Wolfe E-duality for E-differentiable E-invex vector optimization problems with inequality and equality constraints

Najeeb Abdulaleem
Department of Mathematics
Hadhramout University
Al-Mahrah, Yemen
nabbas985@gmail.com

***Abstract*—In this paper, the class of E-differentiable vector optimization problems with both inequality and equality constraints is considered. The so-called vector Wolfe E-dual problem is defined for the considered E-differentiable multiobjective programming problem with both inequality and equality constraints and several E-dual theorems are established under (generalized) E-invexity hypotheses.**

***Index Terms*—E-invex set, E-invex function, E-differentiable function, Wolfe E-duality.**

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I. Introduction

In the classical theory of duality, the theorems on duality in various senses are based on convexity assumptions. Many attempts have been made to weaken these assumptions by introducing various generalized convexity concepts. One of such important generalizations of the convexity notion is the concept of invexity introduced by Hanson [14]. In the case of differentiable scalar optimization problems. Namely, Hanson showed that, instead of the usual convexity assumption, if all functions are assumed to be invex (with respect to the same function η), then the sufficient optimality conditions and weak duality can be proved. Jeyakumar and Mond [15] generalized Hanson's definition to the vectorial case. They defined V-invexity of differentiable vector-valued functions which preserve the sufficient optimality conditions and duality results as in the scalar case and avoid the major difficulty of verifying that the inequality holds for the same function η for invex functions in multiobjective programming problems. Ben-Isreal and Mond [6] have defined quasi-invex function as a generalization of invex functions. Luc and Malivert [17] have extended the study of invexity to set-valued maps and vector optimization problems with set-valued data. Bazaraa et al. [7] have studied necessary conditions for optimality in a nonlinear vector optimization problem. Jeyakumar [16] defined generalized invexity for nonsmooth scalar-valued functions, established an equivalence of saddle points and optima, and studied duality results for nonsmooth problems. The concept of invexity for multiobjective nonlinear programming problems have been introduced and studied extensively in the literature (see, for example, [6], [9], [10], [13], [14], [17], and others).

Recently, the concepts of E-convex sets and E-convex functions were introduced by Youness [22]. This kind of generalized convexity is based on the effect of an operator $E : R^n \to R^n$ on the sets and the domains of functions. However, some results and proofs presented by Youness [22] were incorrect as it was pointed out by Yang [21]. Further, Megahed et al. [19] presented the concept of an E-differentiable convex function which transforms a (not necessarily) differentiable convex function to a differentiable function based on the effect of an operator $E : R^n \to R^n$.

Later, Abdulaleem [1] introduced a new concept of generalized convexity as a generalization of the notion of E-differentiable E-convexity. Namely, he defined the concept of E-differentiable E-invexity in the case of (not necessarily) differentiable vector optimization problems with E-differentiable functions.

In this paper, a class of nonconvex E-differentiable vector optimization problems with both inequality and equality constraints is considered in which the involved functions are E-invex. For such a (not necessarily differentiable) multiobjective programming problem, its Wolfe vector E-dual problem is defined. Then, several Wolfe E-duality results are established between the considered E-differentiable multicriteria optimization problem and its vector E-dual under appropriate E-invexity hypotheses.

II. Preliminaries

Let R^n be the n-dimensional Euclidean space and R^n_+ be its nonnegative orthant. The following convention for equalities and inequalities will be used in the paper. For any vectors $x = (x_1, x_2, ..., x_n)^T$ and $y = (y_1, y_2, ..., y_n)^T$ in R^n, we define:

1) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, ..., n$;
2) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, ..., n$;
3) $x \geqq y$ if and only if $x_i \geqq y_i$ for all $i = 1, 2, ..., n$;
4) $x \geq y$ if and only if $x \geqq y$ and $x \neq y$.

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Definition 1: [1] Let $E : R^n \to R^n$. A set $M \subseteq R^n$ is said to be an E-invex set if and only if there exists a vector-valued function $\eta : M \times M \to R^n$ such that the relation

$$E(u) + \lambda\eta(E(x), E(u)) \in M$$

holds for all $x, u \in M$ and any $\lambda \in [0, 1]$.

Remark 2: If η is a vector-valued function defined by $\eta(z, y) = z - y$, then the definition of an E-invex set reduces to the definition of an E-convex set (see Youness [9]).

Remark 3: If $E(a) \equiv a$, then the definition of an E-invex set with respect to the function η reduces to the definition of an invex set with respect to η (see Mohan and Neogy [22]).

Definition 4: [8] Let $E : R^n \to R^n$ and $f : M \to R$ be a (not necessarily) differentiable function at a given point $u \in M$. It is said that f is an E-differentiable function at u if and only if $f \circ E$ is a differentiable function at u (in the usual sense) and, moreover,

$$(f \circ E)(x) = (f \circ E)(u) + \nabla (f \circ E)(u)(x - u) + \theta(u, x - u)\|x - u\|, \tag{1}$$

where $\theta(u, x - u) \to 0$ as $x \to u$.

Definition 5: [1] Let $E : R^n \to R^n$, $M \subseteq R^n$ be a nonempty open E-invex set with respect to the vector-valued function $\eta : R^n \times R^n \to R^n$ and $f : R^n \to R^k$ be an E-differentiable function on M. It is said that f is a vector-valued E-invex function with respect to η at u on M if, for all $x \in M$,

$$f_i(E(x)) - f_i(E(u)) \geqq \nabla f_i(E(u))\eta(E(x), E(u)),\ i = 1, ..., k. \tag{2}$$

If inequalities (2) hold for any $u \in M$, then f is E-invex with respect to η on M.

Remark 6: From Definition 5, there are the following special cases:

a) If f is a differentiable function and $E(x) \equiv x$ (E is an identity map), then the definition of an E-invex function reduces to the definition of an invex function introduced by Hanson [14].
b) If $\eta : R^n \times R^n \to R^n$ is defined by $\eta(x, u) = x - u$, then we obtain the definition of an E-differentiable E-convex vector-valued function introduced by Megahed et al. [8].
c) If f is differentiable, $E(x) = x$ and $\eta(x, u) = x - u$, then the definition of an E-invex function reduces to the definition of a differentiable convex vector-valued function.
d) If f is differentiable and $\eta(x, u) = x - u$, then we obtain the definition of a differentiable E-convex function introduced by Youness [9].

Definition 7: [1] Let $E : R^n \to R^n$, $M \subseteq R^n$ be an open E-invex set with respect to the vector-valued function $\eta : R^n \times R^n \to R^n$ and $f : R^n \to R^k$ be an E-differentiable function on M. It is said that f is a vector-valued strictly E-invex function with respect to η at u on M if, for all $x \in M$ with $E(x) \neq E(u)$, the inequalities

$$f_i(E(x)) - f_i(E(u)) > \nabla f_i(E(u))\eta(E(x), E(u)),\ i = 1, ..., k, \tag{3}$$

hold. If inequalities (3) are fulfilled for any $u \in M$ ($E(x) \neq E(u)$), then f is strictly E-invex with respect to η on M.

Definition 8: [1] Let $E : R^n \to R^n$, $M \subseteq R^n$ be an open E-invex set with respect to the vector-valued function $\eta : R^n \times R^n \to R^n$ and $f : R^n \to R^k$ be an E-differentiable function on M. It is said that f is a vector-valued pseudo E-invex function with respect to η at u on M if, for all $x \in M$ and $i = 1, ..., k$,

$$f_i(E(x)) < f_i(E(u)) \Longrightarrow \nabla f_i(E(u))\eta(E(x), E(u)) < 0. \tag{4}$$

If (4) holds for any $u \in M$, then f is pseudo E-invex with respect to η on M.

Definition 9: [1] Let $E : R^n \to R^n$, $M \subseteq R^n$ be an open E-invex set with respect to the vector-valued function $\eta : R^n \times R^n \to R^n$ and $f : R^n \to R^k$ be an E-differentiable function on M. It is said that f is a vector-valued quasi E-invex function with respect to η at u on M if, for all $x \in M$ and $i = 1, ..., k$,

$$f_i(E(x)) - f_i(E(u)) \leqq 0 \Rightarrow \nabla f_i(E(u))\eta(E(x), E(u)) \leqq 0. \tag{5}$$

If (5) holds for any $u \in M$, then f is quasi E-invex with respect to η on M.

In this paper, we consider the following (not necessarily differentiable) multiobjective programming problem (VP) with both inequality and equality constraints:

$$\begin{aligned} &\text{minimize } f(x) = (f_1(x), ..., f_p(x)) \\ &\text{subject to } g_j(x) \leqq 0,\ j \in J = \{1, ..., m\}, \\ &\qquad h_t(x) = 0,\ t \in T = \{1, ..., q\}, \\ &\qquad x \in R^n, \end{aligned} \tag{VP}$$

where $f_i : R^n \to R$, $i \in I = \{1, ..., p\}$, $g_j : R^n \to R$, $j \in J$, $h_t : R^n \to R$, $t \in T$, are real-valued functions defined on R^n. We shall write $g := (g_1, ..., g_m) : R^n \to R^m$ and $h := (h_1, ..., h_q) : R^n \to R^q$ for convenience.

For the purpose of simplifying our presentation, we introduce some notations which will be used frequently throughout this paper. Let

$$\Omega := \{x \in R^n : g_j(x) \leqq 0,\ j \in J,\ h_t(x) = 0,\ t \in T\}$$

be the set of all feasible solutions of (VP). Further let us denote by $J(x)$ the set of inequality constraint indices that are active at a feasible solution x, that is, $J(x) = \{j \in J : g_j(x) = 0\}$.

For such multicriterion optimization problems, the following concepts of (weak) Pareto optimal solutions are defined as follows: