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On Generalization for Principally Quasi-Injective S-acts

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Abstract. In this article, the concept of principally quasi-injective acts is extended to the concept of small principally quasi-injective acts and several properties of principally quasi-injective acts are extended to these acts. More specifically, we discovered new characterizations and properties of S-acts in which all subacts are small in the first. Among these characterizations, an S-act N_S will be SP-M-injective act if and only if eachm $\in M_S$ with mS small in M_S and $Hom_S(M,N)m = \ell_N\gamma_S(m)$ and many more. In terms of the projective act as a condition, the relationship between the factors of the injective acts with SP-M-injective is also clarified. Another fascinating finding shows the characterization of (m,1)-small quasi-injective. Secondly, examples are given to illustrate this concept. Finally, conditions are discovered in which subacts inherit the property of being small principally quasi-injective. Furthermore, it is shown that the direct sum of finite SP-M-injective acts is also SP-M-injective. The connection between monoid and small principally quasi-injective acts is explained. Despite the fact that there is no connection between small principally quasi-injective acts and small finitely generated weakly injective acts on act, we discovered that they are equivalent on a monoid S. We elucidated our work's conclusions in the final section.

Keywords. Small principally quasi-injective acts, Small subact, small principally M-injective acts, Principal self-generator,Small finitely generated weakly injective act.

INTRODUCTION

Mathematicians have long been fascinated by the behavior of semigroups. A semigroup action can be thought of as a generalization of the concept of group action in pure mathematics from an algebraic standpoint.Additionally, it's acquainted that within the theoretical computer science and in pure mathematics like algebra, associate degree action of a semigroup on a set may be a rule that associates to every component of the semigroup a transformation of the set in such some way that the product of two components of the semigroup is connected to the composite of the two corresponding transformations.The terminology conveys the concept that the components of the semigroup are acting as transformations of the set.An significant special case is also a monoid action or act, within which the semigroup may be a monoid and thusthe identity component of the monoid acts as the identity transformation of a set.Now, let S be a monoid. A unitary right S-act M over S that denoted by $M_S$could also be a non-empty set with a function $f{:}M\times S \longrightarrow M$ such that $f(m,s) \longmapsto ms$ and also the following properties hold: (1) m•1=m. (2) m(st) = (ms)t for each $m\in M$ and $s,t \in S$.All through this article, S might be a monoid with zero elements and each S-act is unitary right S-act with zero componentΘthat denoted by M_S.It's typical that S-act might be found by different wordings as follows: S-systems, S-sets, S-operands, S-polygons, transition systems, S-automata[1]. For a great deal of insights concerning S-acts and injective acts, we have a tendency to refer the reader to the references [2-15] and [16-24].In [21], Thuyet L.V., and Quynh T.C., introduced the concept of small principally injective modules that may be a generalization to the work of Nicholson and et al. in [16]. But, Wongwai S. in [22], extended the notion and results of principally quasi-injective modules in [17], to small principally quasi-injective modules that motivated us to increase this work and study this notion on S-acts. What is more, it's fascinating to notice that some results on modules stay true in S-acts. The investigation on the generalizations of quasi-injective acts has been of interest to

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many authors. One in every one of them was the author wherever introduced the concept of principally quasi-injective acts in [1]. Our plan of introducing this notion opened a new direction to researchers to supply a basis to find varied generalizations of quasi-injective (and therefore for injective) acts. Besides, the author in [18], studied the generalization of principally quasi-injective acts which is pseudo principally quasi-injective acts over monoids. The great structure of principally quasi-injective acts has led us to extend this notion to a different generalization. A lot of exactly, during this work, we discover a weak kind of PQ-injective (which additionally represents a weak form of quasi-injective) called small principally injective S-act. An S-act M_S is termed principally quasi-injective (PQ-injective) acts if each S-homomorphism from a principal subact of M_S to M_S extends to an S-endomorphism of M_S [1]. A right S-act N_S could be termed small principally M-injective (simply SP-M-injective) if, each S-homomorphism from a small and principal subact of a right S-act M_S to N_S might be expanded to an S-homomorphism from M_S to N_S. A right S-act MS could be termed small principally quasi-injective if it's SP-M-injective. Note that there are some results on principally quasi-injective S-act extended to those S-acts. Also, we are going to use terminology, definitions and notations from previous work freely [1].

The present work consists of two sections. Section two, part one is dedicated to introduce and investigate a brand new quite generalization of principally quasi-injective S-acts, namely small principally quasi-injective acts. Bound categories of subacts that inherit the property of small principally quasi-injective were thought of. Also, the characterizations of this new category of S-acts were investigated. An example was given to demonstrate SP-M-injective acts. Some acknowledged results on Small Principally Quasi-injective for general modules were generalized to S-acts. Within the second part of section two, we've given endomorphism monoid. The third section has processed the conclusions of our work.

RESULTS

1. Small Principally Quasi-injective Act

Definition 1. A subact N of a right S-act M_S is called small (or superfluous) in M_S if for every subact H of M_S, $N \cup H = M_S$ implies $H = M_S$.

Definition 2. Let M_S be a right S-act. If every S-homomorphism from a small and principal subact of M_S to N_S can be extended to an S-homomorphism from M_S to N_S,a right S-act N_S is called small principally M-injective (simply SP-M-injective). If a right S-act $M_S$is SP-M-injective, it is referred to as small principally quasi-injective (simply SPQ-injective).

Proposition 1. Assume that M_S and N_S be right S-act. If and only if each $m \in M_S$ with mS small in M_S and $Hom_S(M,N)m = \ell_N\gamma_S(m)$, then N_S is SP-M-injective act.

Proof: Assume N_S is an SP-M-injective act. Let $\alpha m \in Hom_S(M,N)m$to prove $Hom_S(M,N)m = \ell_N\gamma_S(m)$.We have $\alpha(ms)=\alpha(mt)$ for each $s,t\in S$ with $ms = mt$, so $\alpha m\in \ell_N\gamma_S(m)$. Thus $Hom_S(M,N)m \subseteq \ell_N\gamma_S(m)$. If $x\in \ell_N\gamma_S(m)$ in the other direction, then define $\sigma:mS\to xS$ by $\sigma(ms) = xs$, for $x\in S$. If $ms = mt$,for $s,t \in S$, then $(s,t) \in \gamma_S(m) \subseteq \gamma_S(x)$, hence $xs = xt$, this demonstrates that σ is well-defined, and it's a simpleway to see that σ is an S-homomorphism. Since N is SP-M-injective,there exists an S-homomorphism $\bar{\sigma}:M_S\longrightarrow N_S$that extend σby definition (2.1.2). This means that$\bar{\sigma}i_1= i_2\sigma$, with $i_1:mS\longrightarrow M_s$ and $i_2:xS\longrightarrow N_S$being the inclusion maps . As a result $x = \sigma(m) = \bar{\sigma}(m) \in Hom_S(M,N)m$.As a result,$\ell_N\gamma_S(m) \subseteq Hom_S(M,N)m$ and hence $Hom_S(M,N)m =\ell_N\gamma_S(m)$.Conversely, let $m\in M_S$ with mS small in M_S, and let $\varphi: mS \longrightarrow N_S$ be an S-homomorphism . Then $\varphi(m) \in \ell_N\gamma_S(m)$, so by assumption ($\ell_N\gamma_S(m) = Hom_s(M,N)m$) , $\varphi(m) = \bar{\varphi}(m)$ for some $\bar{\varphi} \in Hom_s(M,N)$. This implies that N_S is SP-M-injective act.

Remark and example 1.

- Assume that $S =\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field, $M_S = S_S$ and $N_S =\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$. N_S is then an SP-M-injective act.

Proof: It is straightforward to demonstrate that A = $\begin{pmatrix}0 & F\\0 & 0\end{pmatrix}$ is the only nonzero small and principal subact of M_S. Let α:A→N_S be S-homomorphism .Since $\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} \in$ A , there are x_{11} , $x_{12} \in$ F such that $\alpha\left(\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\right) = \begin{pmatrix}x_{11} & x_{12}\\0 & 0\end{pmatrix}$. After that, $\alpha\left(\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\right) = \alpha\left[\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right] = \alpha\left(\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\right)\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}x_{11} & x_{12}\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}0 & x_{12}\\0 & 0\end{pmatrix}$. It indicates that x_{11}=0 . Define $\bar{\alpha}: M_s \longrightarrow N_s$ is equal to $\bar{\alpha}\left(\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right) = \begin{pmatrix}x_{12} & 0\\0 & 0\end{pmatrix}$. It is self-evidentthat $\bar{\alpha}$ is an S-homomorphism. Then

$\bar{\alpha}\left(\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\right) = \bar{\alpha}\left[\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\right] = \bar{\alpha}\left(\begin{pmatrix}x_{12} & 0\\0 & 0\end{pmatrix}\right)\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} = \begin{pmatrix}0 & x_{12}\\0 & 0\end{pmatrix}$. This implies that $\bar{\alpha}$ is an extension of α . As a result, N_S is an SP-M-injective act.■

- There is Θ is a small subact in every act. Θ is the only small subactin a semisimple act, in particular. If A is the subact of M_S (because M_S is semisimple, A is a retract of M_S), then there exists B subact of M_S with $A\dot{\cup}B = M_S$. If A is small subact of M_S, then B = M_S and so A = Θ .

The direct sum of finite SP-M-injective act is also SP-M-injective,as explained by the following proposition:

Proposition 2. Assume N_i (1≤ i ≤ n) is an SP-M-injective act. $\bigoplus_{i=1}^{n} N_i$ is then an SP-M-injective act.

Proof: If we prove the proposition for n = 2, then this is enough. Let m∈ M_S with mS small in M_S and $\alpha : mS \longrightarrow N_1 \oplus N_2$ be an S-homomorphism. Since $N_1(N_2)$ is SP-M-injective, then by definition (1.1.2) there exists S-homomorphism $\alpha_1: M_S \longrightarrow N_1$ ($\alpha_2: M_S \longrightarrow N_2$) such that $\alpha_1 i = \pi_1\alpha$ ($\alpha_2 i = \pi_2\alpha$) where $\pi_1(\pi_2)$ is the projection map from $N_1 \oplus N_2$ into $N_1(N_2)$ and i: mS ⟶ M_S is the inclusion map. Put $j_1\alpha_1 = \bar{\alpha}(j_2\alpha_2 = \bar{\alpha}$). Figure (1) clarify it

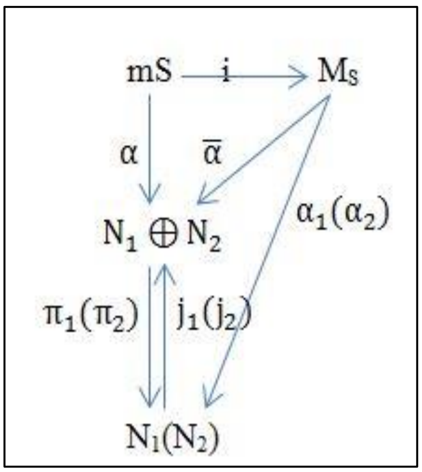

FIGURE 1. Clarifies that $N_1 \oplus N_2$ is SP-M-injective act.

Thus $\bar{\alpha}$extends α.

The following corollary elucidate under which subact of SP-M-injective is also SP-M-injective:

Corollary 1. Retract subactof an SP-M-injective act is also SP-M-injective.

The following theorem reveals characterizations of SP-M-injective act, among these characteristics; the relationship between the factors of the injective act with SP-M-injective in terms of the projective act is demonstrated:

Theorem 1. For projective act M_S, the following conditions are equivalent:

- Every principal subact and a small subact of M_S is projective.
- Every factor act of an SP-M-injective act is also SP-M-injective.
- Every factor act of an injective S-act is also SP-M-injective.

Proof: (1⟶2) Let A_S be an SP-M-injective S-act and mS be small subact in M_S .Let α:mS $\longrightarrow A_S/\rho$ be S-homomorphism, with ρ is a congruence on A_S. Then (1) shows that there is an S-homomorphism $\beta: mS \longrightarrow A_S$ such that $\pi\beta = \alpha$ where $\pi: A_S \longrightarrow A_S/\rho$ is the natural epimorphism. Since A is SP-M-injective S-act, β can be extended to S-homomorphism $\sigma: M_S \longrightarrow A_S$by definition (1.1.2). Put $\varphi = \pi\sigma$, thereafter,φ is the extension of α to M_S .

(2⟶3) Suppose E is injective act and E/ρ is the E's factor. E is SP-M-injective actsince every injective is SP-M-injective act,. Then, E/ρ is SP-M-injective actby (2).

(3⟶1) Assume mS is a small subact of M_S and $f: A_S \longrightarrow B_S$be an S-epimorphism , where $A_S$and $B_S$are two S-act . Then $B_S \cong A_S/\rho$, where the congruence ρ = ker(f). Let$g: mS \longrightarrow B_S$. By corollary (1.6) in [13], every act can be embedding in injective act, so embed A_S in injective act E. Then, since$B_S \cong A_S/\rho$ is a subact of E/ρ , so by (3) g is extended to $\bar{g}: M_S \longrightarrow E/\rho$. Since M_S is projective, so $\bar{g}$ can be lifted to $\alpha: M_S \longrightarrow E$. It is self-evident that $\alpha(mS) \subset A_S$. As a result, g lifted to β. This means that $\beta: mS \longrightarrow A_S$ and f β = g.■

2. The Endomorphism Monoid

Keep in mind that if a right S-act M_S is SP-M-injective, it is referred to as a small principally quasi-injective (simply SPQ-injective). The following proposition exemplifies how the SPQ-injective act is defined:

Proposition 3. Assume that M_S be a right S-act and T=End (M_S). The following conditions are also equivalent:

- M_S is an SPQ-injective.
- $\ell_M\gamma_S(m) = Tm$ for all m ∈ M_S with mS small subact of M_S .
- If $\gamma_S(m) \subset \gamma_S(n)$, where m ,n ∈ M_S and mS small subact of M_S , then Tn ⊂Tm .
- $\ell_M(\gamma_S(m) \cap (aS \times aS)) = \ell_M(aS \times aS) \cup Tm$ for all a ∈ S and m ∈ M_S with mS is small in M_S.

Proof: (1⟺2) By prposition (1.1.3).

(2⟶3) If $\gamma_S(m) \subset \gamma_S(n)$, where m,n ∈$M_S$ and mS small subact of M_S , then $\ell_M\gamma_S(n) \subset \ell_M\gamma_S(m)$. By (2) , we have $\ell_M\gamma_S(n)$ = Tn and $\ell_M\gamma_S(m)$=Tm ,so Tn⊂Tm .

(3⟶4) Let $x \in \ell_M(\gamma_S(m) \cap (aS \times aS))$ with mS is small in M_S . Then, $\gamma_S(ma) \subset \gamma_S(xa)$, if (s,t)∈ $\gamma_s(ma)$for each s,t∈ S, then mas= mat , which implies that (as,at) ∈ $\gamma_S(m) \cap (aS \times aS)$, so xas = xat and hence (s,t) ∈ $\gamma_S(xa)$. By (3), we have Txa⊂ Tma , in particular xa∈Tma , furthermore mS is small in M_S , so xa = σ(ma) for some σ ∈T.Thus x∈ $\ell_M(aS \times aS) \cup Tm$ .This shows that $\ell_M(\gamma_S(m) \cap (aS \times aS)) \subseteq \ell_M(aS \times aS) \cup Tm$. Conversely, let $x \in \ell_M(aS \times aS) \cup Tm$, then x ∈Tm which means that x=σ(m) for some σ ∈ T or x ∈$\ell_M(aS \times aS)$which implies that xas=xatand then σ(xas) = σ(xat) for all s,t ∈ S and a ∈ M_S. We have (as , at) ∈ $\gamma_s(m) \cap (aS \times aS)$for each a ∈ M_S and s,t ∈ S , which implies that mas= mat , since σ is well-define , so σ(mas) = σ(mat) . If x = σ(m) , then xas=xat .Thus x ∈$\ell_M(\gamma_S(m) \cap (aS \times aS))$ and then $\ell_M(aS \times aS) \cup Tm \subseteq \ell_M(\gamma_S(m) \cap (aS \times aS))$.

(4⟶2) By taking a = 1.

Proposition 4. Suppose thatT = End(M_S) where $M_S$is an SPQ-injective act. If $\alpha(M_S)$ small in M_S such thatm ∈M_S and α ∈ T, then $\ell_T(\ker(\alpha) \cap (mS \times mS)) = \ell_T(mS \times mS) \cup T\alpha$.

Proof: Let $\beta \in \ell_T(\ker(\alpha) \cap (mS \times mS))$. Then $\gamma_S(\alpha m) \subset \gamma_S(\beta m)$. Hence $\ell_M\left(\gamma_S(\beta m)\right) \subset \ell_M\left(\gamma_S(\alpha m)\right)$ .Since α(m)S is small in M_S, $T\beta(m) \subset \ell_M(\gamma_S(\beta m)) \subset \ell_M\left(\gamma_S(\alpha m)\right) = T\alpha(m)$. By proposition (2.2.1), we have β(m) = (m) , where σ ∈ T . Thus (β, $\sigma\alpha$) ∈ $\ell_T(mS \times mS)$ and then $\beta \in \ell_T(mS \times mS) \cup T\alpha$. Conversely, let $\beta \in$

$T\alpha \cup \ell_T(mS \times mS)$, this means eitherfor some $\sigma \in T$, we have$\beta = \sigma\alpha$ or $\beta(ms) = \beta(mt)$ for all $s, t \in S$ and $m \in M_S$. Now, if for each $(ms,mt) \in \ker(\alpha) \cap (mS \times mS)$, if $\beta = \sigma\alpha$, then $\alpha(ms) = \alpha(mt)$ and therefore$\sigma\alpha(ms) = \sigma\alpha(mt)$, so $\beta(ms) = \beta(mt)$. As a consequence,$\beta \in \ell_T(\ker(\alpha) \cap (mS \times mS))$. If $\beta(ms) = \beta(mt)$, then, we obtain$\beta \in \ell_T(mS \times mS)$ and therefore,$\beta \in \ell_T(\ker(\alpha) \cap (mS \times mS))$. Thereby,$\ell_T(mS \times mS) \cup T\alpha \subseteq \ell_T(\ker(\alpha) \cap (mS \times mS))$.

Bear in mind that an S-act M_S is referred to as a principally self-generator if an S-homomorphism $f{:}M_S \longrightarrow xS$ exists for every $x \in M_S$, such that $x=f(x_1)$ for $x_1 \in M_S$[1]:

Proposition 5. Assume $M_S$is referred to as a principal act that is a principal self-generator and let T denote the End (M_S). The following conditions are equivalent in this case:

- M_S is an SPQ-injective act.
- $\ell_T(\ker(\alpha) \cap (mS \times mS)) = \ell_T(mS \times mS) \cup T\alpha$with $\alpha(M_S)$ small in $M_S$for all $m \in M$ and $\alpha \in T$.
- $\ell_T(\ker(\alpha)) = T\alpha$where$\alpha(M_S)$ is a small in M_S for all $\alpha \in T$.
- $\ker(\alpha) \subset \ker(\beta)$,where$\alpha,\beta \in T$ with $\alpha(M_S)$ is a small in M_S, as a consequence $T\beta \subset T\alpha$.

Proof: (1⟶2) By the proposition (2.2.2).

(2⟶3) If $M_S = m_0S$, and $m = m_0$ in (2), we get$\ell_T(\ker(\alpha) \cap (M_S \times M_S)) = \ell_T(M_S \times M_S) \cup T\alpha$, which means$\ell_T(\ker(\alpha)) = T\alpha$.

(3⟶4) Assume that$\ker(\alpha) \subset \ker(\beta)$, thereby, by (3) we obtain that $T\beta = \ell_T(\ker(\beta)) \subset \ell_T(\ker(\alpha)) = T\alpha$, therefore, $\beta \in T\alpha$.

(4⟶1) Suppose that $\sigma : mS \rightarrow M_S$is an S-homomorphism with $m \in M_S$ and mS is a small in M_S. There is $\alpha \in T$ such that $m = \alpha(m_0)$ since $M_S$is the principal self-generator. We declare that $\ker(\alpha) \subset \ker(\sigma\alpha)$. For this if $(x,y) \in \ker(\alpha)$, then $\alpha(x)=\alpha(y)$, because σ is well-defined homomorphism , so $\sigma\alpha(x) = \sigma\alpha(y)$ and $(x,y) \in \ker(\sigma\alpha)$. Therefore,$\ker(\alpha) \subset \ker(\sigma\alpha)$and then, mS is a small in M_S. Thereafter, by (4) $T\sigma\alpha \subset T\alpha$. Now, put $\sigma\alpha = \bar{\sigma}\alpha$, where$\bar{\sigma} \in T$. This implies that$M_S$ is an SPQ-injective act and$\bar{\sigma}$extendsσ.■

Lemma 1. If(A) is a small in $N_S$if A_S is a small subact in M_S and $f{:}M_S \longrightarrow N_S$ is an S-homomorphism. Particularly, if A is small in M_S and $M_S \subseteq N_S$, then A_S is small in N_S.

Proof: Assume $f(A) \cup f(B_S) = N_S$. Since A_S is a small subact in M_S, so for $B_S \subseteq M_S$, we get $A_S \cup B_S = M_S$, meaning that $B_S = M_S$. This implies the $A_S \subseteq B_S$. As a result $f(A) \subseteq f(B)$ and $f(B)=N_S$. Thereafter, A_S is a small in $N_S$by definition (1.1.1).

The following theorem is a generalization of theorem (3.4) from[22]:

Theorem 2. Assume $M_S$is an SPQ-injective act and torsion free act over cancellative monoid . Let $m,n \in M_S$ and let mS be small sub act in M_S:

- Tm is an image of Tn if mS is embedding in nS.
- Tn is embedding in Tm if nS is an image of mS.
- $Tm \cong Tn$ if $mS \cong nS$.

Proof: (1) Assumeα: $mS \longrightarrow nS$ is an S-monomorphism, so $\alpha(m) \in nS$, then there exists $s \in S$ for which$\alpha(m) = ns$. Suppose the inclusion maps are$i_1{:}mS \longrightarrow M_s$ and i_2: $nS \longrightarrow M_S$. Because M_S is an SPQ-injective act, therefore, there is an S-homomorphism $\bar{\alpha}$: $M_S \longrightarrow M_S$for which $i_2\alpha = \bar{\alpha} i_1$by definition (2.1.2). This is depicted in figure (2).

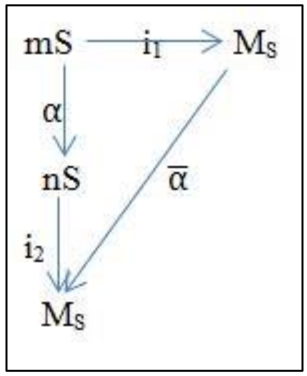

FIGURE 2. Shows that M_S is an SPQ-injective act.

Suppose β: Tn ⟶Tm is determined by $\beta(\sigma(n)) = \sigma(\bar{\alpha}(m))$ for every σ ∈ T. Because $\beta(\sigma(n)) = \sigma\alpha(m) \in \sigma(nS)$. Therefore, for each σn ∈ Tn ,f ∈ T we obtain $\beta(f(\sigma n)) = \beta(f\sigma)(n) = (f\sigma)\bar{\alpha}(m) = f(\sigma(\bar{\alpha}(m))) = f\beta(\sigma n)$. Thereby,β is T-homomorphism. If $\sigma_1 n = \sigma_2 n$, where $\sigma_1,\sigma_2 \in T$, resulting in $\sigma_1 ns_1 = \sigma_2 ns_1$ for which $s_1 \in S$. This leads to $(\sigma_1,\sigma_2) \in \gamma_s(ns_1)$ and thereafter, $(\sigma_1,\sigma_2) \in \gamma_s(\bar{\alpha}m)$. Thus $\sigma_1(\bar{\alpha}m) = \sigma_2(\bar{\alpha}m)$ and since $\bar{\alpha}(m)=(\bar{\alpha}i_1)(m)=i_2\alpha(m)=\alpha(m)$. As a result, $\beta(\sigma_1 n) = (\sigma_2 n)$.As a result β is well-defined. We declare $\gamma_s(\bar{\alpha}m) \subset \gamma_s(m)$, let $(s,t) \in \gamma_s(\bar{\alpha}m)$ which implies that $\bar{\alpha}(ms)=\bar{\alpha}(mt)$. This implies that $\alpha(ms)= \alpha(mt)$. Since α is monomorphism, so ms = mt , then $(s,t) \in \gamma_s(m)$. As a result, by proposition (B.1)(3), we get $Tm \subset T\bar{\alpha}m$. For $\beta m \in T\bar{\alpha}m$, as a result, there is σ ∈ T where $\beta m = \sigma\bar{\alpha}(m) = \beta(\sigma n)$. As a consequence,β is T-epimorphism.

(2) In a similar to the way in (1), assume α: mS →nS is an S-epimorphism. Put $\alpha(ms) = n$, where s ∈ S. α can be extended to $\bar{\alpha} : M_S \to M_S$ where $i_2\alpha = \bar{\alpha}i_1$ because M_S is an SPQ-injective act. Define β: Tn →Tm by $\beta(\sigma(n)) = \sigma(\bar{\alpha}(ms))$ for every σ ∈ T and s ∈ S . From (1), we get β is T-homomorphism. Since α is an epimorphism, so there is s ∈ S where n = (ms). Assume $(\sigma_1 n,\sigma_2 n) \in \ker\beta$, then $\beta(\sigma_1 n) = \beta(\sigma_2 n)$ which implies that $\beta(\sigma_1(\alpha(ms)) = \beta(\sigma_2(\alpha(ms))$, then $\sigma_1(\bar{\alpha}(ms)) = \sigma_2(\bar{\alpha}(ms))$. Thereafter, $\sigma_1(\alpha(ms)) = \sigma_2(\alpha(ms))$. As a result, $\sigma_1 n = \sigma_2 n$ and β is T-monomorphism .

(3) By (1) and (2) , if α: mS⟶nS is S-isomorphism , then β: Tn ⟶Tm is T-isomorphism.

Proposition 6. Assume M_S is a principal act and $M_S \times M_S$ generates $\ker(\alpha)$. T is a right SP-injective monoid if M_S is an SPQ-injective act.

Proof: Suppose σ: ${}_{\alpha}T \longrightarrow T$ is a T-homomorphism and σ(α) = β, where β ∈ T . Let $\ker(\alpha) \subseteq \ker(\beta)$, where, β ∈ T. Then, for any $(x,y) \in \ker(\alpha)$, we have $\alpha(x) = \alpha(y)$. Since $M_S \times M_S$ generates $\ker(\alpha)$,therefore, x = σm , y = σn , where $(m,n) \in M_S \times M_S$ and σ ∈ T. Thereafter, $(\sigma m , \sigma n) \in \ker(\alpha) \subseteq \ker(\beta)$, meaning $\beta(\sigma m) = \beta(\sigma n)$. This implies $\beta(x) = \beta(y)$ with $(x,y) \in \ker(\beta)$. As a result, there is an S-homomorphism $f : \alpha(M) \longrightarrow M_S$ for which $f\alpha = \beta$ by proposition (2.1.3). Therefore, α(M) is a principal and small sub act in M_S because M_S is a principal act. By assumption f can be extended to an S-homomorphism $\bar{f} : M_S \longrightarrow M_S$ where $\bar{f}i=f$ and i is the inclusion map of α(M) into M_S. As a consequence, $\bar{f}i\alpha = f\alpha = \beta$. Define $\bar{\sigma} : T \longrightarrow T$ by $\bar{\sigma}(g) = \bar{f}g$ for every g ∈ T. It is self-evident that $\bar{\sigma}$ is T-homomorphism. Then $\bar{\sigma}(\alpha g)= \bar{f}(\alpha g)= \beta g =\sigma(\alpha g)$. As a result T is a right SP-injective monoid.

Definition 3. An S-act M_S, if for each S-homomorphism from n-generated small sub act of M_S^m to M_S can be extended to S-homomorphism from M_S^m to M_S for which m is a fixed positive integer, M_S is referred to as (m,n)-quasi-injective act.

Definition 4. If for each S-homomorphism from aprincipal and small subact of M_S^m to M_S can be extended to an S-homomorphism from M_S^m to M_S ,where m is a fixed positive integer, an S-act M_S is referred to as (m,1)-small quasi-injective. M_S is (m,1)-small quasi-injective if and only if (n,1)-small quasi-injective for all $n \leq m$.

The proposition that follows is a generalization of proposition (1.1.15) in [1]:

Proposition 7. Assume $M_S$is (m,1)-small quasi-injective with W= Hom(M_S^m,M_S) and let $m_1,m_2,\ldots,m_n$ denote elements of M_S with m_iS and $(m_1,m_2,\ldots,m_n)S$ being small in M_S^m $(1\leq i\leq n)$. After that, you should:

1. Any S-homomorphismα:$m_1S\dot{\cup}m_2S\dot{\cup}\ldots\dot{\cup}m_nS\longrightarrow M_S$ has an extension in W if $Wm_1 \oplus Wm_2 \oplus \ldots \oplus Wm_n$ is direct,
2. $W(m_1,m_2,\ldots,m_n) = Wm_1\dot{\cup}Wm_2\dot{\cup}\ldots\dot{\cup}Wm_n$ if $m_1S \oplus m_2S \oplus \ldots \oplus m_nS$ is direct.

Proof: (1) Assume thatα_i and $\beta$are the restriction of α to m_iS and $(m_1, m_2,\ldots,m_n)S$ respectively, this means$\alpha_i(=\alpha_{|m_iS})$: $m_iS\longrightarrow M_s$ and β: $(m_1,m_2,\ldots,m_n)S \longrightarrow M_S$ with m_iS and $(m_1,m_2,\ldots,m_n)S$ are small in M_S^m . As a consequence, $\overline{\alpha_i}$and $\overline{\beta}$are an extension of α_i and β respectively to $M_S^m$by definition (2.1.8) (since M_S is (m,1)-small quasi-injective act). For each $x \in m_1S\dot{\cup}m_2S\dot{\cup}\ldots\dot{\cup}m_nS$, there is a unique $j \in I=\{1,2,\ldots,n\}$ where $x =m_js_j$, $\overline{\beta}(x) = \overline{\beta}(m_js_j) = \beta(m_j)s_j = \alpha(m_js_j) = \alpha(x)$. This demonstrates$\overline{\beta}$ is an extension of α.

(2) Assume that $x \in Wm_1\dot{\cup}Wm_2\dot{\cup}\ldots\dot{\cup}Wm_n$, so $x = \alpha_i(m_i)$ (where $\alpha_i(=\alpha_{|m_iS})$: $m_iS\longrightarrow M_S$, $\alpha \in T$) . Define an S-homomorphism β:$(m_1,m_2,\ldots,m_n)S \longrightarrow M_S$as follows$\beta((m_1,m_2,\ldots,m_n)s) = \alpha_i(m_i)s = m_is$ where $s \in S$. Since $m_1S\oplus m_2S\oplus\ldots \oplus m_nS$ is direct, resulting inβ is well-defined. For this let $(m_1,m_2,\ldots,m_n)s = (m_1,m_2,\ldots,m_n)t$ for which$s,t\in S$, meaning that $(m_1s,m_2s ,\ldots,m_ns) = (m_1t ,m_2t ,\ldots ,m_nt)$, thereafter, $m_is=m_it$ and $\alpha_i(m_i)s = \alpha_i(m_i)t$.Therefore, $\beta[(m_1,m_2,\ldots,m_n)S]=\beta[(m_1,m_2,\ldots,m_n)t]$ and β is well-defined. Because M_S is (m,1)-small quasi-injective act and $(m_1,m_2,\ldots,m_n)S$ is a small in M_S^m, as a result,β is an extension to $\overline{\beta}$: $M_S\longrightarrow M_S$. Thereby, for $m_j\in M_S$where $j \in\{1,2,\ldots,n\}$and since $m_1S\oplus m_2S\oplus\ldots \oplus m_nS$ is direct, this leads to$m_j = \overline{\beta_j}(m_j) = \beta(m_j) = \beta(m_1,m_2,\ldots,m_n) \in W(m_1,m_2,\ldots,m_n)$. As a consequence, $Wm_1\dot{\cup}Wm_2\dot{\cup}\ldots\dot{\cup}Wm_n\subseteq W(m_1,m_2,\ldots,m_n)$. Inclusion in the opposite direction is always holds.

The following corollary follows from above proposition when m = 1 (that is M_S is an SPQ-injective act):

Corollary 2. Assume $M_S$isa small principally quasi-injective act with T= End(M_S) and let $m_1,m_2,\ldots,m_n$ denote elements of $M_S$wherem_iS and $(m_1,m_2,\ldots,m_n)S$ are small in M_S $(1\leq i\leq n)$. After that, you should:

1. Any S-homomorphism α:$m_1S\dot{\cup}m_2S\dot{\cup}\ldots\dot{\cup}m_nS\longrightarrow M_s$ has an extension in T if $Tm_1 \oplus Tm_2 \oplus \ldots \oplus Tm_n$ is direct.
2. $T(m_1,m_2,\ldots,m_n) = Tm_1\dot{\cup}Tm_2\dot{\cup}\ldots\dot{\cup}Tm_n$ if $m_1S \oplus m_2S \oplus \ldots \oplus m_nS$ is direct.

Definition 5. An S-act M_S if for any S-homomorphism from small finitely generated right ideal of S_S into M_S can be extended to S-homomorphism from S_S into M_S, is called a small finitely generated weakly injective (if this the case , we write SFGW-injective act).

As a result, it is clear that the SPQ-injective act and SFGW-injective have no connection, but they are identical on monoid S, therefore, corollary (2.1.10) will be in as in the following:

Corollary 3. Suppose that S is a SFGW-injective act and let $a_1,a_2,\ldots,a_n$a denote elements of S wherea_iS and $(a_1,a_2,\ldots,a_n)S$ are small in S_S $(1\leq i\leq n)$. Then:

1. Any S-homomorphism α: $a_1S\dot{\cup}a_2S\dot{\cup}\ldots\dot{\cup}a_nS\longrightarrow S_s$ has an extension in S if $Sa_1 \oplus Sa_2 \oplus \ldots \oplus Sa_n$ is direct.
2. $S(a_1,a_2,\ldots,a_n) = Sa_1\dot{\cup}Sa_2\dot{\cup}\ldots\dot{\cup}Sa_n$ if $a_1S \oplus a_2S \oplus \ldots \oplus a_nS$ is direct.

The next proposition represents a generalization of proposition (1.1.18) from[1]:

Proposition 8. Assume that M_S is (m,1)-small quasi-injective act where W=Hom(M_S^m,M_S) , and suppose that A is a sub act of M_S for which $B_1 ,B_2, \ldots,B_n$are small subacts of M_S . If $\oplus_{i=1}^n B_i$ is a direct, then $A \cap \oplus_{i=1}^n B_i = \oplus_{i=1}^n (A \cap B_i)$.

Proof: Assume that $x \in\oplus_{i=1}^n (A \cap B_i)$, meaning there is $j \in I=\{1,2,\ldots n\}$, where $x \in A \cap B_j$ then, we get $x \in A$ and $x \in B_j$ for some $j\in I$, therefore, $x \in A\cap \oplus_{i=1}^n B_i$. As a result,$\oplus_{i=1}^n (A \cap B_i) \subseteq A \cap \oplus_{i=1}^n B_i$. For the other direction,assume $a \in A \cap \oplus_{i=1}^n B_i$leads to $a \in A$ and $a \in\oplus_{i=1}^n B_i$. Thereby, there is $j\in I$ for which $a \in B_j$. Assume that$\pi_j : \oplus_{i=1}^n b_iS \longrightarrow b_jS$is the projection, thereafter, take $\alpha(= \pi_{j|b_jS}) : b_jS \longrightarrow b_jS$. Suppose that $i_1,i_2$are the inclusion maps of b_iS and b_jS into M_S^m and M_S respectively. Because B_i is a small subactof M_S and M_S is (m,1)-small quasi-injective act, therefore, α can be extended to S-homomorphism $\beta :M_S^m \longrightarrow M_S$ (that is there exists $\beta \in W$) by(1) of proposition(2.1.9) and obtained that β extends π_j . Thereby, for $a \in b_jS$, we get $b_j = \pi_j(a) = \beta(a) = \alpha(a)$. As a consequence, $a \in \bigoplus_{i=1}^{n} (A \cap B_i)$ and $A \cap \bigoplus_{i=1}^{n} B_i \subseteq \bigoplus_{i=1}^{n} (A \cap B_i)$.

The following corollary follows from the above proposition when m = 1 (that is M_S is SPQ-injective act):

Corollary 4. Assume that M_S is an SPQ-injective act where T=End(M_S), and let A be a sub act of M_S for which $B_1, B_2, \ldots, B_n$ are being small sub acts of M_S . If $\bigoplus_{i=1}^{n} B_i$ is a direct, then $A \cap \bigoplus_{i=1}^{n} B_i = \bigoplus_{i=1}^{n} (A \cap B_i)$.

CONCLUSIONS

The introduction and analysis of the subject of this article lead to a deeper understanding of the relationship between acts theory and module theory. Furthermore, the relevance of this subject stems from some key points we discovered. As a consequence, we'd like to draw attention to these key points. We discovered novel properties and characterizations for S-acts with small sub acts. To inherit the property of small principally quasi-injective acts, we deduced that a sub act must be retracted. Furthermore, we show that for small principally M-injective, every factor of an injective S-act is SP-M-injective under projective conditions, which is one of the applications for this topic. We discovered and investigated the finite direct sum of S-act for this concept in this article. Furthermore, we discovered that the factor of the injective act can be connected to small principally quasi-injective acts using the projective act condition. A small principally quasi-injective condition is used to obtain the relation of endomorphism monoid with acts. We also deduced that the small act's homo morphic image is small. Small principally quasi-injective act and small principally injective monoid can be linked using the principal act condition. Finally, the finite direct sum of small principally M-injective actsis also small principally M-injective.

This work can be extended to semisimple small injective acts, where a right S-act M_S is said to be the semisimple small-A-injective act, if for any semisimple small subact B of A and any homomorphism from B to M_S extends to A . If an S-act $M_S$is semisimple small M-injective, it is said to be the semisimple small quasi-injective. If a monoid S is semisimple small S-injective, it is said to be the semisimple small injective.

Conflict of Interest: The authors declare that there is no conflict of interest.

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