\partial t}+u_1\frac{du_1}{dx}+u_1\frac{\partial u_2}{\partial x}+u_2\frac{du_1}{dx}+u_2\frac{\partial u_2}{\partial x}-\frac{1}{\text{Re}}\frac{d^2u_1}{dx^2}-\frac{1}{\text{Re}}\frac{\partial^2 u_2}{\partial x^2}=0$$

If we separate the two cases, we obtain the following two equations:

$$\frac{\partial u_2}{\partial t}+u_1\frac{\partial u_2}{\partial x}+u_2\frac{du_1}{dx}+u_2\frac{\partial u_2}{\partial x}-\frac{1}{\text{Re}}\frac{\partial^2 u_2}{\partial x^2}=0 \tag{46}$$

$$u_1\frac{du_1}{dx}-\frac{1}{\text{Re}}\frac{d^2u_1}{dx^2}=0 \tag{47}$$

$$u_1(0)=a \ , \ u_1(1)=-a \quad , \quad 0\le x\le 1$$

By linearizing equation (46), we have:

$$\frac{\partial u_2}{\partial t}+u_1\frac{\partial u_2}{\partial x}+u_2\frac{du_1}{dx}-\frac{1}{\text{Re}}\frac{\partial^2 u_2}{\partial x^2}=0 \tag{48}$$

Equation (47) represents the stationary or steady state viscous Burgers equation. The analytical solution of equation (47) is:

$$u_1(x)=a\left(\left(1-e^{a\,\text{Re}(x-0.5)}\right)/1+e^{a\,\text{Re}(x-0.5)}\right)+O\left(e^{-a\,\text{Re}}\right) \quad , \quad a>0 \tag{49}$$

where $O\left(e^{-a\,\text{Re}}\right)$ is the order of exponentially small error terms [23] in satisfying the boundary conditions . Equation (49) represents the steady state or stationary solution of Burgers equation. Under certain boundary conditions, the solution of the viscous Burgers equation, Eq. (6) approaches a unique stationary solution, $u_1(x)$, if the initial conditions $u(x,0)$ are sufficiently close. The equilibrium solution takes the form of a viscous shock located at the center of the domain.

The rate of convergence is determined by the eigen values m_j of the associated linearized problem.

$$-m_1=O\left(e^{-\text{Re}}\right)>0 \ , \ -b\,\text{Re}>m_2>m_3>\ldots \tag{50}$$

where $\boldsymbol{b}>0$ is a constant independent of $\in$. The solution will approach the steady state approximately as $\boldsymbol{e}^{m_1 \boldsymbol{t}}$, hence for small values of $\in$, this will become an extremely slow process. The equation (47) has another constant solution $u_1 = D$, D is constant. The unsteady state solution of Burgers equation after dimensionalizing and scaling by a is [26]:

$$u(x,t) = -(2a \in / L)(\sinh(ax/L))/(\cosh(ax/L) + \exp(-a^2 \in t/L^2)) \qquad (51)$$

5.1. Stability Analysis in the Case of Constant Amplitude

We assume that the perturbation has the following form [19]:

$$u_2(x,t) = Ae^{ik(x-ct)} \qquad (52)$$

$A > 0$, $k > 0$, $c = c_1 + ic_2$, $i = \sqrt{-1}$

where A is the wave amplitude , k is the wave, number c is the wave velocity . If $c_2 < 0$ the disturbance will decay as $t \to \infty$ and the solution is stable, but if $c_2 > 0$ the disturbance will grow as $t \to \infty$ and the solution is unstable. The case $c_2 = 0$, gives the neutral stability curve, which separates between the stable and unstable regions, c_2 is called the stability indicator [22].

Substitute (52) in (48), and after some mathematical manipulation, we get:

$$-ic_1 + c_2 = -u_1 i - \frac{1}{k}\frac{du_1}{dx} - \frac{k}{\mathrm{Re}}$$

Equating the real and imaginary parts , we have :

$$c_1 = u_1$$

$$c_2 = -\left[\left(k^2 + \mathrm{Re}\frac{du_1}{dx}\right)/ k\,\mathrm{Re}\right] \qquad (53)$$

Now, we shall study the following two cases:

(a) When $u_1 = D$, where D is constant, this leads to $\frac{du_1}{dx} = 0$, substitute in (53), we get:

$$c_2 = -(k/\mathrm{Re}) < 0 \qquad (54)$$

Hence, the constant stationary solution $u_1 = D$ is always stable.

(b) When $u_1(x) = u_1(x) = a((1 - e^{a\,\mathrm{Re}(x-0.5)})/1 + e^{a\,\mathrm{Re}(x-0.5)})$ as shown in Fig.(1), then $\frac{du_1}{dx} = -(2a^2\,\mathrm{Re}\,e^{a\,\mathrm{Re}(x-0.5)}/(1 + e^{a\,\mathrm{Re}(x-0.5)})^2)$

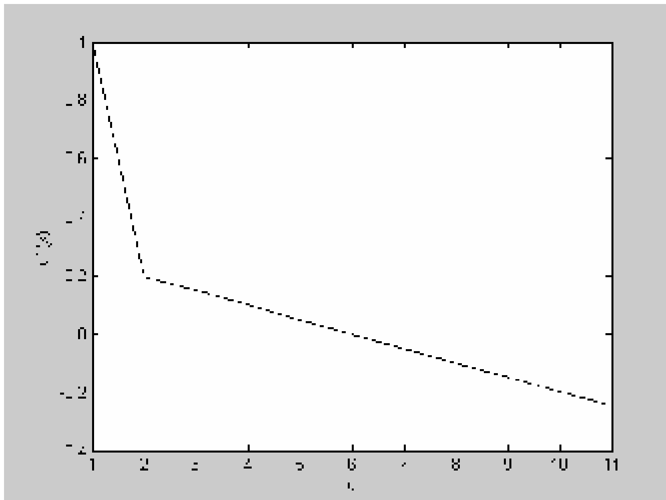

Figure (1) shows the stationary solution $u_1 = u_1(x)$ when
$a = 1$, $\mathrm{Re} = 1$, $0 \le x \le 1$

Here, in above we neglect the error term since it is small [23].

For simplicity, we put $\dfrac{du_1}{dx} = -f(x)$ in (53), we have:

$$c_2 = -\left[\left(k^2 - \mathrm{Re}\, f(x)\right)/\, k\, \mathrm{Re}\right] \tag{55}$$

where $f(x) = \left(2a^2\, \mathrm{Re}\, e^{a\,\mathrm{Re}(x-0.5)} / \left(1 + e^{a\,\mathrm{Re}(x-0.5)}\right)^2\right) > 0$

From Equation (55), we have

(i) If $\boldsymbol{k}^2 < \mathrm{Re}\, \boldsymbol{f}(\boldsymbol{x})$, then $c_2 > 0$ and the solution is unstable.

(ii) If $\boldsymbol{k}^2 > \mathrm{Re}\, \boldsymbol{f}(\boldsymbol{x})$, then $c_2 < 0$ and the solution is stable.

(iii) If $\boldsymbol{k}^2 = \mathrm{Re}\, \boldsymbol{f}(\boldsymbol{x})$, then $c_2 = 0$, which gives the neutral stability curve as shown in Fig. (2):

$$k = \sqrt{\mathrm{Re}\, f(x)} = \sqrt{\left(2(a\,\mathrm{Re})^2\, e^{a\,\mathrm{Re}(x-0.5)} / \left(1 + e^{a\,\mathrm{Re}(x-0.5)}\right)^2\right)} \tag{56}$$

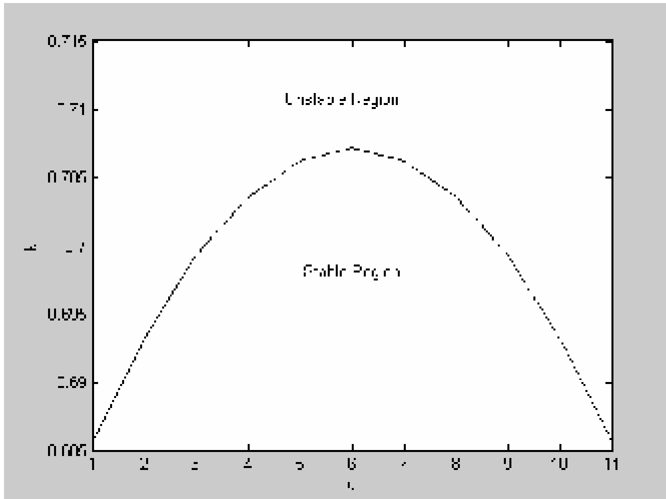

Figure (2)

The neutral stability curve in (56) for the stationary solution $u_1 = u_1(x)$ **when** $a = 1$, $\mathrm{Re} = 1$, $0 \le x \le 1$

5.2. Stability Analysis in the Case of Variable Amplitude

We assume the disturbance to have the following form [19], [1]:

$$u_2(x,t) = A(x)e^{ik(x-ct)} \qquad (57)$$

Substitute (57) in (48), and neglect the imaginary part in the resulting equation, we have:

$$A''(x) - \mathrm{Re}\, u_1 A'(x) - \left(k^2 + \mathrm{Re}\frac{du_1}{dx} + \mathrm{Re}\, kc_2\right)A(x) = 0 \qquad (58)$$

Equation (58) can be written in the following form:

$$A''(x) - \mathrm{Re}\, u_1 A'(x) - l\, A(x) = 0 \qquad (59)$$

$$A(0) = a \quad , \quad A(1) = -a$$

$$l = k^2 + \mathrm{Re}\frac{du_1}{dx} + \mathrm{Re}\, kc_2$$

The characteristic equation of Eq. (59) is:

$$m^2 - \mathrm{Re}\, u_1 m - l = 0 \qquad (60)$$

which has the following solutions:

$$m_1 = \left(\operatorname{Re} u_1 - \sqrt{(\operatorname{Re} u_1)^2 + 4l} \right) / 2 \tag{61}$$

$$m_2 = \left(\operatorname{Re} u_1 + \sqrt{(\operatorname{Re} u_1)^2 + 4l} \right) / 2 \tag{62}$$

According to the sign of l Eq. (59) has the following three analytical solutions:

(i) If $l > 0$ ***i.e.*** $l = \boldsymbol{H}$, $\boldsymbol{H} > 0$, then (61) and (62) become:

$$m_1 = \left(\operatorname{Re} u_1 - \sqrt{(\operatorname{Re} u_1)^2 + 4H} \right) / 2$$

$$m_2 = \left(\operatorname{Re} u_1 + \sqrt{(\operatorname{Re} u_1)^2 + 4H} \right) / 2$$

The general solution of Eq. (59) in this case is:

$$A(x) = Be^{m_1 x} + Ce^{m_2 x}$$

Now, we have the following two cases:

(a) When $u_1 = D$, D is constant by using the boundary conditions, we get:

$$\left.\begin{aligned} &B + C = a \\ &Be^{m_1} + Ce^{m_2} = -a \end{aligned}\right\} \tag{63}$$

By solving the algebraic system (63), we have:

$$\boldsymbol{B} = -\boldsymbol{a}\left(\left(1 + \boldsymbol{e}^{\boldsymbol{m}_2}\right) / \left(\boldsymbol{e}^{\boldsymbol{m}_1} - \boldsymbol{e}^{\boldsymbol{m}_2}\right)\right) \quad , \quad \boldsymbol{C} = \boldsymbol{a}\left(\left(1 + \boldsymbol{e}^{\boldsymbol{m}_1}\right) / \left(\boldsymbol{e}^{\boldsymbol{m}_1} - \boldsymbol{e}^{\boldsymbol{m}_2}\right)\right) \quad , \quad \boldsymbol{e}^{\boldsymbol{m}_1} \; 1 \; \boldsymbol{e}^{\boldsymbol{m}_2}$$

$$A(x) = \left(a / \left(e^{m_1} - e^{m_2}\right)\right)\left(\left(1 + e^{m_1}\right)e^{m_2 x} - \left(1 + e^{m_2}\right)e^{m_1 x}\right) \tag{64}$$

(b) When $\boldsymbol{u}_1 = \boldsymbol{u}_1(\boldsymbol{x}) = \boldsymbol{a}\left(\left(1 - \boldsymbol{e}^{\boldsymbol{a} \operatorname{Re}(\boldsymbol{x} - 0.5)}\right) / \left(1 + \boldsymbol{e}^{\boldsymbol{a} \operatorname{Re}(\boldsymbol{x} - 0.5)}\right)\right)$

$$u_1(0) = a\left(\left(1 - e^{-0.5 a \operatorname{Re}}\right) / \left(1 + e^{-0.5 a \operatorname{Re}}\right)\right) \quad , \quad u_1(1) = a\left(\left(1 - e^{0.5 a \operatorname{Re}}\right) / \left(1 + e^{0.5 a \operatorname{Re}}\right)\right)$$

$$m_1(0) = \left(\operatorname{Re} u_1(0) - \sqrt{(\operatorname{Re} u_1(0))^2 + 4H} \right) / 2 \quad , \quad m_2(0) = \left(\operatorname{Re} u_1(0) + \sqrt{(\operatorname{Re} u_1(0))^2 + 4H} \right) / 2$$

$$m_1(1) = \left(\operatorname{Re} u_1(1) - \sqrt{(\operatorname{Re} u_1(1))^2 + 4H} \right) / 2 \quad , \quad m_2(1) = \left(\operatorname{Re} u_1(1) + \sqrt{(\operatorname{Re} u_1(1))^2 + 4H} \right) / 2$$

By using the boundary conditions, we obtain:

$$B = -a\left(\left(1 + e^{m_2(1)}\right) / \left(e^{m_1(1)} - e^{m_2(1)}\right)\right) \quad , \quad C = a\left(\left(1 + e^{m_1(1)}\right) / \left(e^{m_1(1)} - e^{m_2(1)}\right)\right) \quad , \quad e^{m_1(1)} \neq e^{m_2(1)}$$

In this case the general solution has the form:

$$A(x) = \left(a / \left(e^{m_1(1)} - e^{m_2(1)}\right)\right)\left(\left(1 + e^{m_1(1)}\right)e^{m_2 x} - \left(1 + e^{m_2(1)}\right)e^{m_1 x}\right) \tag{65}$$

(ii) If $l = 0$, then $m_1 = 0$, $m_2 = \operatorname{Re} u_1$

The general solution is:

$$A(x) = B + C\, e^{\operatorname{Re} u_1 x}$$

By using the boundary conditions, we get:

$$\left.\begin{aligned} B + C &= a \\ B + C\,e^{\mathrm{Re}\,u_1} &= -a \end{aligned}\right\} \qquad (66)$$

By solving the system (66), we have:

$$B = -a\left(\left(1+e^{\mathrm{Re}\,u_1}\right)/\left(1-e^{\mathrm{Re}\,u_1}\right)\right) \quad , \quad C = \left(2a/\left(1-e^{\mathrm{Re}\,u_1}\right)\right) \quad , \quad e^{\mathrm{Re}\,u_1} \neq 1$$

(a) When $u_1 = D$, the general solution is:

$$\boldsymbol{A}(\boldsymbol{x}) = -\boldsymbol{a}\left(\left(1+\boldsymbol{e}^{\mathrm{Re}\,\boldsymbol{D}}\right)/\left(1-\boldsymbol{e}^{\mathrm{Re}\,\boldsymbol{D}}\right)\right) + \left(2\boldsymbol{a}/\left(1-\boldsymbol{e}^{\mathrm{Re}\,\boldsymbol{D}}\right)\right)\boldsymbol{e}^{\mathrm{Re}\,\boldsymbol{D}\boldsymbol{x}} \quad , \quad \boldsymbol{e}^{\mathrm{Re}\,\boldsymbol{D}} \neq 1$$

(b) When $u_1 = a\left(\left(1-e^{a\,\mathrm{Re}(x-0.5)}\right)/1+e^{a\,\mathrm{Re}(x-0.5)}\right)$, the general solution is:

$$\boldsymbol{A}(\boldsymbol{x}) = -\boldsymbol{a}\left(\left(1+\boldsymbol{e}^{\mathrm{Re}\,\boldsymbol{u}_1(1)}\right)/\left(1-\boldsymbol{e}^{\mathrm{Re}\,\boldsymbol{u}_1(1)}\right)\right) + \left(2\boldsymbol{a}/\left(1-\boldsymbol{e}^{\mathrm{Re}\,\boldsymbol{u}_1(1)}\right)\right)\boldsymbol{e}^{\mathrm{Re}\,\boldsymbol{u}_1(1)\boldsymbol{x}} \quad , \quad \boldsymbol{e}^{\mathrm{Re}\,\boldsymbol{u}_1(1)} \neq 1$$

The case $l = 0$ is the case of the constant amplitude, which is discussed in (5.1).

(iii) If $l < 0$, let $l = -\boldsymbol{R}$, $\boldsymbol{R} > 0$, then we have the following cases:

(1) When $(\mathrm{Re}\,u_1)^2 > 4R$, then the solution as in the case (i).

(2) When, $(\mathrm{Re}\,u_1)^2 = 4R$ then the general solution is:

$$A(x) = B e^{\frac{\mathrm{Re}\,u_1 x}{2}} + Cx e^{\frac{\mathrm{Re}\,u_1 x}{2}}$$

By using the boundary conditions, we obtain:

$$B = a \quad , \quad C = -a\left(\left(1+e^{\frac{\mathrm{Re}\,u_1}{2}}\right)/e^{\frac{\mathrm{Re}\,u_1}{2}}\right) \quad , \quad A(x) = a\left(1-\left(\left(1+e^{\frac{\mathrm{Re}\,u_1}{2}}\right)/e^{\frac{\mathrm{Re}\,u_1}{2}}\right)x\right)e^{\frac{\mathrm{Re}\,u_1 x}{2}}$$

(a) When $u_1 = D$, then the general solution is:

$$A(x) = a\left(1-\left(\left(1+e^{\frac{\mathrm{Re}\,D}{2}}\right)/e^{\frac{\mathrm{Re}\,D}{2}}\right)x\right)e^{\frac{\mathrm{Re}\,Dx}{2}}$$

(b) When $\boldsymbol{u}_1 = \boldsymbol{a}\left(\left(1-\boldsymbol{e}^{\boldsymbol{a}\,\mathrm{Re}(\boldsymbol{x}-0.5)}\right)/\left(1+\boldsymbol{e}^{\boldsymbol{a}\,\mathrm{Re}(\boldsymbol{x}-0.5)}\right)\right)$, then

$$A(x) = a\left(1-\left(\left(1+e^{\frac{\mathrm{Re}\,u_1(1)}{2}}\right)/e^{\frac{\mathrm{Re}\,u_1(1)}{2}}\right)x\right)e^{\frac{\mathrm{Re}\,u_1 x}{2}}$$

(3) When $(\mathrm{Re}\,u_1)^2 < 4R$, let $(\mathrm{Re}\,u_1)^2 - 4R = -E$, $E > 0$, then

$$\boldsymbol{m}_1 = \left(\mathrm{Re}\,\boldsymbol{u}_1 - \sqrt{\boldsymbol{E}}\,\boldsymbol{i}\right)/2 \quad , \quad \boldsymbol{m}_2 = \left(\mathrm{Re}\,\boldsymbol{u}_1 + \sqrt{\boldsymbol{E}}\,\boldsymbol{i}\right)/2$$

The general solution is:

$$A(x)=B e^{\frac{\operatorname{Re} u_1 x}{2}}\cos\left(\left(\sqrt{E}/2\right)x\right)+C e^{\frac{\operatorname{Re} u_1 x}{2}}\sin\left(\left(\sqrt{E}/2\right)x\right)$$

By using the boundary conditions, we have:

$$\boldsymbol{B}=\boldsymbol{a}\quad,\quad \boldsymbol{a}\boldsymbol{e}^{\frac{\operatorname{Re}\boldsymbol{u}_1}{2}}\left(\cos\left(\sqrt{\boldsymbol{E}}/2\right)+\boldsymbol{C}\sin\left(\sqrt{\boldsymbol{E}}/2\right)\right)=-\boldsymbol{a}$$

For simplicity and to determine the value of c_2, we take $C=1$, $a=1$ and after some mathematical manipulation, we get:

$$E=\left(\sin^{-1}\left(e^{-\operatorname{Re} u_1}-1\right)\right)^2$$

$$E=\left(\sin^{-1}\left(e^{-\operatorname{Re} u_1}-1\right)\right)^2=4R-\left(\operatorname{Re} u_1\right)^2=4\left(k^2+\operatorname{Re}\frac{du_1}{dx}+\operatorname{Re} k c_2\right)-\left(\operatorname{Re} u_1\right)^2\quad\Rightarrow$$

$$c_2=-\left[\left(4k^2+4\operatorname{Re}\frac{du_1}{dx}-\left(\sin^{-1}\left(e^{-\operatorname{Re} u_1}-1\right)\right)^2-\left(\operatorname{Re} u_1\right)^2\right)/4k\operatorname{Re}\right]\tag{67}$$

Equation (67) has the following three cases:

(i) If $4\boldsymbol{k}^2+4\operatorname{Re}\frac{\boldsymbol{du}_1}{\boldsymbol{dx}}<\left(\sin^{-1}\left(\boldsymbol{e}^{-\operatorname{Re}\boldsymbol{u}_1}-1\right)\right)^2+\left(\operatorname{Re}\boldsymbol{u}_1\right)^2$, then $c_2>0$ and the solution is unstable.

(ii) If $4\boldsymbol{k}^2+4\operatorname{Re}\frac{\boldsymbol{du}_1}{\boldsymbol{dx}}>\left(\sin^{-1}\left(\boldsymbol{e}^{-\operatorname{Re}\boldsymbol{u}_1}-1\right)\right)^2+\left(\operatorname{Re}\boldsymbol{u}_1\right)^2$, then $c_2<0$ and the solution is stable.

(iii) If $4\boldsymbol{k}^2+4\operatorname{Re}\frac{\boldsymbol{du}_1}{\boldsymbol{dx}}=\left(\sin^{-1}\left(\boldsymbol{e}^{-\operatorname{Re}\boldsymbol{u}_1}-1\right)\right)^2+\left(\operatorname{Re}\boldsymbol{u}_1\right)^2$, then $c_2=0$, which gives the neutral stability curve:

$$k=\sqrt{\left(\left(\sin^{-1}\left(e^{-\operatorname{Re} u_1}-1\right)\right)^2+\left(\operatorname{Re} u_1\right)^2-4\operatorname{Re}\frac{du_1}{dx}\right)/4}\tag{68}$$

$$\left(\sin^{-1}\left(e^{-\operatorname{Re} u_1}-1\right)\right)^2+\left(\operatorname{Re} u_1\right)^2>4\operatorname{Re}\frac{du_1}{dx}$$

Now, we shall apply the results in above to the following two cases:

(a) When $\boldsymbol{u}_1(\boldsymbol{x})=\boldsymbol{D}$, $\boldsymbol{D}$ is constant, we have

$$c_2=-\left[\left(4k^2-\left(\sin^{-1}\left(e^{-\operatorname{Re} D}-1\right)\right)^2-\left(\operatorname{Re} D\right)^2\right)/4k\operatorname{Re}\right]\tag{69}$$

From equation (69), we have the following three cases:

(i) If $4\boldsymbol{k}^2<\left(\sin^{-1}\left(\boldsymbol{e}^{-\operatorname{Re}\boldsymbol{D}}-1\right)\right)^2+\left(\operatorname{Re}\boldsymbol{D}\right)^2$, then $c_2>0$ and the solution is unstable.

(ii) If $4\boldsymbol{k}^2 > \left(\sin^{-1}\left(\boldsymbol{e}^{-\operatorname{Re}\boldsymbol{D}} - 1\right)\right)^2 + \left(\operatorname{Re}\boldsymbol{D}\right)^2$, then $c_2 < 0$ and the solution is stable.

(iii) If $4\boldsymbol{k}^2 = \left(\sin^{-1}\left(\boldsymbol{e}^{-\operatorname{Re}\boldsymbol{D}} - 1\right)\right)^2 + \left(\operatorname{Re}\boldsymbol{D}\right)^2$, then $c_2 = 0$, which gives the neutral stability curve as shown in Fig. (3):

$$k = \sqrt{\left(\left(\sin^{-1}\left(e^{-\operatorname{Re}D} - 1\right)\right)^2 + \left(\operatorname{Re}D\right)^2\right)/4} \qquad (70)$$

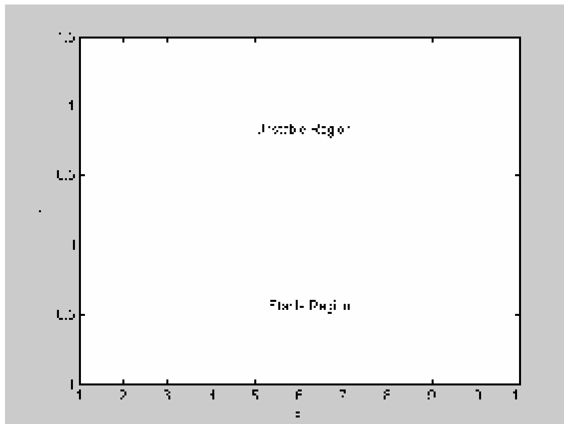

Figure (3)
The neutral stability curve in (70) for the stationary solution
$\boldsymbol{u}_1(\boldsymbol{x}) = \boldsymbol{D}$ **when** $\boldsymbol{D} = 1$, $\operatorname{Re} = 1$, $0 \leq \boldsymbol{x} \leq 1$

(b) When $\boldsymbol{u}_1 = \boldsymbol{u}_1(\boldsymbol{x}) = \left(\left(1 - \boldsymbol{e}^{\operatorname{Re}(\boldsymbol{x}-0.5)}\right)/\left((1 + \boldsymbol{e}^{\operatorname{Re}(\boldsymbol{x}-0.5)})\right)\right)$, we have

$$c_2 = -\left[\left(4k^2 + 4\operatorname{Re}\frac{du_1(x)}{dx} - \left(\sin^{-1}\left(e^{-\operatorname{Re}u_1(x)} - 1\right)\right)^2 - \left(\operatorname{Re}u_1(x)\right)^2\right)/4k\operatorname{Re}\right] \qquad (71)$$

From equation (71), we have the following three cases:

(i) If $4\boldsymbol{k}^2 + 4\operatorname{Re}\dfrac{\boldsymbol{du}_1(\boldsymbol{x})}{\boldsymbol{dx}} < \left(\sin^{-1}\left(\boldsymbol{e}^{-\operatorname{Re}\boldsymbol{u}_1(\boldsymbol{x})} - 1\right)\right)^2 + \left(\operatorname{Re}\boldsymbol{u}_1(\boldsymbol{x})\right)^2$, then $c_2 > 0$ and the solution is unstable.

(ii) If $4\boldsymbol{k}^2 + 4\operatorname{Re}\dfrac{\boldsymbol{du}_1(\boldsymbol{x})}{\boldsymbol{dx}} > \left(\sin^{-1}\left(\boldsymbol{e}^{-\operatorname{Re}\boldsymbol{u}_1(\boldsymbol{x})} - 1\right)\right)^2 + \left(\operatorname{Re}\boldsymbol{u}_1(\boldsymbol{x})\right)^2$, then $c_2 < 0$ and the solution is stable.

(iii) If $4k^2 + 4\operatorname{Re}\frac{du_1(x)}{dx} = \left(\sin^{-1}\left(e^{-\operatorname{Re} u_1(x)} - 1\right)\right)^2 + \left(\operatorname{Re} u_1(x)\right)^2$, then $c_2 = 0$, which gives the neutral stability curve

$$k = \sqrt{\left(\left(\sin^{-1}\left(e^{-\operatorname{Re} u_1(x)} - 1\right)\right)^2 + \left(\operatorname{Re} u_1(x)\right)^2 - 4\operatorname{Re}\frac{du_1(x)}{dx}\right)/4} \qquad (72)$$

$$\left(\sin^{-1}\left(e^{-\operatorname{Re} u_1(x)} - 1\right)\right)^2 + \left(\operatorname{Re} u_1(x)\right)^2 > 4\operatorname{Re}\frac{du_1(x)}{dx}$$

6. Conclusions

The main conclusions from this study in the case of constant amplitude are:

(1) The steady state solution $u_1 = D$, where D is constant, is always stable.

(2) The stationary solution $\boldsymbol{u}_1 = \boldsymbol{u}_1(\boldsymbol{x}) = \boldsymbol{a}\left(\left(1 - \boldsymbol{e}^{\boldsymbol{a}\operatorname{Re}(\boldsymbol{x}-0.5)}\right)/(1 + \boldsymbol{e}^{\boldsymbol{a}\operatorname{Re}(\boldsymbol{x}-0.5)})\right)$ is stable if $\boldsymbol{k}^2 > \left(2(\boldsymbol{a}\operatorname{Re})^2 \boldsymbol{e}^{\boldsymbol{a}\operatorname{Re}(\boldsymbol{x}-0.5)} / \left(1 + \boldsymbol{e}^{\boldsymbol{a}\operatorname{Re}(\boldsymbol{x}-0.5)}\right)^2\right)$ i.e. the solution $u_1 = u_1(x)$ in above is conditionally stable and the neutral stability curve is:

$$\boldsymbol{k} = \sqrt{\left(2(\boldsymbol{a}\operatorname{Re})^2 \boldsymbol{e}^{\boldsymbol{a}\operatorname{Re}(\boldsymbol{x}-0.5)} / \left(1 + \boldsymbol{e}^{\boldsymbol{a}\operatorname{Re}(\boldsymbol{x}-0.5)}\right)^2\right)}$$

The results in the case of variable amplitude are:

(1) The equilibrium solution $u_1 = D$, where D is constant, is stable if:

$4\boldsymbol{k}^2 > \operatorname{Re}\left(\sin^{-1}\left(\boldsymbol{e}^{-\operatorname{Re}\boldsymbol{D}} - 1\right)\right)^2 + \left(\operatorname{Re}\boldsymbol{D}\right)^2$ and the neutral stability curve is:

$$\boldsymbol{k} = \sqrt{\left(\left(\sin^{-1}\left(\boldsymbol{e}^{-\operatorname{Re}\boldsymbol{D}} - 1\right)\right)^2 + \left(\operatorname{Re}\boldsymbol{D}\right)^2\right)/4}$$

(2) The equilibrium state solution $u_1 = u_1(x) = \left(\left(1 - e^{\operatorname{Re}(x-0.5)}\right)/\left(1 + e^{\operatorname{Re}(x-0.5)}\right)\right)$ is stable if $4\boldsymbol{k}^2 + 4\operatorname{Re}\frac{\boldsymbol{du}_1(\boldsymbol{x})}{\boldsymbol{dx}} > \left(\sin^{-1}\left(\boldsymbol{e}^{-\operatorname{Re}\boldsymbol{u}_1(\boldsymbol{x})} - 1\right)\right)^2 + \left(\operatorname{Re}\boldsymbol{u}_1(\boldsymbol{x})\right)^2$ and the neutral stability curve is:

$$k = \sqrt{\left(\left(\sin^{-1}\left(e^{-\operatorname{Re} u_1(x)} - 1\right)\right)^2 + \left(\operatorname{Re} u_1(x)\right)^2 - 4\operatorname{Re}\frac{du_1(x)}{dx}\right)/4}$$

$$\left(\sin^{-1}\left(e^{-\operatorname{Re} u_1(x)} - 1\right)\right)^2 + \left(\operatorname{Re} u_1(x)\right)^2 > 4\operatorname{Re}\frac{du_1(x)}{dx}$$

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