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ON A CERTAIN SUBClASSES OF ANALYTIC FUNCTIONS INVOLVING DIFFERENTIAL OPERATOR

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Abstract

In this paper, a new subclass in the open unit disc of analytic functions is introduced. This subclass $T\mathcal{S}^{m,\alpha}_{\lambda,\beta}(\zeta,\vartheta)$ is mainly defined by a specific differential operator. A coefficient inequality is obtained and other properties like distortion and closure theorems are derived. Moreover, extreme points of differential operator are also given. Additionally, Hadamard products (or convolution) of functions respective to the class are also included.

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1 Introduction

We begin by letting $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk of the complex plane and $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad a_k \geq 1, \tag{1.1}$$

which are analytic in $\mathcal{U}$ and satisfy the following usual normalization conditions $f(0) = f^{'}(0) - 1 = 0$.

The Hadamard product for two analytic functions f defined in (1.1) and

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \qquad b_k \geq 1,$$

is given by

$$f(z) * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

Denote by $\mathcal{T}$ the subclass of $\mathcal{A}$ consisting of functions f of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k. \tag{1.2}$$

This subclass was introduced and studied by Silverman [13].

The following defines the familiar Mittag-Leffler function $E_\alpha(z)$ introduced by Mittag-Leffler [7] and [8] and its generalization $E_{\alpha,\beta}(z)$ introduced by Wiman [16]

$$E_\alpha(z) = \sum_{k=o}^{\infty} \frac{z^k}{\Gamma(\alpha+1)},$$

and

$$E_{\alpha,\beta}(z) = \sum_{k=o}^{\infty} \frac{z^k}{\Gamma(\alpha n+\beta)} \qquad (\alpha \geq 0).$$

where $\alpha, \beta \in \mathbb{C}$, $Re(\alpha) > 0$ and $Re(\beta) > 0$.

As a result, a lot of useful work have been made by many researchers in attempt to explain Mittag-Leffler function and its generalization, see for example [14], [4], [10], [12], and [15].

An important and crucial theory that has contributed significantly in analytic function theory is the differential operator theory. The first study appeared in the year 1900 and since then numerous mathematicians have worked intensively in this way. For recent work see ([1], [6], [9], [5], [3]).

We define the function $Q_{\alpha,\beta}(z)$ by

$$Q_{\alpha,\beta}(z) = z\Gamma(\beta)E_{\alpha,\beta}(z).$$

Now, for $f \in \mathcal{A}$ we define the following differential operator: $D^m_\lambda(\alpha,\beta)f : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$D^0_\lambda(\alpha,\beta)f(z) = f(z) * Q_{\alpha,\beta}(z), \tag{1.3}$$

$$D^1_\lambda(\alpha,\beta)f(z) = (1-\lambda)(f(z) * Q_{\alpha,\beta}(z)) + \lambda z(f(z) * Q_{\alpha,\beta}(z))' \tag{1.4}$$

$$\vdots$$

$$D^m_\lambda(\alpha,\beta)f(z) = D^1_\lambda(D^{m-1}_\lambda(\alpha,\beta)f(z)) \tag{1.5}$$

If f is given by (1.1), then from (1.4) and (1.5) we see that

$$D^m_\lambda(\alpha,\beta)f(z) = z + \sum_{k=2}^{\infty} [1+(k-1)\lambda]^m \frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)} a_k z^k. \tag{1.6}$$

Note that

- when $\alpha = 0$ and $\beta = 1$ we get Al-Oboudi operator [2].
- when $\alpha = 0$, $\beta = 1$, $\lambda = 1$ we get Sălăgean operator [11].
- when $m = 0$ we get $\mathbb{E}_{\alpha,\beta}(z)$ [14].

If $f \in \mathcal{T}$ is given by (1.2) then we have

$$D_\lambda^m(\alpha,\beta)f(z) = z - \sum_{k=2}^{\infty}[1+(k-1)\lambda]^m \frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)} a_k z^k.$$

As a result of full utilization of differential operator we define a new subclass of the class $\mathcal{A}$.

Definition 1.1. *Let $0 < \vartheta \le 1$, $\lambda \ge 0$, $m \in \mathbb{N}_0 = \mathbb{N} \cup 0$ and $\zeta \in \mathbb{C} \diagup \{0\}$. Then,the function f in $\mathcal{A}$ is said to be in the class $\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta)$ if*

$$\left| \frac{1}{\xi} \left(\frac{z\left(D_\lambda^m(\alpha,\beta)f(z)\right)'}{D_\lambda^m(\alpha,\beta)f(z)} - 1 \right) \right| < \vartheta, \qquad z \in \mathcal{U}$$

Now, we define the class $T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta)$ by

$$T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta) = \mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta) \cap \mathcal{T}$$

The object of this paper is to study various properties for functions f belonging to the class $T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta)$

2 Coefficient Inequalities

Theorem 2.1. *Let $f \in \mathcal{T}$. The $f \in T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta)$ if and only if*

$$\sum_{k=2}^{\infty}(k+\vartheta|\zeta|-1)[1+(k-1)\lambda]^m \frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)} a_k \le \vartheta|\zeta| \tag{2.1}$$

for $0 < \vartheta \le 1, \zeta \in \mathbb{C}\backslash \{0\}, \lambda \ge 0$ and $m \in \mathbb{N}_0$.

Proof. Let $f \in T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta)$. Then, we have

$$Re\left\{ \frac{z(D_\lambda^m(\alpha,\beta)f(z))'}{D_\lambda^m(\alpha,\beta)f(z)} - 1 \right\} > -\vartheta|\zeta|$$

Equivalently,

$$Re\left\{ \frac{-\sum_{k=2}^{\infty}(k-1)[1+(k-1)\lambda]^m \frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)} a_k z^k}{z - \sum_{k=2}^{\infty}[1+(k-1)\lambda]^m \frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)} a_k z^k} \right\} > -\vartheta|\zeta| \tag{2.2}$$

since the above inequality is true for all $z \in \mathcal{U}$, choose values of z on the real axis. Upon clearing the denominator in (2.2) and letting $z \to 1^-$ through real values, we obtaing

$$-\sum_{k=2}^{\infty}(k-1)[1+(k-1)\lambda]^m \frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)} a_k$$

$$\geq -\vartheta|\zeta|\left(1-\sum_{k=2}^{\infty}[1+(k-1)\lambda]^m\frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)}a_k\right)$$

Thus, we obtain the desired inequality

$$\sum_{k=2}^{\infty}(k+\vartheta|\zeta|-1)[1+(k-1)\lambda]^m\frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)}a_k\leq\vartheta|\zeta| \tag{2.3}$$

Conversely, Supposed that inequality (2.1) holds true and $|z|=1$, we obtain

$$\begin{aligned}\left|\frac{z\left(D_\lambda^m(\alpha,\beta)f(z)\right)'}{D_\lambda^m(\alpha,\beta)f(z)}-1\right| &= \left|\frac{\sum_{k=2}^{\infty}(k-1)[1+(k-1)\lambda]^m\frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)}a_kz^k}{z-\sum_{k=2}^{\infty}[1+(k-1)\lambda]^m\frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)}a_kz^k}\right| \\ &\leq \frac{\vartheta|\zeta|\left(1-\sum_{k=2}^{\infty}[1+(k-1)\lambda]^m\frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)}a_k\right)}{1-\sum_{k=2}^{\infty}[1+(k-1)\lambda]^m\frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)}a_k} \\ &\leq \vartheta|\zeta|\end{aligned}$$

Hence $f\in T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta)$. □

Corollary 2.1. *If the function f of the form (1.2) is in the class $T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta)$ then*

$$a_k\leq\frac{\vartheta|\zeta|\Gamma(\alpha(k-1)+\beta)}{(k+\vartheta|\zeta|-1)[1+(k-1)\lambda]^m\Gamma(\beta)},\qquad k\geq 2$$

the result is sharp for the function

$$f_k(z)=z-\frac{\vartheta|\zeta|\Gamma(\alpha(k-1)+\beta)}{(k+\vartheta|\zeta|-1)[1+(k-1)\lambda]^m\Gamma(\beta)}z^k,\qquad k\geq 2.$$

Growth and distortion properties for functions belonging to the class $T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta)$ will be given in the following results:

3 Growth and Distortion Theorems

Theorem 3.1. *If f an analytic function given by (1.2) is in the class $T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta)$. Then we have*

$$|z|-\frac{\vartheta|\zeta|\Gamma(\alpha+\beta)}{(1+\vartheta|\zeta|)[1+\lambda]^m\Gamma(\beta)}|z|^2\leq|f(z)|\leq|z|+\frac{\vartheta|\zeta|\Gamma(\alpha+\beta)}{(1+\vartheta|\zeta|)[1+\lambda]^m\Gamma(\beta)}|z|^2.$$

The bounds are sharp, since the equality is attained by the function

$$f(z)=z-\frac{\vartheta|\zeta|\Gamma(\alpha+\beta)}{(1+\vartheta|\zeta|)[1+\lambda]^m\Gamma(\beta)}z^2.$$

Proof. In view of Theorem 2.1, we have

$$\sum_{k=2}^{\infty}(k+\vartheta|\zeta|-1)[1+(k-1)\lambda]^m\frac{\Gamma\beta}{\Gamma(\alpha(k-1)+\beta)}a_k \leq \vartheta|\zeta|$$

and

$$\frac{(1+\vartheta|\zeta|)[1+\lambda]^m\Gamma(\beta)}{\Gamma(\alpha+\beta)}\sum_{k=2}^{\infty}a_k \leq \sum_{k=2}^{\infty}(k+\vartheta|\zeta|-1)[1+(k-1)\lambda]^m\frac{\Gamma\beta}{\Gamma(\alpha(k-1)+\beta)}a_k \leq \vartheta|\zeta|.$$

So, we have

$$\sum_{k=3}^{\infty}a_k \leq \frac{\vartheta|\zeta|\Gamma(\alpha+\beta)}{(1+\vartheta|\zeta|)[1+\lambda]^m\Gamma(\beta)}.$$

Thus for $f \in T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta)$, we obtain

$$\begin{aligned}
|f(z)| &= |z-\sum_{k=2}^{\infty}a_k z^k| \\
&\leq |z|+\sum_{k=2}^{\infty}a_k|z|^2 \\
&\leq |z|+\frac{\vartheta|\zeta|\Gamma(\alpha+\beta)}{(1+\vartheta|\zeta|)[1+\lambda]^m\Gamma(\beta)}|z|^2.
\end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned}
|f(z)| &= |z-\sum_{k=2}^{\infty}a_k z^k| \\
&\geq |z|-\sum_{k=2}^{\infty}a_k|z|^2 \\
&\geq |z|-\frac{\vartheta|\zeta|\Gamma(\alpha+\beta)}{(1+\vartheta|\zeta|)[1+\lambda]^m\Gamma(\beta)}|z|^2.
\end{aligned}$$

This completes the proof. □

Similarly, Following the same method in Theorem 3.1 we can prove the following Theorem.

Theorem 3.2. *If f an analytic function given by (1.2) is in the class $T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta)$. Then we have*

$$1-\frac{2\vartheta|\zeta|\Gamma(\alpha+\beta)}{(1+\vartheta|\zeta|)[1+\lambda]^m\Gamma(\beta)}|z| \leq |f'(z)| \leq 1+\frac{2\vartheta|\zeta|\Gamma(\alpha+\beta)}{(1+\vartheta|\zeta|)[1+\lambda]^m\Gamma(\beta)}|z|.$$

The bounds are sharp , since the equality is attained by the function

$$f(z)=z-\frac{\vartheta|\zeta|\Gamma(\alpha+\beta)}{(1+\vartheta|\zeta|)[1+\lambda]^m\Gamma(\beta)}z^2.$$

Proof. For $f \in T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta)$, we have

$$\begin{aligned} |f^{'}(z)| &= |1-\sum_{k=2}^{\infty} k a_k z^{k-1}| \\ &\leq 1+2\sum_{k=2}^{\infty} a_k |z| \\ &\leq 1+\frac{2\vartheta|\zeta|\Gamma(\alpha+\beta)}{(1+\vartheta|\zeta|)[1+\lambda]^m\Gamma(\beta)}|z|. \end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned} |f'(z)| &= |1-\sum_{k=2}^{\infty} k a_k z^{k-1}| \\ &\geq 1-2\sum_{k=2}^{\infty} a_k |z| \\ &\geq 1-\frac{2\vartheta|\zeta|\Gamma(\alpha+\beta)}{(1+\vartheta|\zeta|)[1+\lambda]^m\Gamma(\beta)}|z|. \end{aligned}$$

This completes the proof. □

4 Extreme Points

The extreme points of the class $T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta)$ will be now determined.

Theorem 4.1. *Let* $f_1(z) = z$ *and*

$$f_k(z) = z - \frac{\vartheta|\zeta|\Gamma(\alpha(k-1)+\beta)}{(k+\vartheta|\zeta|-1)[1+(k-1)\lambda]^m\Gamma(\beta)}z^k, \qquad k \geq 2$$

Then $f \in T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=1}^{\infty} \nu_k f_k(z)$$

where

$$\nu_k \geq 0 \qquad and \qquad \sum_{k=1}^{\infty} \nu_k = 1.$$

Proof. Let $f(z) = \sum_{k=1}^{\infty} \nu_k f_k(z)$ where $\nu_k \geq 0$ and $\sum_{k=1}^{\infty} \nu_k = 1$

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty} \nu_k f_k(z) \\ &= \nu_1 f_1(z) + \sum_{k=2}^{\infty} \nu_k f_k(z) \end{aligned}$$

$$
\begin{aligned}
&= \nu_1 f_1(z) + \sum_{k=2}^{\infty} \nu_k \left\{ z - \frac{\vartheta|\zeta|\Gamma(\alpha(k-1)+\beta)}{(k+\vartheta|\zeta|-1)[1+(k-1)\lambda]^m \Gamma(\beta)} z^k \right\} \\
&= \nu_1 z + \sum_{k=2}^{\infty} \nu_k z - \sum_{k=2}^{\infty} \nu_k \frac{\vartheta|\zeta|\Gamma(\alpha(k-1)+\beta)}{(k+\vartheta|\zeta|-1)[1+(k-1)\lambda]^m \Gamma(\beta)} z^k \\
&= \sum_{k=1}^{\infty} \nu_k z - \sum_{k=2}^{\infty} \nu_k \frac{\vartheta|\zeta|\Gamma(\alpha(k-1)+\beta)}{(k+\vartheta|\zeta|-1)[1+(k-1)\lambda]^m \Gamma(\beta)} z^k \\
&= z - \sum_{k=2}^{\infty} \nu_k \frac{\vartheta|\zeta|\Gamma(\alpha(k-1)+\beta)}{(k+\vartheta|\zeta|-1)[1+(k-1)\lambda]^m \Gamma(\beta)} z^k.
\end{aligned}
$$

Then, we have

$$
\sum_{k=2}^{\infty} (k+\vartheta|\zeta|-1)[1+(k-1)\lambda]^m \frac{\Gamma\beta}{\Gamma(\alpha(k-1)+\beta)} \left\{ \nu_k \frac{\vartheta|\zeta|\Gamma(\alpha(k-1)+\beta)}{(k+\vartheta|\zeta|-1)[1+(k-1)\lambda]^m \Gamma(\beta)} \right\}
$$

$$
= \sum_{k=2}^{\infty} \nu_k \vartheta|\zeta| = \vartheta|\zeta| \left(\sum_{k=1}^{\infty} \nu_k - \nu_1 \right) = (1-\nu_1)\vartheta|\zeta| \leq \vartheta|\zeta|.
$$

The condition (2.1) is satisfied. Thus, $f \in T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta)$.

Conversely, we suppose that $f \in T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta)$, since

$$
a_k \leq \frac{\vartheta|\zeta|\Gamma(\alpha(k-1)+\beta)}{(k+\vartheta|\zeta|-1)[1+(k-1)\lambda]^m \Gamma(\beta)}, \qquad k \geq 2
$$

we set

$$
\nu_k = \frac{(k+\vartheta|\zeta|-1)[1+(k-1)\lambda]^m \Gamma(\beta)}{\vartheta|\zeta|\Gamma(\alpha(k-1)+\beta)} a_k, \qquad \nu_1 = 1 - \sum_{k=2}^{\infty} \nu_k.
$$

Then

$$
\begin{aligned}
f(z) &= z - \sum_{k=2}^{\infty} a_k z^k \\
&= z - \sum_{k=2}^{\infty} \nu_k \frac{\vartheta|\zeta|\Gamma(\alpha(k-1)+\beta)}{(k+\vartheta|\zeta|-1)[1+(k-1)\lambda]^m \Gamma(\beta)} z^k \\
&= (1 - \sum_{k=2}^{\infty} \nu_k) z + \sum_{k=2}^{\infty} \nu_k f_k(z) \\
&= \nu_1 f_1(z) + \sum_{k=2}^{\infty} \nu_k f_k(z) \\
&= \sum_{k=1}^{\infty} \nu_k f_k(z).
\end{aligned}
$$

This completes the assertion of Theorem 4.1. □

Corollary 4.1. *The extreme points of* $T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta)$ *are the functions* $f_1(z) = z$ *and*

$$
f_k(z) = z - \frac{\vartheta|\zeta|\Gamma(\alpha(k-1)+\beta)}{(k+\vartheta|\zeta|-1)[1+(k-1)\lambda]^m \Gamma(\beta)} z^k, \qquad k \geq 2.
$$

5 Closure Theorems

Theorem 5.1. *Let $\upsilon_i \geq 0$ for $i = 1, 2, ..., l$ and $\sum_{i=1}^{l} \upsilon_i = 1$. If the functions f_i defined by*

$$f_i(z) = z - \sum_{k=1}^{\infty} a_{k,i} z^k \qquad (a_{i,k} \geq 0, i = 1, 2, ..., l) \tag{5.1}$$

are in the class $T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta, \vartheta)$ for every $i = 1, 2, ..., l$, then the function $h(z)$ defined by

$$h(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{l} \upsilon_i a_{k,i} \right) z^k$$

in the class $T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta, \vartheta)$.

Proof. Since $f_i(z) \in T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta, \vartheta)$ for every $i = 1, 2, ..., l$ we have

$$\sum_{k=2}^{\infty} (k + \vartheta|\zeta| - 1)[1 + (k-1)\lambda]^m \frac{\Gamma\beta}{\Gamma(\alpha(k-1)+\beta)} a_{k,i} \leq \vartheta|\zeta|.$$

Hence, we get

$$\begin{aligned} &\sum_{k=2}^{\infty} \left\{ (k + \vartheta|\zeta| - 1)[1 + (k-1)\lambda]^m \frac{\Gamma\beta}{\Gamma(\alpha(k-1)+\beta)} \right\} \left(\sum_{i=1}^{l} \upsilon_i a_{k,i} \right) \\ &= \sum_{i=1}^{l} \upsilon_i \left(\sum_{k=2}^{\infty} (k + \vartheta|\zeta| - 1)[1 + (k-1)\lambda]^m \frac{\Gamma\beta}{\Gamma(\alpha(k-1)+\beta)} a_{k,i} \right) \\ &\leq \vartheta|\zeta| \sum_{i=1}^{l} \upsilon_i \leq \vartheta|\zeta|. \end{aligned}$$

This implies that h is in the class $T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta, \vartheta)$. Thus the proof of the theorem is complete. □

Corollary 5.1. *The class $T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta, \vartheta)$ is closed under convex linear combination.*

Proof. Assume that the functions $f_i (i = 1; 2)$ given by (5.1) are in $T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta, \vartheta)$. It suffices to show that the function h defined by

$$h(z) = cf_1(z) + (1-c)f_2(z), \qquad (0 \leq c \leq 1)$$

is in the class $T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta, \vartheta)$. By taking $l = 2, \upsilon_1 = c \quad and \quad \upsilon_2 = 1 - c$ in Theorem 5.1 we obtain the corollary. □

6 Convolution Properties

In this section we will prove that the class $T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta)$ is closed under convolution.

Theorem 6.1. *Let $g(z)$ of the form*

$$g(z) = z - \sum_{k=2}^{\infty} b_k z^k$$

be analytic in $\mathcal{U}$. If $f \in T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta)$ then, the functin $f * g$ is also in the class $T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta)$. Here the symbol "$*$" denoted to the Hadamard product (or convolution).

Proof. Since $f \in T\mathcal{S}_{\lambda,\beta}^{m,\alpha}(\zeta,\vartheta)$, we have

$$\sum_{k=2}^{\infty}(k+\vartheta|\zeta|-1)[1+(k-1)\lambda]^m \frac{\Gamma\beta}{\Gamma(\alpha(k-1)+\beta)} a_k \leq \vartheta|\zeta|.$$

By utilizing the last inequality and the fact that

$$f(z) * g(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k$$

we obtain

$$\sum_{k=2}^{\infty}(k+\vartheta|\zeta|-1)[1+(k-1)\lambda]^m \frac{\Gamma\beta}{\Gamma(\alpha(k-1)+\beta)} a_k b_k \leq \sum_{k=2}^{\infty}(k+\vartheta|\zeta|-1)[1+(k-1)\lambda]^m \frac{\Gamma\beta}{\Gamma(\alpha(k-1)+\beta)} a_k \leq \vartheta|\zeta|$$

and hence, in view of Theorem 2.1, the result follows. □

Conflict of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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