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Research Article

A Subclass of Harmonic Univalent Functions Associated with q-Analogue of Dziok-Srivastava Operator

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We study a class of complex-valued harmonic univalent functions using a generalized operator involving basic hypergeometric function. Precisely, we give a necessary and sufficient coefficient condition for functions in this class. Distortion bounds, extreme points, and neighborhood of such functions are also considered.

1. Introduction

Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc, and let S_H denote the class of functions which are complex valued, harmonic, univalent, and sense preserving in $\mathbb{U}$ normalized by $f(0) = f_z(0) - 1 = 0$. Each $f \in S_H$ can be expressed as $f = h + \overline{g}$, where h and g are analytic in $\mathbb{U}$. We call h the analytic part and g the coanalytic part of f. A necessary and sufficient condition for f to be locally univalent and sense preserving in $\mathbb{U}$ is that $|h'(z)| > |g'(z)|$ in $\mathbb{U}$ (see [1]). In [2], there is a more comprehensive study on harmonic univalent functions. Thus, for $f = h + \overline{g} \in S_H$, we may write

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k \quad (0 \le b_1 < 1). \tag{1}$$

Note that S_H reduces to S, the class of normalized analytic univalent functions, if the coanalytic part of $f = h + \overline{g}$ is identically zero.

The study of basic hypergeometric series (also called q-hypergeometric series) essentially started in 1748 when Euler considered the infinite product $(q;q)_\infty^{-1} = \prod_{k=0}^{\infty}(1-q^{k+1})^{-1}$. In the literature, we were told that the development of these functions was much slower until, in 1878, Heine converted a simple observation that $\lim_{q\to 1}[(1-q^a)/(1-q)] = a$ which returns the theory of ${}_2\phi_1$ basic hypergeometric series to the famous theory of Gauss's ${}_2F_1$ hypergeometric series. The importance of basic hypergeometric functions is due to their application in deriving q-analogue of well-known functions, such as q-analogues of the exponential, gamma, and beta functions. In this paper, we define a class of starlike harmonic functions using basic hypergeometric functions and investigate its properties like coefficient condition, distortion theorem, and extreme points.

For complex parameters a_i, b_j, q $(i = 1, \ldots, r,\ j = 1, \ldots, s,\ b_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\},\ |q| < 1)$, we define the basic hypergeometric function ${}_r\Phi_s\ (a_1, \ldots, a_r; b_1, \ldots, b_s, q, z)$ by

$$\begin{aligned} &{}_r\Phi_s\left(a_1 \ldots, a_r; b_1, \ldots, b_s, q, z\right) \\ &\quad = \sum_{k=0}^{\infty} \frac{(a_1, q)_k \cdots (a_r, q)_k}{(q,q)_k (b_1, q)_k \cdots (b_s, q)_k} z^k, \end{aligned} \tag{2}$$

$(r = s + 1;\ r, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ z \in \mathbb{U})$ where $\mathbb{N}$ denote the set of positive integers and $(a,q)_k$ is the q-shifted factorial defined by

$$(a,q)_k = \begin{cases} 1, & k = 0; \\ (1-a)(1-aq)\left(1-aq^2\right)\cdots\left(1-aq^{k-1}\right), & k \in \mathbb{N}. \end{cases} \tag{3}$$

We note that

$$\lim_{q\to 1^-}\left[{}_r\Phi_s\left(q^{a_1},\dots,q^{a_r};\ q^{b_1},\dots,q^{b_s},q,(q-1)^{1+s-r}z\right)\right] = {}_rF_s\left(a_1,\dots,a_r;b_1,\dots,b_s,z\right), \tag{4}$$

where ${}_rF_s\ (a_1,\dots,a_r;\ b_1,\dots,b_s,z)$ is the well-known generalized hypergeometric function. By the ratio test, one observes that for $|q|<1$ and $r=s+1$ the series defined in (2) converges absolutely in $\mathbb{U}$ so that it represented an analytic function in $\mathbb{U}$. For more mathematical background of basic hypergeometric functions, one may refer to [3, 4].

The q-derivative of a function $h(x)$ is defined by

$$D_q(h(x)) = \frac{h(qx)-h(x)}{(q-1)x}, \quad q\neq 1,\ x\neq 0. \tag{5}$$

For a function $h(z)=z^k$, we can observe that

$$D_q(h(z)) = D_q\left(z^k\right) = \frac{1-q^k}{1-q}z^{k-1} = [k]_q z^{k-1}. \tag{6}$$

Then $\lim_{q\to 1}D_q(h(z)) = \lim_{q\to 1}[k]_q z^{k-1} = kz^{k-1} = h'(z)$, where $h'(z)$ is the ordinary derivative. For more properties of D_q, see [4, 5].

Corresponding to the function ${}_r\Phi_s(a_1,\dots,a_r;b_1,\dots,b_s,q,z)$, consider

$$\begin{aligned}{}_r\mathscr{G}_s&(a_1,\dots,a_r;b_1,\dots,b_s,q,z)\\ &= z\ {}_r\Phi_s(a_1,\dots,a_r;b_1,\dots,b_s,q,z)\\ &= z+\sum_{k=2}^{\infty}\frac{(a_1,q)_{k-1}\cdots(a_r,q)_{k-1}}{(q,q)_{k-1}(b_1,q)_{k-1}\cdots(b_s,q)_{k-1}}z^k.\end{aligned} \tag{7}$$

The authors [6] defined the linear operator $H_s^r(a_1,\dots,a_r;b_1,\dots,b_s;q)f:\mathscr{A}\to\mathscr{A}$ by

$$\begin{aligned}H_s^r&(a_1,\dots,a_r;b_1,\dots,b_s;q)f(z)\\ &= {}_r\mathscr{G}_s(a_1,\dots,a_r;b_1,\dots,b_s,q,z)*f(z)\\ &= z+\sum_{k=2}^{\infty}\Gamma(a_1,q,k)\,a_kz^k,\end{aligned} \tag{8}$$

where $(*)$ stands for convolution and

$$\Gamma(a_1,q,k) = \frac{(a_1,q)_{k-1}\cdots(a_r,q)_{k-1}}{(q,q)_{k-1}(b_1,q)_{k-1}\cdots(b_s,q)_{k-1}}. \tag{9}$$

To make the notation simple, we write

$$H_s^r[a_1,q]f(z) = H_s^r(a_1,\dots,a_r;b_1,\dots,b_s;q)f(z). \tag{10}$$

We define the operator (8) of harmonic function $f=h+\overline{g}$ given by (1) as

$$H_s^r[a_1,q]f(z) = H_s^r[a_1,q]h(z)+\overline{H_s^r[a_1,q]g(z)}. \tag{11}$$

Definition 1. For $0\le\delta<1$, let $S_H^*(a_1,\delta,q)$ denote the subfamily of starlike harmonic functions $f\in S_H^*$ of the form (1) such that

$$\frac{\partial}{\partial\theta}\left(\arg H_s^r[a_1,q]f\right)\ge\delta, \qquad |z|=r<1. \tag{12}$$

Following [7], a function f is said to be in the class $V_{\overline{H}}(a_1,\delta,q) = S_H^*(a_1,\delta,q)\cap V_H$ if f of the form (1) satisfies the condition that

$$\arg(a_k)=\theta_k \quad \arg(b_k)=\vartheta_k \quad (k\ge n+1;\ n\in\mathbb{N}) \tag{13}$$

and if there exists a real number ρ such that

$$\theta_k+(k-1)\phi\equiv\pi\ (\mathrm{mod}\ 2\pi), \quad \vartheta_k+(k-1)\phi\equiv 0, \quad (k\ge n+1;\ n\in\mathbb{N}). \tag{14}$$

By specializing the parameters of $H_s^r[a_1,q]f$, we obtain different classes of starlike harmonic functions, for example,

(i) for $r=s+1, a_2=b_1,\dots,a_r=b_s$, $S_H^*(q,q,\delta)=SH(\delta)$ [8] is the class of sense-preserving harmonic univalent functions f which are starlike of order δ in $\mathbb{U}$; that is, $\partial/\partial\theta(\arg f(re^{i\theta}))\ge\delta$;

(ii) for $r=s+1$, $a_2=b_1,\dots,a_r=b_s$, and $a_1=q^{n+1},q\to 1$, $S_H^*(q^{n+1},q,\delta)=R_H(n,\alpha)$ [9] is the class of starlike harmonic univalent functions with $(\partial/\partial\theta)(\arg D^nf(z))\ge\delta$, where D is the Ruscheweyh derivative (see [10]);

(iii) for $i=\{1,\dots,r\}, j=\{1,\dots,s\}, r=s+1, a_i=q^{\alpha_i}$, and $b_j=q^{\beta_j}$, $q\to 1$, $S_H^*(a_1,q,\delta)=S_H^*(\alpha_1,\delta)$ [11] is the class of starlike harmonic univalent functions with $(\partial/\partial\theta)(\arg H_s^r[\alpha_1]f)\ge\delta$, where $H_s^r[\alpha_1]$ is the Dziok-Srivastava operator (see [12]).

2. Main Results

In our first theorem, we introduce a sufficient coefficient bound for harmonic functions in $S_H^*(a_1,\delta,q)$.

Theorem 2. *Let $f=h+\overline{g}$ be given by* (1). *If*

$$\sum_{k=2}^{\infty}\left(\frac{[k]_q-\delta}{1-\delta}|a_k|+\frac{[k]_q+\delta}{1-\delta}|b_k|\right)\Gamma(a_1,q,k) \le 1-\frac{1+\delta}{1-\delta}|b_1|, \tag{15}$$

where $a_1=1$, $0\le\delta<1$, and $\Gamma(a_1,q,k)$ is given by (9), *then $f\in S_H^*(a_1,\delta,q)$.*

Proof. To prove that $f \in S_H^*(a_1, \delta, q)$, we only need to show that if (15) holds, then the required condition (12) is satisfied. For (12), we can write

$$\frac{\partial}{\partial\theta}\left(\arg H_s^r\left[a_1,q\right]f\left(z\right)\right) = \mathfrak{R}\left\{\frac{zD_q\left(H_s^r\left[a_1,q\right]h\left(z\right)\right)}{H_s^r\left[a_1,q\right]h\left(z\right)+\overline{H_s^r\left[a_1,q\right]g\left(z\right)}} - \frac{\overline{zD_q\left(H_s^r\left[a_1,q\right]g\left(z\right)\right)}}{H_s^r\left[a_1,q\right]h\left(z\right)+\overline{H_s^r\left[a_1,q\right]g\left(z\right)}}\right\} = \mathfrak{R}\frac{A\left(z\right)}{B\left(z\right)}. \tag{16}$$

Using the fact that $\mathfrak{R}(w) \geq \delta$ if and only if $|1-\delta+w| \geq |1+\delta-w|$, it suffices to show that

$$\left|A\left(z\right)+\left(1-\delta\right)B\left(z\right)\right| - \left|A\left(z\right)-\left(1+\delta\right)B\left(z\right)\right| \geq 0. \tag{17}$$

Substituting for $A(z)$ and $B(z)$ in (15) yields

$$\begin{aligned}
&\left|A\left(z\right)+\left(1-\delta\right)B\left(z\right)\right| - \left|A\left(z\right)-\left(1+\delta\right)B\left(z\right)\right| \\
&\geq \left(2-\delta\right)|z| - \sum_{k=2}^{\infty}\left(\left[k\right]_q+1-\delta\right)\Gamma\left(a_1,q,k\right)\left|a_k\right||z|^k \\
&\quad - \sum_{k=1}^{\infty}\left(\left[k\right]_q-1+\delta\right)\Gamma\left(a_1,q,k\right)\left|b_k\right||z|^k \\
&\quad - \delta|z| - \sum_{k=2}^{\infty}\left(\left[k\right]_q-1-\delta\right)\Gamma\left(a_1,q,k\right)\left|a_k\right||z|^k \\
&\quad - \sum_{k=1}^{\infty}\left(\left[k\right]_q+1+\delta\right)\Gamma\left(a_1,q,k\right)\left|b_k\right||z|^k \\
&\geq 2\left(1-\delta\right)|z|\left\{1-\sum_{k=2}^{\infty}\frac{\left[k\right]_q-\delta}{1-\delta}\Gamma\left(a_1,q,k\right)\left|a_k\right|\right. \\
&\quad \left. - \sum_{k=1}^{\infty}\frac{\left[k\right]_q+\delta}{1-\delta}\Gamma\left(a_1,q,k\right)\left|b_k\right|\right\} \\
&= 2\left(1-\delta\right)|z|\left\{1-\frac{1+\delta}{1-\delta}\left|b_1\right|\right. \\
&\quad - \left[\sum_{k=2}^{\infty}\left(\frac{\left[k\right]_q-\delta}{1-\delta}\left|a_k\right|+\frac{\left[k\right]_q+\delta}{1-\delta}\left|b_k\right|\right)\right] \\
&\quad \left. \times\Gamma\left(a_1,q,k\right)\right\}.
\end{aligned} \tag{18}$$

The last expression is nonnegative by (15), and so, $f \in S_H^*(a_1, \delta, q)$. □

Now, we obtain the necessary and sufficient conditions for $f = h + \overline{g}$ given by (14).

Theorem 3. *Let* $f = h + \overline{g}$ *be given by* (11). *Then,* $f \in V_{\overline{H}}(a_1, \delta, q)$ *if and only if*

$$\sum_{k=2}^{\infty}\left(\frac{\left[k\right]_q-\delta}{1-\delta}\left|a_k\right|+\frac{\left[k\right]_q+\delta}{1-\delta}\left|b_k\right|\right)\Gamma\left(a_1,q,k\right) \leq 1-\frac{1+\delta}{1-\delta}\left|b_1\right|, \tag{19}$$

where $a_1 = 1$, $0 \leq \delta < 1$, *and* $\Gamma(a_1, q, k)$ *is given by* (9).

Proof. Since $V_{\overline{H}}(a_1, \delta, q) \subset S_H^*(a_1, \delta, q)$, we only to prove the only if part of the theorem. So that for functions $f \in V_{\overline{H}}(a_1, \delta, q)$, we notice that the condition $(\partial/\partial\theta)(\arg H_s^r[a_1, q]f(z)) \geq \delta$ is equivalent to

$$\frac{\partial}{\partial\theta}\left(\arg H_s^r\left[a_1,q\right]f\left(z\right)\right)-\delta = \mathfrak{R}\left\{\frac{zD_q\left(H_s^r\left[a_1,q\right]h\left(z\right)\right)}{H_s^r\left[a_1,q\right]h\left(z\right)+\overline{H_s^r\left[a_1,q\right]g\left(z\right)}} - \frac{\overline{zD_q\left(H_s^r\left[a_1,q\right]g\left(z\right)\right)}}{H_s^r\left[a_1,q\right]h\left(z\right)+\overline{H_s^r\left[a_1,q\right]g\left(z\right)}} - \delta\right\} \geq 0. \tag{20}$$

That is,

$$\mathfrak{R}\left[\frac{\left(1-\delta\right)z+\sum_{k=2}^{\infty}\left(\left[k\right]_q-\delta\right)\Gamma\left(a_1,q,k\right)\left|a_k\right|z^k-\sum_{k=1}^{\infty}\left(\left[k\right]_q+\delta\right)\overline{\Gamma\left(a_1,q,k\right)}\left|b_k\right|\overline{z}^k}{z+\sum_{k=2}^{\infty}\Gamma\left(a_1,q,k\right)\left|a_k\right|z^k+\sum_{k=1}^{\infty}\overline{\Gamma\left(a_1,q,k\right)}\left|b_k\right|\overline{z}^k}\right] \geq 0. \tag{21}$$

The previous condition must hold for all values of z in $\mathbb{U}$. Upon choosing ϕ according to (14), we must have

$$\frac{(1-\delta)-(1+\delta)|b_1|}{1+|b_1|+\sum_{k=2}^{\infty}(|a_k|+|b_k|)\Gamma(a_1,q,k)r^{k-1}} - \frac{\sum_{k=2}^{\infty}\left(\left([k]_q-\delta\right)|a_k|+\left([k]_q+\delta\right)|b_k|\right)\Gamma(a_1,q,k)r^{k-1}}{1+|b_1|+\sum_{k=2}^{\infty}(|a_k|+|b_k|)\Gamma(a_1,q,k)r^{k-1}} \geq 0. \tag{22}$$

If condition (19) does not hold, then the numerator in (22) is negative for r sufficiently close to 1. Hence, there exist $z_0 = r_0$ in (0,1) for which the quotient of (22) is negative. This contradicts the fact that $f \in V_{\overline{H}}(a_1,\delta,q)$, and this completes the proof. □

The following theorem gives the distortion bounds for functions in $V_{\overline{H}}(a_1,\delta,q)$ which yield a covering result for this class.

Theorem 4. *If $f \in V_{\overline{H}}(a_1,\delta,q)$, then*

$$\begin{aligned} |f(z)| &\leq (1+|b_1|)r + \frac{1}{\Gamma(a_1,q,2)}\left(\frac{1-\delta}{(q+1)-\delta}-\frac{1+\delta}{(q+1)-\delta}\right)r^2, \quad |z|=r<1, \\ |f(z)| &\geq (1+|b_1|)r - \frac{1}{\Gamma(a_1,q,2)}\left(\frac{1-\delta}{(q+1)-\delta}-\frac{1+\delta}{(q+1)-\delta}\right)r^2, \quad |z|=r<1, \end{aligned} \tag{23}$$

where

$$\Gamma(a_1,q,2) = \frac{(1-a_1)\cdots(1-a_r)}{(1-q)(1-b_1)\cdots(1-b_s)}, \qquad [2]_q = (q+1). \tag{24}$$

Proof. We will only prove the right hand inequality. The proof for the left hand inequality is similar.

let $f \in V_{\overline{H}}(a_1,\delta,q)$. Taking the absolute value of f, we obtain

$$\begin{aligned} |f(z)| &\leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k \\ &\leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2. \end{aligned} \tag{25}$$

That is,

$$\begin{aligned} |f(z)| &\leq (1+|b_1|)r + \frac{1-\delta}{\Gamma(a_1,q,2)\left([2]_q-\delta\right)} \times \sum_{k=2}^{\infty}\left(\frac{[2]_q-\delta}{1-\delta}|a_k|+\frac{[2]_q-\delta}{1-\delta}|b_k|\right) \times \Gamma(a_1,q,2)r^2 \\ &\leq (1+|b_1|)r + \frac{1-\delta}{\Gamma(a_1,q,2)((q+1)-\delta)} \times \left[1-\frac{1+\delta}{1-\delta}|b_1|\right]r^2 \\ &= (1+|b_1|)r + \frac{1}{\Gamma(a_1,q,2)} \times \left[\frac{1-\delta}{(q+1)-\delta}-\frac{1+\delta}{(q+1)-\delta}|b_1|\right)r^2. \end{aligned} \tag{26}$$

□

Corollary 5. *Let f be of the form* (1) *so that $f \in V_{\overline{H}}(a_1,\delta,q)$. Then,*

$$\left\{w : |w| < \frac{2\Gamma(a_1,q,2)-1-(\Gamma(a_1,q,2)-1)\delta}{((q+1)-\delta)\Gamma(a_1,q,2)} - \frac{2\Gamma(a_1,q,2)-1-(\Gamma(a_1,q,2)-1)\delta}{((q+1)+\delta)\Gamma(a_1,q,2)}|b_1|\right\} \subset f(\mathbb{U}). \tag{27}$$

Next, one determines the extreme points of closed convex hull of $V_{\overline{H}}(a_1,\delta,q)$ denoted by clco $V_{\overline{H}}(a_1,\delta,q)$.

Theorem 6. *Set*

$$\lambda_k = \frac{1-\delta}{\left([k]_q-\delta\right)\Gamma(a_1,q,k)}, \qquad \mu_k = \frac{1-\delta}{\left([k]_q+\delta\right)\Gamma(a_1,q,k)}. \tag{28}$$

For b_1 fixed, the extreme points for clco $V_{\overline{H}}(a_1,\delta,q)$ *are*

$$\left\{z+\lambda_k x z^k+\overline{b_1 z}\right\} \cup \left\{z+\overline{b_1 z+\mu_k x z^k}\right\}, \tag{29}$$

where $k \geq 2$ and $|x| = 1-|b_1|$.

Proof. Any function $f \in$ clco $V_{\overline{H}}(a_1,\delta,q)$ may be expressed as

$$f(z) = z + \sum_{k=2}^{\infty}|a_k|e^{i\alpha_k}z^k + \overline{b_1 z} + \overline{\sum_{k=2}^{\infty}|b_k|e^{i\beta_k}z^k}, \tag{30}$$

where the coefficients satisfy the inequality (15). Set $h_1(z) = z$, $g_1(z) = b_1 z, h_k(z) = z + \lambda_k e^{i\alpha_k} z^k$, and $g_k(z) = b_1(z) + e^{i\beta_k} z^k$, for $k = 2, 3, \ldots$. Writing $X_k = |a_k|/\lambda_k, Y_k = |b_k|/\mu_k$, $k = 2, 3, \ldots$ and $X_1 = 1 - \sum_{k=2}^{\infty} X_k$; $Y_1 = 1 - \sum_{k=2}^{\infty} Y_k$, we have

$$f(z) = \sum_{k=1}^{\infty} \left(X_k h_k(z) + Y_k g_k(z)\right). \tag{31}$$

In particular, set

$$f_1(z) = z + \overline{b_1 z}, \qquad f_k(z) = z + \lambda_k x z^k + \overline{b_1 z} + \overline{\mu_k y z^k}, \quad \left(k \geq 2, |x| + |y| = 1 - |b_1|\right). \tag{32}$$

Therefore, the extreme points of clco $V_{\overline{H}}(a_1, \delta, q)$ are contained in $\{f_k(z)\}$. To see that f_1 is not an extreme point, note that f_1 may be written as a convex linear combination of functions in clco $V_{\overline{H}}(a_1, \delta, q)$ as follows:

$$\begin{aligned} f_1(z) = {} & \frac{1}{2}\left\{f_1(z) + \lambda_2 \left(1 - |b_1|\right) z^2\right\} \\ & + \frac{1}{2}\left\{f_1(z) - \lambda_2 \left(1 - |b_1|\right) z^2\right\}. \end{aligned} \tag{33}$$

If both $|x| \neq 0$ and $|y| \neq 0$, we will show that it can also be expressed as a convex linear combination of functions in clco $V_{\overline{H}}(a_1, \delta, q)$. Without loss of generality, assume that $|x| \geq |y|$. Choose $\epsilon > 0$ small enough so that $\epsilon < |x|/|y|$. Set $A = 1 + \epsilon$ and $B = 1 - |\epsilon x/y|$. We then see that both

$$\begin{aligned} t_1(z) &= z + \lambda_k A x z^k + \overline{b_1 z} + \overline{\mu_k y B z^k}, \\ t_2(z) &= z + \lambda_k (2 - A) x z^k + \overline{b_1 z} + \overline{\mu_k y (2 - B) z^k}, \end{aligned} \tag{34}$$

are in clco $V_{\overline{H}}(a_1, \delta, q)$ and note that

$$f_n(z) = \frac{1}{2}\left\{t_1(z) + t_2(z)\right\}. \tag{35}$$

The extremal coefficient bound shows that the functions of the form (29) are extreme points for clco $V_{\overline{H}}(a_1, \delta, q)$, and so the proof is complete. □

Following the earlier works in [2, 13], we refer to the γ-neighborhood of the functions f defined by (1) to be the set of functions F for which

$$N_\gamma(f) = \left\{F = z + \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} \overline{B_k z^k} : \sum_{k=2}^{\infty} k\left(|a_k - A_k| + |b_k - B_k|\right) + |b_1 - B_1| \leq \gamma\right\}. \tag{36}$$

We define the q-γ-neighborhood of a function $f \in S_H$ as follows:

$$\begin{aligned} N_\gamma^q(f) = \Bigg\{ & F = z + \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} \overline{B_k z^k} \\ & : \sum_{k=2}^{\infty} [k]_q \left(|a_k - A_k| + |b_k - B_k|\right) \\ & + |b_1 - B_1| \leq \gamma \Bigg\}. \end{aligned} \tag{37}$$

In our case, let us define the generalized q-γ-neighborhood of f to be the set

$$\begin{aligned} N_\gamma^q(f) = \Bigg\{ F : & \sum_{k=2}^{\infty} \Gamma(a_1, q, k) \left[\left([k]_q - \delta\right) |a_k - A_k| \right. \\ & \left. + \left([k]_q + \delta\right) |b_k - B_k|\right] \\ & + (1 + \delta) |b_1 - B_1| \leq (1 - \delta) \gamma \Bigg\}. \end{aligned} \tag{38}$$

Theorem 7. *Let f be given by* (1). *If f satisfies the conditions*

$$\begin{gathered} \sum_{k=2}^{\infty} [k]_q \left([k]_q - \delta\right) |a_k| \Gamma(a_1, q, k) \\ + \sum_{1=2}^{\infty} [k]_q \left([k]_q + \delta\right) |b_k| \Gamma(a_1, q, k) \leq 1 - \delta, \\ 0 \leq \delta < 1, \\ \gamma \leq \frac{1 - \delta}{(q + 1) - \delta} \left(1 - \frac{1 + \delta}{1 - \delta} |b_1|\right), \end{gathered} \tag{39}$$

then $N_\gamma^q(f) \subset S_H^*(a_1, \delta, q)$.

Proof. Let f satisfy (39) and let

$$F(z) = z + \overline{B_1 z} + \sum_{k=2}^{\infty} \left(A_k z^k + \overline{B_k z^k}\right) \tag{40}$$

belong to $N_\gamma^q(f)$. We have

$$\begin{aligned} & (1 + \delta) |B_1| + \sum_{k=2}^{\infty} \Gamma(a_1, q, k) \left[\left([k]_q - \delta\right) |A_k| \right. \\ & \qquad \left. + \left([k]_q + \delta\right) |B_k|\right] \\ & \leq (1 + \delta) |B_1 - b_1| + (1 + \delta) |b_1| \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=2}^{\infty}\Gamma\left(a_1,q,k\right)\left[\left([k]_q-\delta\right)\left|A_k-a_k\right|\right.\\
&\qquad\left.+\left([k]_q+\delta\right)\left|B_k-b_k\right|\right]\\
&+\sum_{k=2}^{\infty}\Gamma\left(a_1,q,k\right)\left[\left([k]_q-\delta\right)\left|a_k\right|+\left([k]_q+\delta\right)\left|b_k\right|\right]\\
&\leq(1-\delta)\gamma+(1+\delta)\left|b_1\right|\\
&+\frac{1}{[2]_q-\delta}\sum_{k=2}^{\infty}\Gamma\left(a_1,q,k\right)\left[[k]_q\left([k]_q-\delta\right)\left|a_k\right|\right.\\
&\qquad\left.+[k]_q\left([k]_q+\delta\right)\left|b_k\right|\right]\\
&\leq(1-\delta)\gamma+(1+\delta)\left|b_1\right|+\frac{1}{(q+1)\ -\delta}\\
&\times\left[(1-\delta)-(1+\delta)\left|b_1\right|\right]\leq 1-\delta.
\end{aligned}
\tag{41}
$$

Hence,

$$
\begin{gathered}
\gamma\leq\frac{1-\delta}{(q+1)-\delta}\left(1-\frac{1+\delta}{1-\delta}\left|b_1\right|\right),\\
F\in S_H^*\left(a_1,\delta,q\right).
\end{gathered}
\tag{42}
$$

□

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