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SOME PROPERTIES ON A CLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY q-ANALOGUE OF RUSCHEWEYH OPERATOR

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Abstract. A subclass of harmonic univalent functions is successfully introduced in this study through utilization of q-analogue of Ruscheweyh operator. In this paper, some results including coefficient conditions, extreme points and growth bounds are obtained for the above mentioned harmonic univalent functions.

1. Introduction and preliminaries

A very crucial and important function amongst several important branches of complex analysis is called the harmonic function. Clunie and Sheil Small [4] introduced the first study of complex-values, harmonic mappings defined on a domain $D \subset \mathbb{C}$. This function was also studied by several researchers such as Silverman [14], Silverman and Silvia [15] and Jahangiri [8].

Let $\mathcal{U} = \{z \in \mathbb{C} : |z|<1\}$ be the open unit disk of the complex plane and $\mathcal{S}_H$ denote the class of functions $f = h + \bar{g}$ that are harmonic ,univalent and sense-preserving in $\mathcal{U}$ which normalized by $f(0) = f'(0) - 1 = 0$ where h and g belong to the class $\mathcal{A}$ of all analytic functions in $\mathcal{U}$ take the form

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad and \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k \quad (0 \leq b_1<1).$$

Also, we call h the analytic part and g the co-analytic part of f.
Thus for each f in $\mathcal{S}_H$ takes the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k}\,\overline{z^k}. \tag{1.1}$$

A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathcal{U}$ is that $|h^{'}(z)|>|g^{'}(z)|$ in $\mathcal{U}$ (See Clunie and Sheil-Small [4]). Note that $\mathcal{S}_H$ reduces to S, the class of normalized analytic univalent functions if the co-analytic part of $f = h + \overline{g}$ is identically zero.

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In [6], [7], for function $f \in \mathcal{A}$ and $0<q<1$ Jackson defined the q-derivative operator D_q as follows:

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \quad (z \neq 0)$$

and $D_q f(0) = f'(0)$ and $D_q^2 f(z) = D_q(D_q f(z))$. In case $f(z) = z^k$ for k is a positive integer, the q-derivative of $f(z)$ is given by

$$D_q z^k = \frac{z^k - (zq)^k}{z(1-q)} = [k]_q z^{k-1},$$

where $[k]_q$ defined by

$$[k]_q = \frac{1-q^k}{1-q}.$$

As $q \to 1$ and $k \in \mathbb{N}$, $[k]_q \to k$.

The authors in [1] defined the q- analogue of Ruscheweyh operator $\mathcal{R}_q^\lambda$ by

$$\mathcal{R}_q^\lambda f(z) = z + \sum_{k=2}^{\infty} \frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!} a_k z^k, \tag{1.2}$$

where $[k]_q!$ defined by :

$$[k]_q! = \begin{cases} [k]_q[k-1]_q......[1]_q, & k = 1, 2, ...; \\ 1; & k = 0. \end{cases}$$

All the details about q- calculus used in this paper can be found in [3] and [5].

Also, as $q \longrightarrow 1$ we have

$$\begin{aligned} \lim_{q \longrightarrow 1} \mathcal{R}_q^\lambda f(z) &= z + \lim_{q \longrightarrow 1} \left[\sum_{k=2}^{\infty} \frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!} a_k z^k \right] \\ &= z + \sum_{k=2}^{\infty} \frac{(k+\lambda-1)!}{(\lambda)!(k-1)!} a_k z^k \\ &= \mathcal{R}^\lambda f(z), \end{aligned}$$

where $\mathcal{R}^\lambda f(z)$ is Russcheweyh differential operator which was defined in [12] and has been studied by several authors, for example [9] and [13].

Now we define the operator $\mathcal{R}_q^\lambda f(z)$ in (1.2) of harmonic function $f = h + \overline{g}$ given by (1.1) as

$$\mathcal{R}_q^\lambda f(z) = \mathcal{R}_q^\lambda h(z) + \overline{\mathcal{R}_q^\lambda g(z)} \qquad z \in \mathcal{U},$$

where

$$\mathcal{R}_q^\lambda h(z) = z + \sum_{k=2}^{\infty} \frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!} a_k z^k,$$

and

$$\mathcal{R}_q^\lambda g(z) = \sum_{k=1}^{\infty} \frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!} b_k z^k.$$

Involving the operator $\mathcal{R}_q^\lambda f(z)$ we introduce the class of harmonic univalent functions as follows.

Definition 1.1 For $0 \leq \vartheta < 1$, the function $f = h + \overline{g}$ is in the class $S_H^*(\lambda, q, \vartheta)$ if satisfy the inequality

$$Re\left\{\frac{zD_q(\mathcal{R}_q^\lambda h(z)) - \overline{zD_q(\mathcal{R}_q^\lambda g(z))}}{\mathcal{R}_q^\lambda h(z) + \overline{\mathcal{R}_q^\lambda g(z)}}\right\} \geq \vartheta. \qquad |z| = r < 1. \tag{1.3}$$

Note that $S_H^*(0, q, \vartheta) = S_H(\vartheta)$ is the class of sense-preserving harmonic univalent functions which are starlike of order ϑ in $\mathcal{U}$ defined by Jahangiri [8].

Let $S_{\overline{H}}^*(\lambda, q, \vartheta)$ denote the subclass of $S_H^*(\lambda, q, \vartheta)$ consisting of harmonic functions $f = h + \overline{g}$, where h and g are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \qquad g(z) = \sum_{k=1}^{\infty} |b_k| z_k. \quad (|b_1| < 1)$$

The main objective in this paper is to investigate number of properties for subclasses of harmonic functions. Particularly the coefficient bound, growth theorem and extreme points. Recently , several subclasses of S_H have been studied by numerous researchers see for example [2],[4], [10], [11], and [16]

2. Main Results

In our first theorem, we begin with a sufficient coefficient condition for functions f in $S_H^*(\lambda, q, \vartheta)$.

Theorem 2.1. *Let $f = h + \overline{g}$ given by (1.1). If*

$$\sum_{k=2}^{\infty} \left[\frac{[k]_q - \vartheta}{1 - \vartheta}|a_k| + \frac{[k]_q + \vartheta}{1 - \vartheta}|b_k|\right]\left(\frac{[k + \lambda - 1]_q!}{[\lambda]_q![k - 1]_q!}\right) \leq 1 - \frac{1 + \vartheta}{1 - \vartheta}|b_1|, \tag{2.1}$$

where $a_1 = 1, 0 \leq \vartheta < 1$, then f is sense-preserving, harmonic, univalent in $\mathcal{U}$, and $f \in S_H^*(\lambda, q, \vartheta)$.

Proof. If $|z_1| \neq |z_2| < q$, then

$$\begin{aligned}
\left|\frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)}\right| &\geq 1 - \left|\frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)}\right| \\
&= 1 - \left|\frac{\sum_{k=1}^{\infty} b_k(z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k(z_1^k - z_2^k)}\right| \\
&> 1 - \frac{\sum_{k=1}^{\infty}[k]_q|b_k|}{1 - \sum_{k=2}^{\infty}[k]_q|a_k|} \\
&\geq 1 - \frac{\sum_{k=1}^{\infty}[([k]_q + \vartheta)\left(\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}\right)/(1 - \vartheta)]|b_k|}{1 - \sum_{k=2}^{\infty}[([k]_q - \vartheta)\left(\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}\right)/(1 - \vartheta)]|a_k|} \\
&\geq 0,
\end{aligned}$$

which proves univalence. Note that f is sense-preserving in $\mathcal{U}$. This is because

$$\begin{aligned}
|D_q h(z)| &\geq 1-\sum_{k=2}^{\infty}[k]_q|a_k||z|^{k-1} \\
&> 1-\sum_{k=2}^{\infty}\frac{([k]_q-\vartheta)}{1-\vartheta}\left(\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}\right)|a_k| \\
&\geq \sum_{k=1}^{\infty}\frac{([k]_q+\vartheta)}{1-\vartheta}\left(\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}\right)|b_k| \\
&> \sum_{k=1}^{\infty}\frac{([k]_q+\vartheta)}{1-\vartheta}\left(\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}\right)|b_k||z|^{k-1} \geq \sum_{k=1}^{\infty}[k]_q|b_k||z|^{k-1} \\
&\geq |D_q g(z)|.
\end{aligned}$$

Then we have $\lim_{q\to 1}[|D_q h(z)| \geq |D_q g(z)|] = [|h'(z)| \geq |g'(z)|]$.
We show that if (2.1) holds for the coefficients of $f = h + \overline{g}$, the required condition (1.3) is satisfied. From (1.3), we can write

$$Re\left\{\frac{zD_q(\mathcal{R}_q^{\lambda}h(z)) - \overline{zD_q(\mathcal{R}_q^{\lambda}g(z))}}{\mathcal{R}_q^{\lambda}h(z) + \overline{\mathcal{R}_q^{\lambda}g(z)}}\right\} = Re\left\{\frac{A(z)}{B(z)}\right\},$$

where

$$\begin{aligned}
A(z) &= zD_q(\mathcal{R}_q^{\lambda}h(z)) - \overline{zD_q(\mathcal{R}_q^{\lambda}g(z))} \\
&= z+\sum_{k=2}^{\infty}[k]_q\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k z^k - \sum_{k=1}^{\infty}[k]_q\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}\overline{b_k}\overline{z^k},
\end{aligned}$$

and

$$B(z) = \mathcal{R}_q^{\lambda}h(z) + \overline{\mathcal{R}_q^{\lambda}g(z)} = z + \sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_k z^k + \sum_{k=1}^{\infty}\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}\overline{b_k}\overline{z^k}.$$

Using the fact that $Re(w) \geq \vartheta$ if and only if $|1-\vartheta+w| \geq |1+\vartheta-w|$,it suffices to show that

$$|A(z)+(1-\vartheta)B(z)| - |A(z)-(1+\vartheta)B(z)| \geq 0. \tag{2.2}$$

Substituting for $A(z)$ and $B(z)$ in (2.2), we get

$$\begin{aligned}
&|A(z)+(1-\vartheta)B(z)|-|A(z)-(1+\vartheta)B(z)|\\
&=\left|(2-\vartheta)z+\sum_{k=2}^{\infty}([k]_q-\vartheta+1)\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_kz^k-\sum_{k=1}^{\infty}([k]_q+\vartheta-1)\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}\overline{b_kz^k}\right|\\
&-\left|-\vartheta z+\sum_{k=2}^{\infty}([k]_q-\vartheta-1)\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}a_kz^k-\sum_{k=1}^{\infty}([k]_q+\vartheta+1)\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}\overline{b_kz^k}\right|\\
&\geq(2-\vartheta)|z|-\sum_{k=2}^{\infty}([k]_q-\vartheta+1)\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}|a_k||z|^k-\sum_{k=1}^{\infty}([k]_q+\vartheta-1)\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}|b_k||z|^k\\
&-\vartheta|z|-\sum_{k=2}^{\infty}([k]_q-\vartheta-1)\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}|a_k||z|^k-\sum_{k=1}^{\infty}([k]_q+\vartheta+1)\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}|b_k||z|^k\\
&\geq 2(1-\vartheta)|z|\left\{1-\sum_{k=2}^{\infty}\frac{([k]_q-\vartheta)}{1-\vartheta}.\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}|a_k||z|^{k-1}\right.\\
&\qquad\left.-\sum_{k=1}^{\infty}\frac{([k]_q+\vartheta)}{1-\vartheta}.\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}|b_k||z|^{k-1}\right\}\\
&=2(1-\vartheta)|z|\left\{1-\frac{1+\vartheta}{1-\vartheta}|b_1|-\left(\sum_{k=2}^{\infty}\left[\frac{[k]_q-\vartheta}{1-\vartheta}|a_k|+\frac{[k]_q+\vartheta}{1-\vartheta}|b_k|\right]\right)\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}\right\}.
\end{aligned}$$

By using the enquiringly (2.1), we see that the last expression is non-negative. This implies that $f\in S^*_H(\lambda,q,\vartheta)$.

Now, we obtain the necessary and sufficient condition for a function belongs to the class $S^*_{\overline{H}}(\lambda,q,\vartheta)$.

Theorem 2.2. *Let $f=h+\overline{g}$ given by (1.1). Then $f\in S^*_{\overline{H}}(\lambda,q,\vartheta)$ if and only if*

$$\sum_{k=2}^{\infty}\left[\frac{[k]_q-\vartheta}{1-\vartheta}|a_k|+\frac{[k]_q+\vartheta}{1-\vartheta}|b_k|\right]\left(\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}\right)\leq 1-\frac{1+\vartheta}{1-\vartheta}|b_1|, \tag{2.3}$$

where $a_1=1, 0\leq\vartheta<1$.

Proof. Since $S^*_{\overline{H}}(\lambda,q,\vartheta)\subseteq S^*_H(\lambda,q,\vartheta)$, we only need to prove the "only if "part of the theorem. To this end, for functions $f\in S^*_{\overline{H}}(\lambda,q,\vartheta)$, we notice that the condition (1.3) is equivalent to

$$Re\left\{\frac{zD_q(\mathcal{R}_q^\lambda h(z))-\overline{zD_q(\mathcal{R}_q^\lambda g(z))}}{\mathcal{R}_q^\lambda h(z)+\overline{\mathcal{R}_q^\lambda g(z)}}-\vartheta\right\}\geq 0.$$

That is

$$Re\left[\frac{(1-\vartheta)z-\sum_{k=2}^{\infty}([k]_q-\vartheta)\left(\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}\right)|a_k|z^k-\sum_{k=1}^{\infty}([k]_q+\vartheta)\left(\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}\right)|b_k|\overline{z}^k}{z-\sum_{k=2}^{\infty}\left(\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}\right)|a_k|z^k+\sum_{k=1}^{\infty}\left(\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}\right)|b_k|\overline{z}^k}\right]\geq 0.$$

The above condition must hold for all values of z in $\mathcal{U}$. Upon choosing the values of z on the positive real axis where $0 \le z = r<1$, we must have

$$\frac{(1-\vartheta)-(1+\vartheta)b_1-\left(\sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}\left[([k]_q-\vartheta)|a_k|+([k]_q+\vartheta)|b_k|\right]r^{k-1}\right)}{1+|b_1|+\sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}[|a_k|+|b_k|]r^{k-1}} \ge 0. \tag{2.4}$$

If the condition (2.3) does not hold, then the numerator in (2.4) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0,1)$ for which the quotient of (2.4) is negative. This contradicts the required condition for $f \in S^*_{\overline{H}}(\lambda, q, \vartheta)$ and so the proof is complete .

Next, we determine the extreme points of $S^*_{\overline{H}}(\lambda, q, \vartheta)$

Theorem 2.3. *$f \in S^*_{\overline{H}}(\lambda, q, \vartheta)$ if and only if*

$$f(z) = \sum_{k=1}^{\infty}(X_k h_k + Y_k g_k) \tag{2.5}$$

where

$$h_1(z) = z, h_k(z) = z - \frac{(1-\vartheta)[\lambda]_q![k-1]_q!}{([k]_q-\vartheta)[k+\lambda-1]_q!}z^k; \quad (k \ge 2),$$

$$g_k(z) = z + \frac{(1-\vartheta)[\lambda]_q![k-1]_q!}{([k]_q-\vartheta)[k+\lambda-1]_q!}\overline{z}^k; \quad (k \ge 2),$$

$$\sum_{k=1}^{\infty}(X_k + Y_k) = 1, \quad X_k \ge 0 \quad \textit{and} \quad Y_k \ge 0.$$

In particular, the extreme points of $S^*_{\overline{H}}(\lambda, q, \vartheta)$ are h_k and g_k.

Proof. Note that for f of the form (2.5),we can write

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty}(X_k h_k + Y_k g_k) \\ &= \sum_{k=1}^{\infty}(X_k+Y_k)z - \sum_{k=2}^{\infty}\frac{(1-\vartheta)[\lambda]_q![k-1]_q!}{([k]_q-\vartheta)[k+\lambda-1]_q!}X_k z^k + \sum_{k=1}^{\infty}\frac{(1-\vartheta)[\lambda]_q![k-1]_q!}{([k]_q+\vartheta)[k+\lambda-1]_q!}Y_k\overline{z}^k. \end{aligned}$$

Then

$$\begin{aligned} \sum_{k=2}^{\infty}\frac{([k]_q-\vartheta)[k+\lambda-1]_q!}{(1-\vartheta)[\lambda]_q![k-1]_q!}|a_k| + \sum_{k=1}^{\infty}\frac{([k]_q+\vartheta)[k+\lambda-1]_q!}{(1-\vartheta)[\lambda]_q![k-1]_q!}|b_k| &= \sum_{k=2}^{\infty}X_k + \sum_{k=1}^{\infty}Y_k \\ &= 1 - X_1, \\ &\le 1, \end{aligned}$$

so $f \in S^*_{\overline{H}}(\lambda, q, \vartheta)$. Conversely, suppose that $f \in S^*_{\overline{H}}(\lambda, q, \vartheta)$. Set

$$X_k = \frac{([k]_q-\vartheta)[k+\lambda-1]_q!}{(1-\vartheta)[\lambda]_q![k-1]_q!}|a_k|, 0 \le X_k \le 1, k = 2, 3, ...$$

$$Y_k = \frac{([k]_q+\vartheta)[k+\lambda-1]_q!}{(1-\vartheta)[\lambda]_q![k-1]_q!}|b_k|, 0 \le Y_k \le 1, k = 1, 2, ...$$

and

$$X_1 = 1 - \sum_{k=2}^{\infty}X_k - \sum_{k=1}^{\infty}Y_k.$$

Then, f can be written as

$$\begin{aligned} f(z) &= z-\sum_{k=2}^{\infty}|a_k|z^k+\sum_{k=1}^{\infty}|b_k|\overline{z}^k \\ &= z-\sum_{k=2}^{\infty}\frac{(1-\vartheta)[\lambda]_q![k-1]_q!}{([k]_q-\vartheta)[k+\lambda-1]_q!}X_k z^k+\sum_{k=1}^{\infty}\frac{(1-\vartheta)[\lambda]_q![k-1]_q!}{([k]_q+\vartheta)[k+\lambda-1]_q!}Y_k\overline{z}^k. \\ &= z+\sum_{k=2}^{\infty}(h_k(z)-z)X_k+\sum_{k=1}^{\infty}(g_k(z)-z)Y_k \\ &= \sum_{k=2}^{\infty}h_k(z)X_k+\sum_{k=1}^{\infty}g_k(z)Y_k+z\left(1-\sum_{k=2}^{\infty}X_k-\sum_{k=1}^{\infty}Y_k\right) \\ &= \sum_{k=1}^{\infty}(h_k(z)X_k+g_k(z)Y_k), \end{aligned}$$

as required. Then the proof is completed.
The following theorem gives the growth bounds for functions $f\in S^*_{\overline{H}}(\lambda,q,\vartheta)$ which yields a covering result for this class.

Theorem 2.4. *If $f\in S^*_{\overline{H}}(\lambda,q,\vartheta)$ then*

$$|f(z)|\le(1+|b_1|)r+\frac{1}{[\lambda+1]_q}\left(\frac{1-\vartheta}{[2]_q-\vartheta}-\frac{1+\vartheta}{[2]_q-\vartheta}|b_1|\right)r^2,\quad |z|=r<1,$$

and

$$|f(z)|\ge(1-|b_1|)r-\frac{1}{[\lambda+1]_q}\left(\frac{1-\vartheta}{[2]_q-\vartheta}-\frac{1+\vartheta}{[2]_q-\vartheta}|b_1|\right)r^2,\quad |z|=r<1.$$

Proof. The left-hand inequality was proved where as the proof for the right hand inequality will be omitted for being similar. Let $f\in S^*_{\overline{H}}(\lambda,q,\vartheta)$. Taking the absolute value of f, we obtain

$$\begin{aligned} |f(z)| &= \left|z-\sum_{k=2}^{\infty}|a_k|z^k+\sum_{k=1}^{\infty}|b_k|\overline{z}^k\right| \\ &\ge (1-|b_1|)r-\sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k \\ &\ge (1-|b_1|)r-\sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2 \\ &= (1-|b_1|)r-\frac{(1-\vartheta)}{([2]_q-\vartheta)[\lambda+1]_q} \\ &\times \left[\sum_{k=2}^{\infty}\left(\frac{([2]_q-\vartheta)[\lambda+1]_q}{1-\vartheta}|a_k|+\frac{([2]_q-\vartheta)[\lambda+1]_q}{1-\vartheta}|b_k|\right)r^2\right] \\ &\ge (1-|b_1|)r-\frac{(1-\vartheta)}{([2]_q-\vartheta)[\lambda+1]_q}\left(1-\frac{1+\vartheta}{1-\vartheta}|b_1|\right)r^2 \\ &= (1-|b_1|)r-\frac{1}{[\lambda+1]_q}\left(\frac{1-\vartheta}{[2]_q-\vartheta}-\frac{1+\vartheta}{[2]_q-\vartheta}|b_1|\right)r^2. \end{aligned}$$

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