Appropriate Probability Density Function of Convex Bodies

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Abstract: We investigate under the notion of Large Deviation Principle & Concentration of Measure as a technique, the ability of estimating the probability density function of any random vector in the space $\mathbb{R}^n$. We found that an appropriate probability distribution for any convex body in the space is sub – Gaussian.

المستخلص: تحت مفهوم مبدأ الانحراف الأعظم و تركيز الحجم, بحثنا عن امكانية تقدير دالة الكثافة الاحتمالية للجسم المحدب في الفضاء $\mathbb{R}^n$. و وجدنا أن دالة الكثافة الاحتمالية المناسبة لأي جسم محدب هي دالة جاوس للتوزيع الاحتمالي.

Key words: Probability density function, concentration of measure, large deviation principle, Gaussian distribution.

I- INTRODUCTION

In the space of probability measure, the law of large numbers scales the probability that, $\frac{S_n}{n} \to E[X \quad], a.s$, where $\{X_i, i \in \mathbb{N}\}$ is a sequence of random variable which is independent and identically distributed. The quantity S_n stands for the n^{th} sum of $X_i; i = 1, \dots, n$. So, and according to the forgoing convergence we can conclude that

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - E[X]\right| \geq \varepsilon\right) = 0 \dots\dots (1)$$

The point $\frac{S_n}{n}$ is some fixed point on the probability space. If we investigate on where it placed we can deal with two probabilities: The first one take $\frac{S_n}{n}$ to be in some Borel A subset of the $\sigma - algebra$ of the probability space $\left(\frac{S_n}{n} \in A\right)$. The second one takes $\frac{S_n}{n}$ to be near some fixed point x ,that is, $\left(\frac{S_n}{n} \in dx\right)$. To describe the measure of these probabilities we need to have a distribution function. R.C.Srivastava [16] was described the estimation of the distribution function (for short) and he state that, $\hat{F}_n(X) = \frac{[numbers\ of\ X_i'\ s\ \ \leq x]}{n}$. We will stop here for a second. The numerator of $\hat{F}_n(X)$ stands for a level set which supporting the $\hat{F}_n(X)$. Maarten Loffler and Jeff M. Philips[12] was investigate the shape of the level set. So to produce a density function $(for\ short(DF))$, we need a level set for which $X_i's \leq$ some known quantity, which is confirm with the principle of well organized. This level set demands some mild conditions. By the other hand, to generate the probability density function (pdf for short) for a certain (DF) we also need a sequence of ε_n numbers which $\varepsilon_n \to 0\ as\ n \to \infty$.

1.1 Lemma (Parzen): Let $\{h(n)\}$ be a sequence of numbers such that $h(n) \to 0$ as $n \to \infty$ and $K(y)$ be a Borel function which positive definite, symmetric and $\|K\|_\infty = 1$, $\sup K(y) < \infty$ and $\lim_{y\to\infty} |yK(y)| \to 0$:

i- The estimate:
$\hat{f}_n(X) = \frac{1}{h(n)} \int_{-\infty}^{\infty} K\left(\frac{x-y}{h(n)}\right) d\hat{F}_n(y) = \frac{1}{nh(n)} \sum_{i=1}^{n} K\left(\frac{x-y}{h(n)}\right) \dots\dots (2)$
Is asymptotically at all points of X at which pdf is continuous.

ii- Its variance is given by $\sigma^2\left(\hat{f}_n(x)\right) = \frac{1}{n} var\left(\frac{1}{h(n)} K\left(\frac{x-X}{h(n)}\right)\right)$ and satisfies the following equation:

$$\lim_{n\to\infty} nh(n)\sigma^2\left(\hat{f}_n(x)\right) = f(x)\int_{-\infty}^{\infty} K^2(y)\,dy$$

At all points x of continuity of $f(x)$.

iii- If $nh(n) \to \infty\ as\ n \to \infty$, then $\hat{f}_n(X)$ is consistent,

iv- It is asymptotically normal, that is

$$\lim_{n\to\infty} P\left\{\left(\frac{\left(\hat{f}_n(x) - f(x)\right)}{\sigma\left(\hat{f}_n(x)\right)}\right) \leq u\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{x^2}{2}} dx \dots\dots (3)$$

v- It is uniformly consistent, that is, for every

$$P\left\{\sup_x \left|\hat{f}_n(X) - f(x)\right| < \varepsilon\right\} \to 1\ as\ n \to \infty$$

If the (pdf) is uniformly continuous.

Form Parzen Lemma, we can conclude that for a Borel function, to guarantee the estimation $\hat{f}_n(X)$, it demand to satisfies that it has a density $\|K\|_\infty = 1$ with zero mean $\lim_{y\to\infty} |yK(y)| = 0$ and identity covariance $(cov\ (X) = K^2 Id)$ with domain $\lim_{y\to\infty} |yK(y)| \to 0$. By the other hand the same Lemma obtains the sufficient conditions for appropriate (pdf).

As we mentioned before, to measure $\left(\frac{S_n}{n} \in A\right)$ or $\left(\frac{S_n}{n} \in dx\right)$ and with respect to Parzen Lemma, we can deal with the large deviation principle (for short). So from the law of large number to large deviation, we check that in (i – iii) of Parzen Lemma the formula of the plausible probability density function in the sense of large deviation principle. So for a level set we can describe the notion of convex set to be the mother set of any random variable.

Amir Dembo with Ofer Zeitoumi [1] had removed from a sequence of random variable, to generate the (pdf) using large deviation as technique in a sense that if $X_1, \dots, X_n$ is independent, standard normal, real – valued sequence of random variable with empirical mean $\hat{S}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and zero mean and variance $\frac{1}{n}$, then $P(|\hat{S}_n| \geq \delta) \to 0$, and for any Borel set A, then $P(|\hat{S}_n| \geq \delta)\overrightarrow{n \to \infty}\frac{1}{\sqrt{2\pi}}\int_A e^{-\frac{x^2}{2}} dx$. Then we conclude that $\frac{1}{n}\log P(|\hat{S}_n| \geq \delta)\overrightarrow{n \to \infty} -\frac{s^2}{2}$, where $\hat{S}_n = s$ in A.

Under the notion of Concentration of Measure we can relate any probability measure μ of any probability space to another probability measure σ with an affine transformation map $T: \mathbb{R}^n \to \mathbb{R}^n$, in a sense of convergence .

1.2 Theorem: Let X be a normed space and let T be a subset of S^{n-1}. Then for every $\varepsilon > 0$, if $E_T^* \leq C\varepsilon E(X)$, there is a linear operator $A: \mathbb{R}^n \to X$ with; $(1-\varepsilon) \leq \|At\| \leq (1+\varepsilon)$; for all $t \in T$, $C > 0$ is a universal constant.

This topic shall investigate the convex set as container set of random variable, and its log – concave function as a probability density; for the purpose of $P(x \in K)$. Our main result Corollary (4.7) gets an appropriate pdf for the distribution of random vector in a convex body in the space.

Our paper will organize as follows. In 2nd section we will investigate the notion of Large Deviation Principle & Concentration of Measure as an important technique to estimate the appropriate pdf. In the third section, we will describe the notion of the log – concave function and its appropriateness to be as probability density function for any random vector. We know that every log – concave function can create a convex body. The fourth section takes the notion of convex bodies in the space as appropriate body to concentrate with in the space, and it contains our main result Corollary (4.7). At the end we give a brief discussion.

II- Appropriation of Concentration of Measure & Large Deviation Theorem in Estimation of Probability Density Function

From measure theory point of view, given a sequence of probability measures μ_n, we say that μ_n converge (weakly) to μ or simply $\mu_n \to \mu$ if $\lim_{n\to\infty} \int f\, d\mu_n = \int f d\mu$ for every $f \in C_b(E)$ (space of bounded continuous function). For a good reference of the topic refer to {[1],[7],[9],[13]}. The theory of large deviation calculates the exponential rate of decay of this convergence on a closed interval under probability measure. Fraydoun Rezakhanlou[7] had explained an appropriate definition for LDP; Throughout, the probability measure on a measure space $(E, \mathcal{B})$, where E is a polish (separable complete metric space) and $\mathcal{B}$ is the corresponding $\sigma - lagebra$ of the Borel sets, to motivate the definition of LDP we recall two facts:

i- By definition, a sequence of probability measures $\{P_n\}_{n\in\mathbb{N}}$ converges weakly to a probability measure P if and only if $\lim_{n\to\infty} \int f dP_n = \int f dP$, or equivalently:
  - For every open set U : $\lim_{n\to\infty} \inf P_n(U) \geq P(U)$, or
  - For every closed set C: $\lim_{n\to\infty} \inf P_n(C) \leq P(C)$

ii- If $a_1, \dots, a_k \in \mathbb{R}$, then: $\lim_{n\to\infty} n^{-1}\log\left(\sum_{i=1}^{k} e^{-na_i}\right) = -\inf_i a_i$.

2.1 Definition: Let $\{P_n\}_{n\in\mathbb{N}}$ be a family of probability measure on a polish space E and let $I: E \to [0,\infty)$ be a function:

i- We then say that the family $\{P_n\}_{n\in\mathbb{N}}$ satisfies a large deviation principle with rate function I, if the following conditions satisfied:
  - For every $a \geq 0$ the set $\{x: I(x) \leq a\}$ is compact.
  - For every open set U: $\lim_{n\to\infty} \inf\frac{1}{n}P_n(U) \geq -\inf_{x\in U} I(x)$
  - For every closed set C: $\lim_{n\to\infty} \sup\frac{1}{n}P_n(C) \leq -\inf_{x\in C} I(x)$.

ii- We say that the family $\{P_n\}_{n\in\mathbb{N}}$ satisfied a weakly LDP with rate function I, if I is lower semi – continuous and the forgoing statements are true.

As we mentioned before to deal with the theory of probability measure, we need a sequence $(\varepsilon_n \to 0 \;\; as \; n \to \infty)$ and a level set.

2.2 Definition: Fix a metric space M. A function $I: M \to [0,\infty)$ is called
  - A rate function if its lower semi – continuous, which means that the level set $\{x \in M: I(x) \leq a\}$ are closed for any $a \geq 0$.
  - A good rate function if the level set are compact for any $a \geq 0$.

2.3 Definition: A sequence of random variable $X_1, X_2, \dots$ with values in a metric space, is said to satisfy a large deviation principle with:
  - Speed $a_n \to \infty$ and,
  - Rate function I

If, for all Borel set $A \subset M$,

- $\lim_{n\to\infty} sup \frac{1}{a_n} \log P\{X_n \in A\} \leq -\inf_{X\in cl\ A} I(X)$
- $\lim_{n\to\infty} inf \frac{1}{a_n} \log P\{X_n \in A\} \geq -\inf_{X\in Int\ A} I(X)$

Notation: the condition $a_n \to \infty$ stands in [15] for $nh(n) \to \infty$.

Definition (2.3) show the bounds for the LDP. By the other hand, it state that the random variable X_n or its pdf satisfy LDP if the following limit exist:

$$\lim_{n\to\infty} -\frac{1}{n} \ln P_{X_n} = I(x) \dots\dots (4)$$

This implies that $P_{X_n} \approx e^{-nI(x)}$.

From static mechanic point of view, the rate function can be considered as the difference between energy function and some kind of entropy. We will get that in Cramer's Theorem later. By the other hand, the rate function is called large deviation rate, so we can think of it as $I(x) \approx \frac{(.)}{(.)}$.

2.4 Definition: A rate function I is a lower semi – continuous mapping $I: X \to [0, \infty)$, such that the level set $\Psi_I(\alpha) \coloneqq \{x: I(x) \leq \alpha\}$ is closed subset of X. A good rate function is a rate function for which all the level sets $\Psi_I(\alpha)$ are compact subset in X. The effective domain of I, denoted D_I , namely $\mathrm{D}_I(\alpha) \coloneqq \{x: I(x) \leq \alpha\}$.

So, up to this definition the rate function works to measure the degree of concentrated around some (point of event), and it can be evaluated numerically according to the moment generating function, see [13] for insure, and that appear obviously in Cramer's Theorem. Also in the same theorem, we can see the convexity of the rate function.

2.5 Theorem (Cramer's): Assume that $\int e^{x.v} \mu(dx) < \infty$, for every $v \in \mathbb{R}^d$. Then the sequence $\{P_n\}$ satisfies LDP with rate function:

$$I(x) = \sup_{x\in\mathbb{R}^d}\big(x.v - \lambda(v)\big) \dots\dots (5)$$

Where, $\lambda(v) = \log \int e^{x.v} \mu(dx)$.

2.6 Lemma: For $r \in P_n$, the rate function $I_q(r)$ measure the discrepancy between r and q in the sense that $I_q(r) > 0$ and $I_q(r) = 0$ if and only if $r = q$. Thus $I_q(r)$ attains its infimum of 0 over P_n at the unique measure $r = q$. In addition, I_q is strictly convex on P_n, that is, for $0 < \lambda < 1$and only $\mu \neq v$ in P_n;

$$I_q(\lambda\mu + (1-\lambda)v) \leq \lambda I_q(\mu) + (1-\lambda) I_q(v) \dots\dots (6)$$

From the above clarifications we can think of the level set as convex with rate function as support function.

For more properties of the rate function we have,

2.7 Theorem: Suppose that Y_n satisfies LDP on X with rate function I. The following conditions hold:

a- The infimum of I over X equals 0, and the set of $x \in X$ for which $I(x) = 0$ is nonempty and compact.

b- Denote $\mathcal{E}$ to be the nonempty, compact set of $x \in X$ for which $I(x) = 0$ and let A be a Borel subset of X such that $\bar{A} \cap \mathcal{E} = \emptyset$. Then $I(\bar{A}) > 0$, and for some $C < \infty$

$$P_n\{Y_n \in A\} \leq C \exp\left[-\frac{nI(\bar{A})}{2}\right] \to 0 \ as\ n \to \infty \dots\dots (7)$$

We can see that up to Equation (7), that $\left|P_n\{Y_n \in A\} - C \exp\left[-\frac{nI(\bar{A})}{2}\right]\right| \leq \varepsilon$ for some $\varepsilon \in [0,1]$. With the same Equation we can remove from LDP to the notion of *Concentration of measure*. The concentration of measure plays an important role in the estimation of pdf of any random variable which distributed uniformly on anybody in the space $\mathbb{R}^n$.

2.8 Definition: Let X be $\mathbb{R}^n$ with norm $\|.\|$. Let $k = k(X) \leq n$ be the largest integer such that:

$$\mu_{G_{n,k}}\left(\left\{E; \frac{M}{2}|x| \leq \|x\| \leq 2M|x|, for\ all\ x \in E\right\}\right) > 1 - \left(\frac{k}{n+k}\right) \dots\dots (8)$$

With Equation (8) we can see that the probability of $x \in E$ has a density that proportional to the function $k \mapsto \left(1 - \left(\frac{k}{n+k}\right)\right)^{n-1} \approx e^{-C(k)}$ for large n. Here $C(k)$ is depends only on k.

2.9 Theorem:

i- If, for some orthogonal transformation $u_1, \dots, u_t \in \mathbb{R}^n$ and all $x \in \mathbb{R}^n$, $|x| \leq \frac{1}{t}\sum_{i=1}^{t} \|u_i x\| \leq C|x|$. Then $(\mathbb{R}^n, \|.\|)$ Contains, for each $\varepsilon > 0$, a subspace of dimension $k = \left[\frac{\eta\varepsilon^2}{c^2}\frac{n}{t}\right]$ on which the norm is $(1+\varepsilon)$ equivalent to a multiple of the Euclidean norm, $\eta > 0$ is a universal constant, Moreover, the collection of all subspaces of dimension k having this property has probability $\geq 1 - \exp\left(-\frac{C(\varepsilon)}{C^2 t} n\right)$, here $C(\varepsilon) > 0$ depends only on ε and the probability measure is the normalized Haar measure on the relevant Grassmanian.

ii- Conversely, there exist an absolute constant $C > 0$ such that if for some $1 < t < n$ and some $\varepsilon > 0$, the collection of all $\left(\frac{n}{C^2\varepsilon^2 t} + 1\right) - dimensional$ subspace V of $(\mathbb{R}^n, \|.\|)$, satisfying $|x| \leq \|x\| \leq 2|x|$ for all has probability larger than $1 - \frac{1}{2C^2\varepsilon^2 t}$.

So, we conclude that $\left(1 - \frac{1}{2C^2\varepsilon^2 t}\right)^{1-n} \approx \exp\left(-\frac{C(\varepsilon)n}{C^2 t}\right)$. Form another point of view:

2.10 Theorem: Let $\|.\|$ be a norm on $(\mathbb{R}^n)$, for every $0 < \beta < 1$, there exist a subspace Y of dimension $[\beta n]$ of $(\mathbb{R}^n, \|.\|)$ For $\bar{d}_Y$ is bounded by a constant depending on β, t only. Moreover, the collection of all $[\beta n] - subspace$ satisfying the conclusion has probability tending to one as n tends to ∞.

Notation: $\bar{d}_Y$ stands for the natural distance of Y to the natural Euclidean space.

The shape of a level set under concentration of measure is had a ball design

2.11 Theorem: For all $0 < \alpha < 1$ and all $\varepsilon > 0$;

$$\min\{\sigma_n(A_\varepsilon): \sigma_n(A) = \alpha\} \ldots\ldots (9)$$

Is attained for a spherical cap $C = \{x \in S^{n-1}: d(x, x_0) \leq r\}$ with $x_0 \in S^{n-1}$ and $r > 0$ such that $\sigma_n(C) = \alpha$.

2.12 Theorem (Concentration on S^{n-1}): Let $f: S^{n-1} \to \mathbb{R}$ be $1 - Lipschitz$ function. Then for any $\varepsilon > 0$;

$$P\{|f(x) - Ef(x)| \geq L_{S^{n-1}}\varepsilon\} \leq 2\exp(-C\varepsilon^2 n) \ldots\ldots (10)$$

III- Log – concave Measure: Geometric Properties and its Density Function

Brunn – Minkowski inequality plays an important role in the theory of convex body and log – concave measure, which it states that the well-known measure on a convex body is the log – concave, and that the distribution under this log – concave has sub – exponential tail. B. Klartag and V.D. Milman in [6] had investigated widely the connection between the convex bodies and log – concave function. Jon A.Wellner [10], and Laszlo Lovasz, Santosh. Vempala[11] was described the geometric properties of the log – concave measure.

A function $f: \mathbb{R}^n \to [0, \infty)$ is log – concave, if for every $x, y \in \mathbb{R}^n$ and $0 < \lambda < 1$, then $f(\lambda x + (1 + \lambda)y) \geq f^\lambda(x) f^{1-\lambda}(y)$. If $0 < \int f < \infty$, then f has moments of all order. We denote the isotropic constant of a log – concave function f as:

$$L_f = \left(\frac{\sup_{x\in\mathbb{R}^n} f(x)}{\int_{\mathbb{R}^n} f(x)dx}\right)^{\frac{1}{n}} (\det Cov(f))^{\frac{1}{2n}} \ldots\ldots (11)$$

Also, if $f: \mathbb{R}^n \to [0, \infty)$ is log – concave with $0 < \int f < \infty$ and M is the median of the Euclidean norm $|.|$ with respect to f, then by Borel's lemma we get:

$$\int_{\mathbb{R}^n} |x|^2 f(x)\frac{dx}{\int f} = M^2 \Rightarrow \int_{\mathbb{R}^n} |x|^2 \mu(dx) = M^2$$

We say that $f: \mathbb{R}^n \to [0, \infty)$ is in isotropic position if $\sup_{x\in\mathbb{R}^n} f = \int f(x) = 1$, and $Cov(f)$ is scalar matrix. In other words $Cov(f) = L_f^2 Id$. For observation $\int |x|^2 \mu(dx) = L_f^2$. In the same way, for every log – concave function $f: \mathbb{R}^n \to [0, \infty)$ the level set is $K_f(t) = \{x \in \mathbb{R}^n: f(x) \geq t\}$.

From another point of view for a log – concave function to be a density function for an isotropic random variable then $f \approx gaussian$ density function.

3.1 Theorem: Let X be an isotropic random vector in $\mathbb{R}^n$ with a log – concave density. Let $1 \leq k \leq n^{c_1}$ be an integer. Then there exist a subset $\mathcal{E} \subseteq G_{n,k}$ with $\mu_{n,k}(\mathcal{E}) \geq 1 - C\exp(-n^{-c_2})$ such that for any $E \in \mathcal{E}$, the following holds: Denote by f_E the density of the random vector $proj_E(x)$. Then,

$$\left|\frac{f_E(x)}{\gamma(x)} - 1\right| \leq \frac{C}{n^{c_3}} \ldots\ldots (12)$$

For all $x \in E$ with $|x| \leq n^{c_4}$. Here $\gamma(x) = (2\pi)^{-\frac{k}{2}} \exp\left(-\frac{|x|^2}{2}\right)$ is the standard Gaussian density in E, and $C, c_1, c_2, c_3, c_4 > 0$ are universal constants.

Notation: We say that the random variable X in $\mathbb{R}^n$ is in isotropic position or isotropic if the following holds: $E(X) = 0$, $Cov(X) = Id$. In other words $X \sim N(0,1)$.

3.2 Corollary: *Let $f: \mathbb{R}^n \to [0, \infty)$ be log – concave function which is in isotropic position, then for the distribution function of any random variable $x \in K_f$, the appropriate density function is equivalent to a Gaussian density in such way that:*

$$x \sim N(0, L_f^2) \ldots\ldots (13)$$

Now let $d_{TV} = 2\sup_{A\subset\Omega}[prob\{X \in A\} - prob\{Y \in A\}]$ denote the total variance distance, which X and Y are some random variable in some measure space Ω.

3.3 Theorem: There exist a sequence $\varepsilon_n \to 0$, $\delta_n \to 0$ as $n \to \infty$ for which the following hold: Let $n \geq 1$, and let X be a random vector in $\mathbb{R}^n$ with an isotropic, log – concave density. Then there exists a subset $\Theta \subset S^{n-1}$ with $\sigma_{n-1}(\Theta) \geq 1 - \delta_n$ such that for all $\varphi \in \Theta$;

$$d_{TV}(\langle x, \varphi\rangle, Z) \leq \varepsilon_n \ldots\ldots (14)$$

Where, $Z \sim N(0,1)$ is the standard normal distribution.

Now, we had the following corollary.

3.4 Corollary: *Let X be a random vector in $\mathbb{R}^n$ and satisfies LDP, with an isotropic log – concave density function $f: \mathbb{R}^n \to [0, \infty)$.set $\varepsilon \in [0,1], \delta \in [0,1]$. Then with probability greater than $1 - \delta_n$there exist a subspace $G_{n,k} \subset \mathbb{R}^n$of dimension $k = k(\varepsilon, n, C)$ such that for every $A \subset G_{n,k}$ and for all $y \in A$, X has a density function of the form :*

$$f_A(X) = \frac{1}{(2\pi L_f^2)^{\frac{k}{2}}} \int_A e^{-\frac{d_{TV}(x,y)}{2L_f^2}} \mu_{n,k} d(x) \ldots\ldots (15)$$

Such that: $\left| P(\langle x, y\rangle \in A) - \frac{1}{(2\pi L_f^2)^{\frac{k}{2}}} \int_A e^{-\frac{d_{TV}(x,y)}{2L_f^2}} \mu_{n,k} d(x) \right| \le \varepsilon_n \ldots\ldots (16)$

And here $X \sim N(0, L_f^2)$, *and* $\varepsilon_n \to 0, \delta_n \to 0$ *as* $n \to \infty$.

The principles of concentration of measure for the log – concave function appear in the next lemma.

3.5 Lemma: Let $n \ge 2$ and let $g, f: \mathbb{R}^n \to [0, \infty)$be continuous log – concave functions, $C^2 - smooth$ on $[0, \infty)$, such that $f(0) > 0, g(0) > 0, \int f < \infty, \int g < \infty$. Assume that for any $t \ge 0$:

$$|f(t) - g(t)| \le e^{-5n} \min\{f(0), g(0)\}$$

Then,

$$(1 - e^{-n}) t_n(g) \le t_n(f) \le (1 + e^{-n}) t_n(g)$$

Also, in the sense of LDP we have;

3.6 Lemma: Let $X \in \mathbb{R}^n$ be a random point from an isotropic log – concave distribution. Then for any $R > 1$

$$P(|X| > R) \le e^{-R}$$

One the most important probability properties of the log – concave measure appear in the following theorem

3.7 Theorem (Prekopa (1971,1973), Rinolt (1976)): Suppose P is a probability measure on $\mathcal{B}_n$($(\mathbb{R}^n, \mathcal{B}_n)$ is a probability space) such that the affine hull of $supp(P)$ has dimension d. Then P is log – concave if and only if there is a log – concave (density) function f on $\mathbb{R}^n$ such that;

$$P(B) = \int_B f(x)dx \qquad for\ all\ B \in \mathcal{B}_n$$

IV- Concentration of Measure and Probability Density Function of Convex Bodies

The distribution of a linear functional on a convex set has a sub – exponential tail, that is, if $K \subset \mathbb{R}^n$ of volume 1 and a linear functional $\varphi: \mathbb{R}^n \to \mathbb{R}$ we have

$$Vol_n\{x \in K: |\varphi(x)| \ge t\|x\|_{L_1(K)}\} \le e^{-ct}\ (for\ all\ t > 1)$$

Where, $\|x\|_{L_1(K)} = \int_K |\varphi(x)|\, dx$ and $c > 0$ is universal and Vol_n stands for Lebesgue measure on $\mathbb{R}^n$.

In [2] B.Klartag, clarify this with more assumption, that is if $\varphi: \mathbb{R}^n \to \mathbb{R}$ be a non – zero, linear functional and non – negative with $\|\varphi\|_{L_1(K)} = 1$, and if x is a random variable that is distributed uniformly in $K \subset \mathbb{R}^n$(convex body), then $\varphi(x)$ has density proportional to e^{-t} for some $0 \le t \le n - 1$. For example if $\mathcal{E} \subset \mathbb{R}^n$ is an ellipsoid with volume 1 and φ with assumption as before, then φ has density has density function proportional to $\exp\left(\frac{-at^2}{2}\right)$ for some $a > 0$.

B. Klartag[4] and R. Eldan, B. Klartag[15] was described the concentration of measure on convex bodies and a central limit theorem for convex bodies. According to Dvortzky's theorem, the geometric structure of the support of the suggested distribution may be approximated by a regular body like, Euclidean ball or ellipsoid.

4.1 Theorem (Borell's inequality): Let $G(t), t \in T$ be a centered Gaussian process indexed by the countable set T, and such that $\sup_{t\in T} G(t) < \infty$ almost surely. Then $E \sup_{t\in T} G(t) < \infty$, and for every $r \ge 0$ we have,

$$P\{\sup_{t\in T} G(t) - E \sup_{t\in T} G(t) \ge r\} \le 2e^{-\frac{r^2}{2\sigma}} \ldots\ldots (17)$$

Where, $\sigma^2 = \sup_{t\in T} E(G^2(t)) \le \infty$.

4.2 Definition: A Borell measure μ in $\mathbb{R}^n$ is called log – concave, if for any compact sets $A, B \subset \mathbb{R}^n$, and any $0 \le \lambda \le 1$;

$$\mu(\lambda A + (1-\lambda)B) \ge (\mu A)^{\lambda} (\mu B)^{1-\lambda}$$

Here, $\lambda A + (1-\lambda)B = \{\lambda x + (1-\lambda)y : x \in A, y \in B\}$.

4.3 Definition: A random vector X is called log – concave, if its density has the form :

$$f_X(x) = \exp(-u(y)) \ldots\ldots (18)$$

Where, $u: \mathbb{R}^n \to [0, \infty)$ is a convex function.

4.4 Corollary: *Every random variable X which satisfies LDP is log – concave and the rate function I is convex function.*

4.5 Theorem (Dvortzk's Theorem): Let K be an origin – symmetric convex body in $\mathbb{R}^n$ such that the ellipsoid of maximum volume contained in K is the unite Euclidean ball B_2^n. Fix $\varepsilon \in (0,1)$. Let E be the random space of dimension $d = c\varepsilon^{-2} \log n$ down from the Grassmanian $G_{n,d}$ according to the Haar measure, then there exist $R \ge 0$ such that with high probability (say 0.99), we have;

$$(1-\varepsilon)B(R) \subseteq K \cap E \subseteq (1+\varepsilon)B(R)$$

4.6 Theorem: Let $\|x\|_Y$ be a norm on $\mathbb{R}^n$. Let $g \in \mathbb{R}^n$ be the standard Gaussian vector. Set ;

$$b(Y) = \max_{x\in S^{n-1}} \|x\|_Y$$

Then, a random subspace $E \subset \mathbb{R}^n$ of dimension $k = c\varepsilon^2 \frac{E\|g\|_Y}{b(Y)}$ satisfies $\forall x \in E$;

$$(1-\varepsilon)\|x\|_2 \le \|g\|_Y \le (1+\varepsilon)\|g\|_2$$

Let G be an $n \times k$ Gaussian matrix. Set $E = G\mathbb{R}^K$. Then E is uniformly distributed over the Grassmanian $G_{n,k}$.

Notation: We note that for an isotropic convex body in $\mathbb{R}^n$ we have $\|K\| = 1$ with isotropic constant L_K.

Now we had our main result for the paper.

4.7 Corollary: *Let* $K \subset \mathbb{R}^n$ *be an isotropic convex body with log – concave measure. Let* $\varepsilon \in [0,1]$. *Suppose that* $S^{n-1} \subset K$ *with* $|x| \le c(\varepsilon)\|x\|_K$, *let E be a random subspace of dimension* $d = d(\varepsilon, n)$ *such that* $E = GS^{n-1}$, *and*

$$(1-\varepsilon)|x| \le \|x\|_{K\cap E} \le (1+\varepsilon)|x| \ldots\ldots (19)$$

Then, for every $x \in (K \cap E)$, *there is a distribution density function* $f_{K\cap E}$ *of the form ;*

$$f_{x\in(K\cap E)}(x) := \frac{1}{(2\pi L_K^2)^{\frac{d}{2}}} \int_{K\cap E} e^{-\frac{d_{TV}(x,y)}{2L_K^2}} \mu_{n,d}(dx) \ldots\ldots (20)$$

For every $y \in E$, *such that*

$$sup_{x\in\mathbb{R}^n} |P(x \in (K \cap E)) - f_{K\cap E}(x)| \le \varepsilon \ldots\ldots (21)$$

1. ***Discussion:*** *The notion of convex bodies assists powerfully in the study of scattering data in the space* $\mathbb{R}^n$. *Many topics had been discuss the problem of optimization under more and more conditions. A convex body in a space describes the geometric structure of the random data, in the sense of probability measure to*

guarantee that the same points of data will be in the same group (Cluster). Under the concentration of measure we will do the same.

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