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q-Analogue of Ruscheweyh Operator and its Applications to Certain Subclass of Uniformly starlike Functions

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Abstract. Motivating by q-analogue theory, we define here q-analogue of Ruscheweyh differential operator. Also, we introduce a new subclass of uniformly starlike functions with negative coefficients and obtain the coefficient bounds, distotion theorem for functions belonging to this class.

Keywords:q-analogue of Ruscheweyh differential operator, uniformly starlike function.
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INTRODUCTION

The theory of q-analogues is an active field of research to day since it has assumed of significant importance for studying q-calculus (q-differentiation and q-integration).This theory of q-analogues or q-extensions of classical formulas and functions based on the observation that

$$\lim_{q\to 1}\frac{1-q^{\alpha}}{1-q}=\alpha.$$

Therefore, the number $\frac{(1-q^{\alpha})}{(1-q)}$ is sometimes called the basic number $[\alpha]_q$.

The theory of q- analogues find applications in a number of areas including combinatorial theory, q-deformed superalgebras, hypergeometric orthogonal polynomials, dynamical systems and physics. This paper is an attempt to give a connection of q-analogues theory and geometric function theory by obtaining the q-analogue of Ruscheweyh differential operator using basic hypergeometric function and defining a new
class of uniformly starlike functions.

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{k=2}^{\infty} a_k z^k, (1)$$

which are analytic and univalent in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. Also, denote by T the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z)=z-\sum_{k=2}^{\infty} a_k z^k, \quad (a_k\geq 0) \qquad (1)$$

A function f is uniformly convex (uniformly starlike) in $\mathbb{U}$. if f is in $CV(S^*)$ and has the property that for every circular arcγ contained in $\mathbb{U}$, with center ξ also in $\mathbb{U}$, the arc $f(\gamma)$ is convex(starlike) with respect to $f(\xi)$. The class of uniformly convex functions is denoted by $UCV$and the class of uniformly starlike functions by US^*. Analytically,

$$f(z)\in UCV \Leftrightarrow Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\left|\frac{zf''(z)}{f'(z)}\right|, z\in\mathbb{U}.$$

In [1,2], Ronning introduced a new class of starlike functions related to UCV defined as

$$f(z)\in S_p \Leftrightarrow Re\left\{\frac{zf'(z)}{f(z)}\right\}>\left|\frac{zf'(z)}{f(z)}-1\right|, z\in\mathbb{U}.$$

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Note that, $f \in UCV \Leftrightarrow zf' \in S_p$. These classes have been extensively studied by Goodman [3,4], Kanas and srivastava[5], Ma and Minda [6] . Moreover, Shams, Kulkarni and Jahangiri [7] introduced the subclass $US_p^*(\alpha,\beta)$ of uniformly β-starlike functions of order α,consisting of functions which satisfy

$$Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} > \beta\left|\frac{zf'(z)}{f(z)} - 1\right|, z \in \mathbb{U},$$

for some $0 \leq \alpha < 1, \beta \geq 0$. Later, the class was studied by Nishiwaki and Owa[8].

Let $f, g \in \mathcal{A}$, where f is defined by (2) and g is given by

$$g(z) = z - \sum_{k=2}^{\infty} b_k z^k.$$

Then, the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k.$$

One of the most important summation formulas for hypergeometric series is given by the binomial Series is

$${}_2F_1(a,c,c;z) = {}_1F_0(a,-;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k = (1-z)^{-a}, \quad a \in \mathbb{R},$$

where $|z| < 1$. The q-analogue of this formula is defined by

$${}_1\phi_0(a,-;q;z) = \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)} z^k, \quad |z| < 1, |q| < 1, (2)$$

which was derived by Cauchy (1843), Heine (1847) and by other mathematicians. $(a,q)_k$ is the q-analogue of the Pochhammer symbol $(a)_k$ defined by

$$(a,q)_k = \begin{cases} 1, & k = 0; \\ (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{k-1}), & k \in \mathbb{N}. \end{cases}$$

It is clear that

$$\lim_{q\to 1} \frac{(q^a;q)_k}{(1-q)^k} = (a)_k.$$

By using the ratio test, one recognize that, if $|q| < 1$, the q series (3) (also called basic hypergeometric series) converges absolutely for $|z| < 1$. For more details concerning the q-theory may refer to [9,10].

Replacing a by $(\lambda + 1, \lambda > -1)$ in (3), we now define the q-analogue of Ruscheweyh differential operator $\mathcal{D}_q^\lambda f: \mathcal{A} \to \mathcal{A}$ as follows:

$$\begin{aligned} \mathcal{D}_q^\lambda f(z) &= {}_1\phi_0(a,-;q;z) * f(z) \\ &= z + \sum_{k=2}^{\infty} \frac{(\lambda+1;q)_{k-1}}{(q;q)_{k-1}} a_k z^k, \quad \lambda > -1, |z| < 1, |q| < 1. \end{aligned} \quad (3)$$

Observe that if $\lambda + 1 = q^{\mu+1}, \mu > -1$, we have

$$\lim_{q\to 1} \mathcal{D}_q^\lambda f(z) = z + \lim_{q\to 1}\left[\sum_{k=2}^{\infty} \frac{(q^{\mu+1};q)_{k-1}}{(q;q)_{k-1}} a_k z^k\right] = z + \sum_{k=2}^{\infty} \frac{(\mu+1)_{k-1}}{k!} a_k z^k = \mathcal{D}^\mu f(z),$$

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where $\mathcal{D}^{\mu} f(z)$ is Ruscheweyh differential operator[11].

For $-1 < \alpha \leq 1$, $\beta \geq 0$, we let $S_p^*(q, \alpha, \beta)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1) and satisfying the analytic criterion

$$Re\left\{\frac{z\left(\mathcal{D}_q^{\lambda} f(z)\right)'}{\mathcal{D}_q^{\lambda} f(z)} - \alpha\right\} > \beta\left|\frac{z\left(\mathcal{D}_q^{\lambda} f(z)\right)'}{\mathcal{D}_q^{\lambda} f(z)} - 1\right|, z \in \mathbb{U}, (4)$$

where $\mathcal{D}_q^{\lambda} f(z)$ is given by (4). We also let $TS_p^*(q, \alpha, \beta) = S_p^*(q, \alpha, \beta) \cap T$.

BASIC PROPERTIES

In this section we obtain a necessary and sufficient condition for functions f in the classes $S_p^*(q, \alpha, \beta)$ and $TS_p^*(q, \alpha, \beta)$.

Theorem 1. A function $f(z)$ of the form (1) is in $S_p^*(q, \alpha, \beta)$ if

$$\sum_{k=2}^{\infty} [k(1+\beta) - (\alpha+\beta)] \frac{(\lambda+1;q)_{k-1}}{(q;q)_{k-1}} |a_k| \leq 1 - \alpha, (5)$$

where $-1 < \alpha \leq 1$, $\beta \geq 0$ and $0 \leq \gamma \leq 1$

Proof. It suffices to show that

$$\beta\left|\frac{z(\mathcal{D}_q^{\lambda} f(z))'}{(\mathcal{D}_q^{\lambda} f(z)} - 1\right| - Re\left\{\frac{z(\mathcal{D}_q^{\lambda} f(z))'}{\mathcal{D}_q^{\lambda} f(z)} - 1\right\} \leq 1 - \alpha.$$

We have

$$\beta\left|\frac{z(\mathcal{D}_q^{\lambda} f(z))'}{\mathcal{D}_q^{\lambda} f(z)} - 1\right| - Re\left\{\frac{z(\mathcal{D}_q^{\lambda} f(z))'}{\mathcal{D}_q^{\lambda} f(z)} - 1\right\} \leq (1+\beta)\left|\frac{z(\mathcal{D}_q^{\lambda} f(z))'}{\mathcal{D}_q^{\lambda} f(z)} - 1\right|$$

$$\leq \frac{(1+\beta)\sum_{k=2}^{\infty}(k-1)\frac{(\lambda+1;q)_{k-1}}{(q;q)_{k-1}}|a_k|}{1 - \sum_{k=2}^{\infty}\frac{(\lambda+1;q)_{k-1}}{(q;q)_{k-1}}|a_k|}.$$

This last expression is bounded above by $(1-\alpha)$ if

$$\sum_{k=2}^{\infty} [k(1+\beta) - (\alpha+\beta)] \frac{(\lambda+1;q)_{k-1}}{(q;q)_{k-1}} |a_k| \leq 1 - \alpha,$$

hence the proof is complete.

Theorem 2. *A necessary and sufficient condition for $f(z)$ of the form (2) to be in the class $TS_p^*(q, \alpha, \beta)$ is that*

$$\sum_{k=2}^{\infty} [k(1+\beta) - (\alpha+\beta)] \frac{(\lambda+1;q)_{k-1}}{(q;q)_{k-1}} |a_k| \leq 1 - \alpha,$$

where $-1 < \alpha \leq 1$, $\beta \geq 0$.

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Proof. In view of Theorem 1, we need only to prove the necessary condition. If $f \in TS_p^*(q,\alpha,\beta)$ and z is real then

$$\frac{1-\sum_{k=2}^{\infty} k\frac{(\lambda+1;q)_{k-1}}{(q;q)_{k-1}}a_k z^{k-1}}{1-\sum_{k=2}^{\infty}\frac{(\lambda+1;q)_{k-1}}{(q;q)_{k-1}}a_k z^{k-1}}-\alpha$$

$$\geq \beta\left|\frac{\sum_{k=2}^{\infty}(k-1)\frac{(\lambda+1;q)_{k-1}}{(q;q)_{k-1}}|a_k|}{1-\sum_{k=2}^{\infty}\frac{(\lambda+1;q)_{k-1}}{(q;q)_{k-1}}|a_k|}\right|$$

Letting $z \to 1$ along the real axis, we obtain the desired inequality

$$\sum_{k=2}^{\infty}[k(1+\beta)-(\alpha+\beta)]\frac{(\lambda+1;q)_{k-1}}{(q;q)_{k-1}}|a_k| \leq 1-\alpha.$$

The next theorem shows that the class $TS_p^*(q,\alpha,\beta)$ is closed under linear convex combination.

Theorem 3. *Let $f_i(z), i=1,2$ defined by (2) in the class $TS_p^*(q,\alpha,\beta)$. Then, the function $F(z)$is defined by*

$$F(z)=(1-\eta)f_1(z)+\eta f_2(z)=z-\sum_{k=2}^{\infty}c_k z^k,$$

where $(c_k=(1-\eta)a_{k,1}+\eta a_{k,2})$,$0\leq\eta<1$ is also in the class $TS_p^*(q,\alpha,\beta)$.

Proof. Since

$$f_i(z)=z-\sum_{k=2}^{\infty}a_{k,i}z^k, \quad a_{k,i}\geq 0, i=1,2,$$

in $TS_p^*(q,\alpha,\beta)$, so that

$$F(z)=z-\sum_{k=2}^{\infty}[\eta a_{k,1}+(1-\eta)a_{k,2}]z^k, \quad 0\leq\eta\leq 1.$$

An easy computation with the aid of Theorem 2 gives

$$\sum\nolimits_{\mathrm{k}=2}^{\infty}[k(1+\beta)-(\alpha+\beta)]\frac{(\lambda+1;q)_{k-1}}{(q;q)_{k-1}}\eta a_{k,1}$$

$$+\sum\nolimits_{\mathrm{k}=2}^{\infty}[k(1+\beta)-(\alpha+\beta)]\frac{(\lambda+1;q)_{k-1}}{(q;q)_{k-1}}(1-\eta)a_{k,2}$$

$$\leq \eta(1-\alpha)+(1-\eta)(1-\alpha)\leq 1-\alpha,$$

which implies that $F\in TS_p^*(q,\alpha,\beta)$. Hence $TS_p^*(q,\alpha,\beta)$ is closed under linear convex combination.

Theorem 4. Let the hypotheses of Theorem 2be satisfied. Then, for $z\in\mathbb{U}$ and $-1\leq\alpha\leq 1,\beta\geq 0$, we have

$$|\mathcal{D}_q^{\lambda}f(z)|\leq|z|-\frac{1-\alpha}{[2-\alpha+\beta]}|z|^2 \text{and} |\mathcal{D}_q^{\lambda}f(z)|\geq|z|+\frac{1-\alpha}{[2-\alpha+\beta]}|z|^2$$

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Proof. By using Theorem 2, one can verify that

$$[2-\alpha+\beta]\sum_{k=2}^{\infty}\frac{(\lambda+1;q)_{k-1}}{(q;q)_{k-1}}|a_k| \le \sum_{k=2}^{\infty}[k(1+\beta)-(\alpha+\beta)]\frac{(\lambda+1;q)_{k-1}}{(q;q)_{k-1}}|a_k|$$

$$\le 1-\alpha,$$

then

$$\sum_{k=2}^{\infty}\frac{(\lambda+1;q)_{k-1}}{(q;q)_{k-1}}|a_k| \le \frac{1-\alpha}{[2-\alpha+\beta]}.$$

Thus we obtain

$$\begin{aligned}|\mathcal{D}_q^{\lambda}f(z)| &\le |z| + \sum_{k=2}^{\infty}\frac{(\lambda+1;q)_{k-1}}{(q;q)_{k-1}}|a_k||z|^k \\ &\le |z| + \sum_{k=2}^{\infty}\frac{(\lambda+1;q)_{k-1}}{(q;q)_{k-1}}|a_k||z|^2 \\ &\le |z| + \frac{1-\alpha}{[2-\alpha+\beta]}|z|^2.\end{aligned}$$

The proof of the second assertion is similar to the proof of the first assertion.

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REFERENCES

1. F. Rønning, “Uniformly convex functions and a corresponding class of starlike functions”, *Pros. Amer. Math. Soc*., vol. 118, 1993, pp. 189-196.
2. F. Rønning, “Integral representations for bounded starlike functions”, *Annal. Polon. Math*., vol. 60, 1995, pp. 289-297.
3. A. W. Goodman, “On uniformly convex functions”, *Ann. polon. Math*., vol. 56, 1991, pp. 87-92.
4. A. W. Goodman, “On uniformly starlike functions,” *Journal of Mathematical Analysis and Applications*, vol. 155, no. 2, 1991, pp. 364–370.
5. S. Kanas and H. M. Srivastava, “Linear operators associated with k-uniformly convex functions,” *Integral Transforms and Special Functions*, vol. 9, no. 2, 2000, pp. 121–132.
6. W. C. Ma and D. Minda, “Uniformly convex functions,” *Annales Polonici Mathematici*, vol. 57, no. 2, 1992, pp. 165–175.
7. S. Shams, S. R. Kulkarni, and J. M. Jahangiri, “Classes of uniformly starlike and convex functions,” *International Journal of Mathematics and Mathematical Sciences*, vol. 2004, no. 55, 2004, pp. 2959–2961.
8. J. Nishiwaki and S. Owa, “Certain classes of analytic functions concerned with uniformly starlike and convex functions,” *Applied Mathematics and Computation*, vol. 187, no. 1, 2007, pp. 350–355.
9. G.Gasper and M. Rahman, *Basic Hypergeometric Series,* Cambridge, England: Cambridge University Press, 1990.
10. H. Exton, *q-Hypergeometric functions and applications*, Ellis Horwood Limited, Chichester, 347 pages, (1983).
11. S. Ruscheweyh, *New criteria for univalent functions*, Proceedings of the America Mathematical Society, vol. 49, 1975, pp. 109-115.

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