PAIRWISE COMPACT IN INTUITIONISTIC DOUBLE TOPOLOGICAL SPACES

Prof.Dr. Hadi J.Mustafa , Rewayda Razaq Mohsin *Enas Yehya Abdullah*

Math-Dep College of Math. Science *Math-Dep. College of Education*

University of Kufa-Najaf , Iraq

Abstract

The concept of intuitionistic topological space was introduced by Coker .The aim of this paper is to discuss the relation between bitopological spaces and double-topological spaces and give a notion of pairwise compact for double topological spaces .

1-Introduction

The concept of a fuzzy topology introduced by Change[2] , after the introduction of fuzzy sets by Zadeh . Later this concept was extended to intuitionistic fuzzy topological spaces by Coker in [4] . In [5] Coker studied continuity, connectedness,compactness and separation axioms in intuitionistic fuzzy topological spaces. In this paper, we follow the suggestion of J.G. Garcia and S.E. Rodabaugh [7] that (double fuzzy set) is a more appropriate name than (intuitionistic fuzzy set) ,and therefore adopt the term (double-set) for the intuitionistic set , and (double-topology) for the intuitionistic topology of Dogan Coker , (this issue), we denote by **Dbl-Top** the construct (concrete texture over set) whose objects are pairs (X,τ) where τ is a double-topology on X .In section three, we discuss making use of this relation between bitopological spaces and double- topological spaces , we generalize a notion of compactness for double-topological space in section four .

2-Preliminaries

Throughout the paper by X we denote a non-empty set . In this section we shall present various fundamental definitions and propositions. The following definition is obviously inspired by Atanassov [1].

2.1.Definition. [3] A double-set (DS in brief) A is an object having the form A=<x,A_1,A_2>,

Where A_1 and A_2 are subsets of X satisfying $A_1 \cap A_2 = \phi$. The set A_1 is called the set of members of A , while A_2 is called the set of non- members of A .

throughout the remainder of this paper we use the simpler A=(A_1,A_2) for a double-set.

2.2.Remark. Every subset A of X is may obviously be regarded as a double-set having the form A'=(A,A^c),

where A^c=X-A is the complement of A in X .

we recall several relations and operations between DS' s as follows:

2.3.Definition. [3] Let the DS's A and B on X be the form A=(A_1,A_2),

B=(B_1 , B_2), respectively . Furthermore, let $\{A_j : j \in J\}$ be an arbitrary family of DS's in X, where $A_j = (A_j^{(1)}, A_j^{(2)})$. Then

(a) $A \subseteq B$ *if and only if* $A_1 \subseteq B_1$ *and* $A_2 \supseteq B_2$;

(b) $A = B$ *if and only if* $A \subseteq B$ *and* $B \subseteq A$;

(c) $\overline{A} = (A_2, A_1)$ denotes the complement of A ;

(d) $\bigcap A_j = (\bigcap A_j^{(1)}, \bigcup A_j^{(2)})$;

(e) $\bigcup A_j = (\bigcup A_j^{(1)}, \bigcap A_j^{(2)})$;

(f) $[]A = (A_1, A_1^c)$;

(g) $\langle\rangle A = (A_2^c, A_2)$;

(h) $\underset{\sim}{\phi} = (\phi, X)$ *and* $\underset{\sim}{X} = (X, \phi)$.

In this paper we require the following :

(i) $()A = (A_1, \phi)$ *and* $)(A = (\phi, A_2)$.

Now, we recall the image and preimage of DS' s under a function .

2.4.Definition. [3,8] Let $x \in X$ be a fixed element in X. Then:

(a)The DS given by $\underset{\sim}{x} = (\{x\}, \{x\}^c)$ is called a double–point (DP in brief X) .

(b)The DS $\underset{\approx}{x} = (\phi, \{x\}^c)$ is called a vanishing double-point (VDP in brief X) .

2.5.Definition. [3,8]

(a) Let $\underset{\sim}{x}$ be a DP in X and A=(A_1,A_2) be a DS in X . Then $\underset{\sim}{x} \in A$ iff $x \in A_1$.

Let $\underset{\approx}{x}$ be a VDP in X and A=(A_1,A_2) a DS in X . Then $\underset{\approx}{x} \in A$ iff $x \notin A_2$.

It is clear that $\underset{\sim}{x} \in A \Leftrightarrow \underset{\sim}{x} \subseteq A$ and that $\underset{\approx}{x} \in A \Leftrightarrow \underset{\approx}{x} \subseteq A$.

2.6.Definition. [5] A double-topology (DT in brief) on a set X is a family τ of DS's in X satisfying the following axioms :

T1: $\phi, X \in \tau$,

T2: $G_1 \bigcap G_2 \in \tau$, for any $G_1, G_2 \in \tau$,

T3: $\bigcup G_j \in \tau$ **,** for any arbitrary family $\{G_j : j \in J\} \subseteq \tau$.

In this case the pair (X, τ) is called a double-topological space (DTS in brief), and any DS in τ is known as a double open set (DOS in brief). The complement $\overline{A}$ of a DOS A in a DTS is called a double closed set (DCS in brief) in X .

2.7.Definition. [5] Let (X, τ) be an DTS and A = (A_1,A_2) be a DS in X.

Then the interior and closure of A are defined by :

$$\text{int}(A) = \bigcup\{G : G \text{ is a DOS in } X \text{ and } G \subseteq A\},$$

$$cl(A) = \bigcap(H : H \text{ is a DCS in } X \text{ and } A \subseteq H\},$$

respectively .

It is clear that cl(A) is a DCS in and int(A) a DOS in X . Moreover , A is a DCS in X iff cl(A) = A , and A is a DOS in X iff int (A) = A.

2.8. Example. [5] Any topological space (X, τ_0) gives rise to a DT of the form $\tau = \{A' : A \in \tau_0\}$ by identifying a subset A in X with its counterpart A'=(A , A^c), as in Remark 2.2.

3- The Constructs Dbl-Top and Bitop :

We begin recalling the following result which associates a bitopology with a double topology.

3.1.Proposition. [5] Let (X,τ) be a DTS.

(a) $\tau_1 = \{A_1 : \exists A_2 \subseteq X \text{ with } A = (A_1, A_2) \in \tau\}$ is a topology on X.

(b) $\tau_2^* = \{A_2 : \exists A_1 \subseteq X \text{ with } A = (A_1, A_2) \in \tau\}$ is the family of closed sets of the topology $\tau_2 = \{A_2^c : \exists A_1 \subseteq X \text{ with } A = (A_1, A_2) \in \tau\}$ on X.

(c) Using (a) and (b) we may conclude that (X, τ_1, τ_2) is a bitopological space.

3.2.Proposition. Let *(X,u,v)* be a bitopological space. Then the family

$$\{(U, V^c) : U \in u,\ V \in v, U \subseteq V\}$$

is a double topology on X .

Proof . The condition $U \subseteq V$ ensures that $U \cap V^c = \phi$,while the given family contains $\underset{\sim}{\phi}$ because $\phi \in u, v$, and it contains $\underset{\sim}{X}$ because $X \in u, v$. Finally this family is closed under finite intersections and arbitrary unions by Definition 2.3 (d,e) and the corresponding properties of the topologies *u* and *v* .

3.3. Definition. Let *(X,u,v)* be a bitopological space. Then we set

$$\tau_{uv} = \{(U, V^c) : U \in u,\ V \in v, U \subseteq V\}$$

and call this the double topology on X associated with *(X,u,v)* .

3.4.Proposition. If *(X,u,v)* is a bitopological space and τ_{uv} the corresponding DT on X, then

$$(\tau_{uv})_1 = u \text{ and } (\tau_{uv})_2 = v .$$

Proof. $U \in u$ implies $(U, \phi) \in \tau_{uv}$ since $U \subseteq X \in v$, so $u \subseteq (\tau_{uv})_1$. Conversely, take $U \in (\tau_{uv})_1$. Then $(U, B) \in \tau_{uv}$ for some $B \subseteq X$. Now $U \in u$, hence $(\tau_{uv})_1 \subseteq u$, and the first equality is proved . □

The proof of the second equality may be obtained in a similar way , and we omit the details.

4- Piarwise Compact in Double- Topological Spaces .

In this section we define double compact set and we use the link between bitopological space and double topological space to established some theorems .

4.1. Definition. By an double open cover of a subset A of a double topological space (X,τ), we mean a collection $C = \{G_j : j \in J\}$ of double open subsets of X such that $A \subset \bigcup\{G_j : j \in J\}$ then we say that C covers A . In particular , a collection C is said to be an open cover of the space X iff $X = \bigcup\{(G_j^1, G_j^2) : j \in J\}$ of double open subsets of X .

4.2.Definition. A double-set A of DTS in (X,τ) is said to be double compact set iff for every double open cover has double finite sub cover , that is iff for every collection $\{G_j : j \in J\}$ of DOS's for which

$A \subset \bigcup\{G_j : j \in J\}$ *for* $A = (A_1, A_2)$ such that $(A_1, A_2) \subset (G^1_{j1}, G^2_{j1}) \bigcup ... \bigcup (G^1_{jn}, G^2_{jn})$.

4.3.Definition. Let (X, τ) be double topological space and let Y be a double subset of X. The τ- double relative topology for Y is the collection τ_Y given by $\tau_Y = \{G \bigcap Y : G \in \tau\}$.The double topological space (Y, τ_Y) is called double subspace of (X, τ).

4.4.Proposition. Let Y be a subspace of double topological spaces X and let $A \subset Y$, then A is double compact set relative to X iff A is double compact set relative to Y.

Proof : Let A be double compact set relative to X and let $\{V_j : j \in J\}$ be a collection of DS's , double open relative to Y . Which covers A so that $(A_1, A_2) \subset \{(V^1_j, V^2_j) : j \in J\}$ then there exists G_j double open set's relative to X such that $V_j = Y \bigcap G_j$, for every $j \in J$. It follows that $(A_1, A_2) \subset \{(G^1_j, G^2_j) : j \in J\}$ so that $\{G_j : j \in J\}$ is open cover of A relative to X . Since A is double compact set relative to X , there exist finitely many indices $j_1, ... j_n$ such that

$$(A_1, A_2) \subset (G^1_{j1}, G^2_{j1}) \bigcup ... \bigcup (G^1_{jn}, G^2_{jn})$$

Since $A \subset Y$ we have

$$(A_1, A_2) \subset Y \bigcap \{(G^1_{j1}, G^2_{j1}) \bigcup ... \bigcup (G^1_{jn}, G^2_{jn})\}$$

$$= (Y \bigcap (G^1_{j1}, G^2_{j1})) \bigcup ... \bigcup (Y \bigcap (G^1_{jn}, G^2_{jn}))$$

Since $Y \bigcap G_{ji} = V_{ji} \quad (i = 1,2,...,n)$ we obtain $(A_1, A_2) \subset (V^1_{j1}, V^2_{j1}) \bigcup ... \bigcup (V^1_{jn}, V^2_{jn})$ this shows that A is double compact set relative to Y .

Conversely , let A be double compact set relative to Y and let $\{G_j : j \in J\}$ a collection of DOS's of X which cover A , so that $(A_1, A_2) \subset \{(G^1_j, G^2_j) : j \in J\} \ldots (1)$

Hence $A \subset Y$,(1) implies that $A \subset Y \bigcap [\bigcup\{(G^1_j, G^2_j) : j \in J\}] =$

$\bigcup\{Y \bigcap (G^1_j, G^2_j) : j \in J\}$, hence

$Y \bigcap (G^1_j, G^2_j)$ is double open relative to Y , the collection $\{Y \bigcap G_j : j \in J\}$ is double open cover of A relative to Y . Since A is double compact relative to Y we must have .

$$(A_1, A_2) \subset (Y \bigcap (G^1_{j1}, G^2_{j1})) \bigcup ... \bigcup (Y \bigcap (G^1_{jn}, G^2_{jn})) ...(2)$$

Some choice of finitely many indices $j_1, ... j_n$, but (2) implies that $(A_1, A_2) \subset (G^1_{j1}, G^2_{j1}) \bigcup ... \bigcup (G^1_{jn}, G^2_{jn})$ it follows that A is double compact relative to X .

4.5.Proposition. In DTS (X, τ), double close subsets of compact sets are double compact set .

Proof : Let Y be a double compact subset of a topological space X and let F be a double subset of Y, double closed relative to X . To show that F is double compact . let $C = \{G_j : j \in J\}$ be an open cover of F then the collection $D = \{G_j\} \bigcup \{X - F\}$ forms an open cover of Y. Since Y compact , there is a finite subcollection D' *of* D which covers Y , and hence covers F . If X-F is member of D' we may remove it from $D^{'}$ and still retain an open finite cover of F . Hence F is double compact .

4.6. Definition. A collection C of double set's said to have the double finite intersection property (DIFP) or to be finitely common iff the intersection of members of each finite subcollection of C is non-empty .

4.7. Proposition. Double topological space (X,τ) is double compact iff every collection of double closed subset's of X has a non-empty intersection .

Proof : Let X be double compact set and let $\{F_j = (F_j^1, F_j^2) : j \in J\}$ be collection of double closed set's of X with FIP and suppose if possible $\bigcap\{F_j : j \in J\}$

$= \bigcap\{(F_j^1, F_j^2) : j \in J\} = \underset{\sim}{\phi} = (\phi, X)$

$\Rightarrow (\bigcap F_j^1, \bigcup F_j^2) = (\phi, X)$, then

$[\bigcap\{F_j : j \in J\}]^{'}$

$= [\bigcap\{(F_j^1, F_j^2) : j \in J\}]^{'} = \underset{\sim}{X} = (X, \phi)$, or

$\bigcup\{\overline{F_j} : j \in J\} = \bigcup\{(F_j^2, F_j^1) : j \in J\} = \underset{\sim}{X}$

$\Rightarrow (\bigcup F_j^2, \bigcap F_j^1) = (\phi, X) \Rightarrow \bigcup\{F_j^2 : j \in J\} = X$

This means that $\{\overline{F_j} : j \in J\}$ is a double open cover of X . Since F_j's are DCS's . Since X is double compact set . We have

$\bigcup\{\overline{F_{ji}} : i = 1,2,...,n\} = \underset{\sim}{X} \Rightarrow$

$[\bigcap\{F_{ji} : i = 1,2,...,n\}]^{'} = \underset{\sim}{X}$

Which implies that

$\bigcap\{F_{ji} : i = 1,2,...,n\} = \underset{\sim}{\phi}$ and this contradicts that FIP of F . Hence we must have $\bigcap\{F_j : j \in J\} \neq \phi$.

Conversely let every collection of DCS's of X with the FIP have non-empty intersection and let $C = \{G_j : j \in J\} = \{(G_j^1, G_j^2) : j \in J\}$ be a double open cover of X so that

$\underset{\sim}{X} = \bigcup\{(G_j^1, G_j^2) : j \in J\} \Rightarrow \bigcup\{G_j^1 : j \in J\} = X$

Hence taking complements

$\underset{\sim}{\phi} = [\bigcup\{(G_j^1, G_j^2) : j \in J\}]^{'} = \bigcap\{(G_j^2, G_j^1) : j \in J\}$ thus

$\{(G_j^2, G_j^1) : j \in J\}$ is a collection of DCS's with empty intersection and so by hypothesis this collection does not have the FIP , hence there exists a finite number of G_{ji} , $i = 1,2,...,n$ such that $\underset{\sim}{\phi} = \bigcap\{(G_{ji}^2, G_{ji}^1) : i = 1,2,...,n\}$

$= \bigcup\{(G_{ji}^1, G_{ji}^2) : i = 1,2,...,n\}]^{'}$

$\Rightarrow \underset{\sim}{X} = \bigcup\{(G_{ji}^1, G_{ji}^2) : i = 1,2,...,n\}$, hence X is double compact .

4.8. Definition. [6] A cover H of bitopological space (X,u,v) is pairwise open if $H \subset u \bigcup v$ $with H \bigcap u$ containing a non-empty set and with $H \bigcap v$ containing a non-empty set .

4.9. Definition. Let A be piarwise open subsets of a topological space X and let

$C=\{G_j : j\in J\}$ be a collection of pairwise open subsets of X such that

$A\subset\bigcup\{G_j : j\in J\}$. We then say that C pairwise covers A . By a pairwise sub cover of a pairwise open cover C of A, we mean a pairwise open sub collection $C^{`}$ of C such that $C^{`}$ pairwise covers A . A pairwise open cover of A is said to be finite if it consists of finite number of pairwise open sets .

4.10. Definition. The DTS (X,τ) is called pairwise compact if every pairwise open cover of X has a finite subcover .

4.12. Proposition. If *(X,u,v)* is pairwise compact then (X,τ_{uv}) is pairwise compact .

Proof : Let H be pairwise open cover of X and such that H $\bigcap u\neq\phi$ and H $\bigcap v\neq\phi$ and let A , B subset's in X ,such that $A\in u$, $B\in v$, since X is pairwise compact then $\{G_j : j\in J\}$, $\{H_j : j\in J\}$ are respectively an open cover of A , B such that $A\subset\{G_j : j\in J\}$, $B\subset\{H_j : j\in J\}$, there exists a finite sub cover such that $A\subset G_{j1}\bigcup...\bigcup G_{jn}$, $B\subset H_{j1}\bigcup...\bigcup H_{jn}$,

take $U=(A,\phi)\in\tau_{uv}$, $V=(\phi,B^c)\in\tau_{uv}$

Then $U=(A,\phi)\subset(G_{j1},\phi)\bigcup...\bigcup(G_{jn},\phi)$ and

$$V=(\phi,B^c)\subset\bigcap(\phi,H_j^c)=(\bigcap\phi,\bigcup H_j^c)$$

so that $H_{j1}^c\bigcup...\bigcup H_{jn}^c\subset B^c$ then τ_{uv} is pairwise compact .

This suggests the following definition for general double topologies .

.13. Proposition. If (X,τ) is pairwise compact then (X,τ_1,τ_2) is pairwise compact .

Proof : Let A be subset in X , since (X,τ) is pairwise compact then for open cover $\{G_j : j\in J\}$ of $A=(C,D)\in\tau$, there exist a finite open sub cover such that $(C,D)\subset(G_{j1}^1,G_{j1}^2)\bigcup...\bigcup(G_{jn}^1,G_{jn}^2)$ by property $\bigcup A=(\bigcup A_1,\bigcap A_2)$

$$C\subset G_{j1}^1\bigcup...\bigcup G_{jn}^1 \quad , \quad G_{j1}^2\bigcap...\bigcap G_{jn}^2\subset D$$

So that $(G_{j1}^2)^c\bigcup...\bigcup(G_{jn}^2)^c\subset D^c$

$$\therefore (C,D^c)\subset\bigcup\{(G_j^1,(G_j^2)^c) : j\in J\}$$

$\therefore (X,\tau_1,\tau_2)$ is pairwise compact .

4.14. Colloroly. The bitopological space *(X,u,v)* is pairwise compact iff (X,τ_{uv}) is pairwise compact .

Proof : Necessity follows from proposition 4.12 and sufficiency from proposition 4.13 and 3.4 .

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الملخص:

في هذا البحث قدمنا تعريف التراص الزوجي للفضاء التبولوجي المضاعف وناقشنا العلاقة بين فضاء التبولوجي المضاعف وفضاء التبولوجي الثنائي من خلال تعميم تعريف التراص الزوجي.