REGIONAL GRADIENT STRATEGIC SENSORS
CHARACTERIZATIONS

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ABSTRACT
In the present paper, the characterizations of regional gradient strategic sensors notions have been given for different cases of regional gradient observability. The results obtained are applied to two dimensional linear infinite distributed system in Hilbert space where the dynamic is governed by strongly continuous semi-group. Various cases of regional strategic sensors are considered and analyzed in connection with regional gradient strategic sensors concepts. Also, we show that there is a various sensors which are not gradient strategic in usual sense for the considered systems, but may be regionally gradient strategic of this system.

Keywords: $\omega_G$-strategic sensors; exactly $\omega_G$-observability; weakly $\omega_G$-observability; diffusion systems.

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Mathematical approach for parabolic distributed parameter systems governed by semi-group operator in a Hilbert state space.
1. INTRODUCTION

The analysis of distributed parameter systems refers to a set of concepts such as controllability, observability, detectability [13-14, 18]. The study of these concepts can be made via actuators and sensors structures see [14-17], these concepts give an important link between a system and it's environment [15-18], so that the concepts of actuators and strategic sensors for a class of distributed parameter systems are introduced in order that controllability and observability can be achieved [14-18]. The regional analysis is one of the most important notion of system theory [20-22], it consist to reconstruction the state observation on a sub-region \( \omega \) of spatial domain \( \Omega \) in finite time [19-23, 25-28], this concepts introduced and developed by El-Jai et al. An important extended to the asymptotic case for infinite time by El-Jai and Al-Saphory in several works [1-7]. The study of regional gradient observability for a diffusion system has been given in [27-28] where one is interested in knowledge of the state gradient only in a critical sub-region of the system domain without the knowledge of the state itself. Moreover, the applications are motivated by many real world see [10-12, 22]. Commercial buildings are responsible for a significant fraction of the energy consumption and greenhouse gas emissions in the U.S. and worldwide. Consequently, the design, optimization and control of energy efficient buildings can have a tremendous impact on energy cost and greenhouse gas emission. Mathematically, building models are complex, multi-scale, multi-physics, highly uncertain dynamical systems with wide varieties of disturbances [10].

![Fig. 1: Room control model with sensor, in flow and out flow](image)

In this paper we use a model problem to illustrate that distributed parameter control based on PDEs, combined with high performance computing can be used to provide practical insight into important issues such as optimal sensor/actuator placement (may be best or strategic sensors/ actuators) and optimal supervisory building control. In order to illustrate some of the ideas, we consider the problem illustrated by a single room shown in (Figure 1). This model one can reformulated [11] as spatial case of more general model of distributed parameter systems and represented in the next section (see Figure 2). In addition, the characterization of regional strategic sensors have been given for various types of regional observability in [7].

The purpose of this paper is to extended these results in [7] to the case of regional gradient sensors. Thus, we give a characterization of regional gradient strategic sensors for different cases of regional gradient observation. Therefore, we study and analyze the relationship between the regional gradient strategic sensors and the regional exactly gradient observability. So, the outline of this paper is organized as follows:

Section 2 is present problem statement and basic definitions with characterization of the regional gradient observability. The mathematical concepts of regional gradient strategic sensors in a various situations are studied and developed in section 3. In the last section we gives an application about different sensors locations.

2. REIONAL GRADIENT OBSERVABILITY

In this section, we are interested to recall the notion of regional gradient observability and give original results related to particular systems as in [27-28].

2.1 Problem Statement

Let \( \Omega \) be a regular bounded open subset of \( \mathbb{R}^n \), with a smooth boundary \( \partial \Omega \) and \( \omega \) be a non-empty given sub-region of \( \Omega \). Let \([0, T], T > 0\) be a time of measurement interval. We denoted \( Q = \Omega \times ]0, T[ \) and \( \Sigma = \partial \Omega \times ]0, T[ \). Consider the following distributed parabolic defined by

\[
\begin{aligned}
\frac{\partial x}{\partial t}(\xi, t) &= Ax(\xi, t) + Bu(t) & \text{in } Q \\
x(\xi, 0) &= x_0(\xi) & \text{in } \Omega \\
x(\eta, t) &= 0 & \text{in } \Sigma
\end{aligned}
\]  

(1)

with the measurements given by the output function

\[ y(\cdot, t) = Cx(\cdot, t) \]  

(2)

We have

\[ A = \sum_{i,j}^n \frac{\partial}{\partial \eta_j}(a_{ij} \frac{\partial}{\partial \eta_i}), \text{ with } a_{ij} \in \mathcal{D}(\bar{Q}). \]
Suppose that $-A$ is elliptic, i.e., there exists $\alpha > 0$ such that

$$\sum_{i,j=1}^n a_{ij} \xi_i \xi_j \geq \alpha \sum_{j=1}^n |\xi_j|^2,$$

almost everywhere (a.e) on $Q$, $\forall \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$.

This operator is a second order linear differential operator, which generates a strongly continuous semi-group $S_A(t)_{t \geq 0}$ on the Hilbert space $X = H^1(\Omega)$ and is self-adjoint with compact resolvent. The operator $B \in L(R^p, X)$ and $C \in L(X, R^q)$, depend on the structure of actuators and sensors \[18\]. The space $X, U$ and $O$ be separable Hilbert spaces where $X$ is the state space, $U = L^2(0, T, R^p)$ is the control space and $O = L^2(0, T, R^q)$ is the observation space where $p$ and $q$ are the numbers of actuators and sensors (see Figure 2).

**Fig. 2: The domain of $\Omega$, the sub-region $\omega$, various sensors locations**

Under the given assumption, the system (1) has a unique solution \[24\]:

$$x(\xi, t) = S_A(t) x_0(\xi) + \int_0^t S_A(t - \tau) Bu(\tau) d\tau$$

(3)

The problem is to provide sufficient conditions to ensure that, how to extend the results in [7], so that to give a sufficient conditions of regional gradient strategic sensors which achieve the observability in sub-region $\omega$ using various regional gradient sensors.

### 2.2 Definitions And Characterizations

The regional gradient observability concept and reconstruction gradient state have been introduced by Zerrik E. et al. in ref.s \[27-28\] and recently this concept is developed to the regional asymptotic case by Al-Saphory R \[1-7\]. Consider the autonomous system to (1) given by

$$\begin{cases}
\frac{\partial x}{\partial t}(\xi, t) = A x(\xi, t) & \text{in } Q \\
x(\xi, 0) = x_0(\xi) & \text{in } \Omega \\
x(\eta, t) = 0 & \text{in } \Sigma
\end{cases}$$

(4)

The solution of (4) is given by the following form,

$$x(\xi, t) = S_A(t) x_0(\xi), \quad \forall t \in [0, T]$$

(5)

The measurements are obtained through the output function

$$y(., t) = C x(\xi, t)$$

- We first recall a sensors is defined by any couple $(D, f)$, where $D$ is spatial support represented by a nonempty part of $\Omega$ and $f$ represents the distribution of the sensing measurements on $D$.

Depending on the nature of $D$ and $f$, we could have various type of sensors. A sensor may be pointwise if $D = \{b\}$ with $b \in \Omega$ and $f = \delta(.-b)$, where $\delta$ is the Dirac mass concentrated at $b$. In this case the operator $C$ is unbounded and the output function (2) can be written in the form \[13-14\]

$$y(t) = x(b, t)$$

- It may be zonal when $D \subset \Omega$ and $f \in L^2(D)$. The output function (2) can be written in the form

$$y(t) = \int_D x(\xi, t) f(\xi)$$

Now, we define the operator

$$K: x \in X \rightarrow Kx = C S_A(.) x \in O$$

(6)

Thus, we get that

$$y(., t) = K(t) x(., 0)$$
The definition 2.4 is equivalent to say that the system (4) is observable, if the definition 2.3
where \( K \) is bounded linear operator [8].

\[ K^*y^* = \int_0^1 S_\tau y(s) \mathcal{C}y^*(s)ds \]  \hspace{1cm} (7)

Consider the operator

\[ \Phi: \{H^1(\Omega) \to (L^2(\Omega))^n \}
\]

\[ x \to \Phi x = (\frac{\partial x}{\partial n_1}, \ldots, \frac{\partial x}{\partial n_n}) \]  \hspace{1cm} (8)

It's adjoint \( \Phi^* \) is given by

\[ \Phi^*: \{(L^2(\Omega))^n \to H^1(\Omega) \}
\]

\[ x \to \Phi^* x = \nu \]  \hspace{1cm} (9)

where \( \nu \) is a solution of the Dirichlet problem

\[ \begin{align*}
\Delta \nu &= -\text{div}(x) & \text{in} \, \Omega \\
\nu &= 0 & \text{in} \, \partial \Omega
\end{align*} \]

For a nonempty subset \( \omega \) of \( \Omega \), we consider the operators

\[ \chi_\omega: \{(L^2(\Omega))^n \to (L^2(\omega))^n \}
\]

\[ x \to \chi_\omega x = x|_\omega \]  \hspace{1cm} (10)

and

\[ \tilde{\chi}_\omega: (L^2(\omega) \to L^2(\omega))
\]

\[ x \to \tilde{\chi}_\omega x = x|_\omega \]  \hspace{1cm} (11)

where \( x|_\omega \) is the restriction of \( x \) to \( \omega \) [9].

Their adjoints are respectively denoted by \( \chi_\omega^* \) and \( \tilde{\chi}_\omega^* \) are defined by

\[ \chi_\omega^*: \{(L^2(\omega))^n \to (L^2(\Omega))^n \}
\]

\[ x \to \chi_\omega^* x = \begin{cases} x|_\omega & \text{in} \, \omega \\ 0 & \text{in} \, \Omega \setminus \omega \end{cases} \]  \hspace{1cm} (12)

and

\[ \tilde{\chi}_\omega^*: (L^2(\omega) \to L^2(\Omega))
\]

\[ x \to \tilde{\chi}_\omega^* x = \begin{cases} x|_\omega & \text{in} \, \omega \\ 0 & \text{in} \, \Omega \setminus \omega \end{cases} \]  \hspace{1cm} (13)

The idea of gradient observability is based on the existence of an operator \( H: \partial \to (L^2(\omega))^n \) such that \( Hy = \nabla x_0 \). This is a natural extension of the observability concept [8]. Then we defined the operator \( H = \chi_\omega \nabla K^* \) from \( \partial \) into \( (L^2(\omega))^n \) as in [27]. Now, let us denoted the system (4) together with the output (2) by (4)-(2).

**Definition 2.1:** The system (4)-(2) is said to be regionally exactly observable on a sub-region \( \omega \) (exactly \( \omega \)-observable), if

\[ \text{Im} \tilde{\chi}_\omega K^* = L^2(\omega) \]

**Definition 2.2:** The system (4)-(2) is said to be regionally weakly observable on \( \omega \) (weakly \( \omega \)-observable), if

\[ \text{Im} \tilde{\chi}_\omega K^*(\cdot) = L^2(\omega) \]

**Definition 2.3:** The system (4)-(2) is said to be regionally exactly gradient observable on \( \omega \) (exactly \( \omega_G \)-observable), if

\[ \text{Im} \chi_\omega \nabla K^* = (L^2(\omega))^n \]

**Definition 2.4:** The system (4)-(2) is said to be regionally weakly gradient observable on \( \omega \) (weakly \( \omega_G \)-observable), if

\[ \text{Im} \chi_\omega \nabla K^*(\cdot) = (L^2(\omega))^n \]

We see that if a system is weakly \( \omega_G \)-observable then there is one to one relationship between the output and the initial gradient, viz., if \( y \) is given and \( x_0 \) satisfies \( y = CS(\cdot)x_0 \), then \( \nabla x_0 \) is a unique.

**Remark 2.5:** We can deduced that, the definition 2.4 is equivalent to say that the system (4)-(2) is weakly \( \omega_G \)-observable, if
\[ \ker K(t) V^* \chi_\omega^* = \{0\} \]

Then, the following characterization can extend to the regional gradient case as in ref. [26].

**Proposition 2.6:** The system (4)-(2) is exactly \( \omega_G \)-observable if and only if there exist \( c > 0 \) such that for all \( x^* \in (L^2(\omega))^n \), such that,

\[
\| x^* \|_{(L^2(\omega))^n} \leq c \| K(t) V^* \chi_\omega^* \|_0 \tag{14} \]

**Proof:** The proof of this property is deduced from the usual results on observability considering \( \chi_\omega \nabla K^* \). Let \( V, W \) and \( X \) be a reflexive Banach space and let \( F \in L(V, X), G \in L(W, X) \), then the following conditions are equivalent:

1. \( \text{Im} F \subset \text{Im} G \).
2. There exist \( c > 0 \) such that \( \| F^* x^* \|_V \leq c \| G^* x^* \|_W \), \( \forall x^* \in G^* \)

Now, by applying the above result we obtain the equivalent condition for exactly \( \omega_G \)-observable as:

Let \( V = X = (L^2(\omega))^n, W = \Omega, F = \text{Id}_{(L^2(\omega))^n}, \) and \( G = \chi_\omega \nabla K^* \).

Now, since the system is exactly \( \omega_G \)-observable we have \( \text{Im} F \subset \text{Im} G \), which is equivalent to that fact there exist \( c > 0 \), such that

\[
\| F^* x^* \|_{(L^2(\omega))^n} \leq c \| G^* x^* \|_W, \quad \forall x^* \in G^*. \]

**Remark 2.7:** We have:

(1) The regional state reconstruction will be more precise than the whole domain if we estimate the state in the whole the domain.

(2) From (14) there exists a reconstruction error operator that gives the estimation \( \tilde{x}_0 \) of the initial state \( x_0 \) in \( \omega \), and then, if we put \( \tilde{e} = x_0 - \tilde{x}_0 \), we have

\[
\| e \|_{(L^2(\omega))^n} \leq c \| \tilde{e} \|_{(L^2(\omega))^n} \tag{16} \]

\[
\Rightarrow \| x_0 - \tilde{x}_0 \|_{(L^2(\omega))^n} \leq \| x_0 - \tilde{x}_0 \|_{(L^2(\omega))^n} \]

where \( x_0 \) is the exact state of the system and \( \tilde{x}_0 \) is the estimated state of the system.

**Proposition 2.8:** If the system is exactly \( \omega \)-observable then it is exactly \( \omega_G \)-observable.

**Proof:** Since the system is exactly \( \omega \)-observable there exist \( \gamma > 0 \) such that \( \forall x_0 \in L^2(\omega), \) we have

\[
\| x_0 \|_{L^2(\omega)} \leq \gamma \| K \chi_\omega^* x_0 \|_{L^2(\omega)}, \quad \forall \gamma > 0 \]

and then

\[
(L^2(\omega))^n = \{ \forall x_0 = g_1 \int_\omega |g_1|^2 < \infty, \ g_i = \frac{\partial x_0}{\partial t^i}, \forall i = 1, 2, \ldots \}. \tag{15} \]

To prove \( \| x_0 \|_{L^2(\omega)}^2 \leq c \| KV^* \chi_\omega^* x_0 \|_{L^2(\omega), \omega} \) then from (15) and since a system exactly \( \omega \)-observable, then there exist \( \gamma > 0 \) and \( c > 0 \) such that \( \gamma = \frac{1}{c} \). By choosing \( \gamma = \frac{1}{c} \), we can get

\[
c = \frac{\| K \chi_\omega^* x_0 \|_{0}}{\| KV^* \chi_\omega^* x_0 \|_{0}} \tag{16} \]

Then, we can get

\[
\| x_0 \|_{L^2(\omega)}^2 \leq \| x_0 \|_{L^2(\omega)} \leq \gamma \| K \chi_\omega^* x_0 \|_0 \tag{17} \]

And by substituting (16) in (17), we obtain

\[
\| x_0 \|_{L^2(\omega)}^2 \leq \| K \chi_\omega^* x_0 \|_0 \]

Therefore this system is exactly \( \omega_G \)-observable with \( \gamma = 1. \) \( \square \)
Remark 2.9: From the above proposition we can get the following result:
If the system is exactly \( \omega \)-observable then it is exactly \( \omega^1 \)-observable in \( \omega^1 \) for all \( \omega^1 \subset \omega \) (exactly \( \omega^2 \)-observable).

3. REGIONAL GRADIENT STRATEGIC SENSORS

The purpose of this section is to give the characterization for sensors in order that the system (4)-(2) which is observable in \( \omega \).

3.1 \( \omega_G \)-Strategic Sensors

Definition 3.1: A sensor \((D,f)\) is regional gradient strategic on \( \omega \) (\( \omega_G \)-strategic) if the observed system is weakly \( \omega_G \)-observable.

Definition 3.2: A sensor \((D_i,f_i)\) is \( \omega_G \)-strategic if there exist at least one sensor \((D_i,f_i)\) which \( \omega_G \)-strategic.

We can deduce that the following result:

Corollary 3.3: A sensor is \( \omega_G \)-strategic if the observed system is exactly \( \omega_G \)-observable.

Proof: Let the system exactly \( \omega_G \)-observable, then, we have

\[
\text{Im} \chi_\omega \nabla K^* = (L^2(\omega))^n
\]

From decomposition sub-space of direct sum in Hilbert space, we represent \((L^2(\Omega))^n\) by the unique form [13]

\[
\text{ker} \chi_\omega + \text{Im} \chi_\omega \nabla K^* = (L^2(\Omega))^n
\]

We obtain

\[
\text{ker} K(t) \nabla \chi_\omega = \{0\}
\]

This is equivalent to [9]

\[
\text{Im} \chi_\omega \nabla K^*(\cdot) = (L^2(\omega))^n
\]

Finally, we can deduce this system is weakly \( \omega_G \)-observable and therefore this sensor is \( \omega_G \)-strategic.

Corollary 3.4: A sensor is \( \omega_G \)-strategic if and only if the operator \( N_\omega = HH^* \) is positive definite.

Proof: Since a sensor is \( \omega_G \)-strategic this mean that the system is weakly \( \omega_G \)-observable, let \( x^* \in (L^2(\omega))^n \) such that

\[
< N_\omega x^*, x^* > (L^2(\omega))^n = 0 \quad \text{then} \quad \|H^* x^*\|_\omega = 0
\]

and therefore \( x^* \in \text{ker} H^* \), thus, \( x^* = 0 \), i.e., \( N_\omega \) is positive definite.

Conversely, let \( x^* \in (L^2(\omega))^n \) such that

\[
H^* x^* = 0, \quad \text{then} \quad < H^* x^*, H^* x^* >_\omega = 0
\]

and thus,

\[
< N_\omega x^*, x^* > (L^2(\omega))^n = 0
\]

Hence \( x^* = 0 \) thus the system is weakly \( \omega_G \)-observable and therefore a sensor is \( \omega_G \)-strategic.

Remark 3.5: From the previous results, we obtain that:

(1) If the system is exactly \( \omega_G \)-observable then the system is weakly \( \omega_G \)-observable and therefore this sensor is \( \omega_G \)-strategic.

(2) A sensor which is regional gradient strategic sensor in \( \omega^1 \) (\( \omega_G^1 \)-strategic) for a system where \( \omega^1 \subset \Omega \), is regional gradient strategic sensor in \( \omega^2 \) (\( \omega_G^2 \)-strategic) for any \( \omega^2 \subset \omega^1 \).

(3) The concept of exact \( \omega_G \)-observability is more restrictive than weak \( \omega_G \)-observability.

Now, assume that the operator \( A \) has a complete set of eigenfunction in \( H^1(\Omega) \), denoted by \( \varphi_{\alpha j} \), which is orthonormal in \( L^2(\omega) \) and the associated with the eigenvalue \( \lambda_{\alpha j} \) of multiplicities \( n_{\alpha j} \), then the concept of regional gradient strategic on \( \omega \) can be characterized by the following result:

Theorem 3.6: Assume that \( \sup n_{\alpha j} = r < \infty \), then the suite of sensors \((D_i,f_i)\) is \( \omega_G \)-strategic if
(1) \( q \geq r \)

(2) \( \text{rank } G_n = r_n \quad \forall n \geq 1, \text{where} \)

Where \( G_n = (G_n)_{ij} \) for \( 1 \leq i \leq q, 1 \leq j \leq r_n \), and

\[
(G_n)_{ij} = \begin{cases} 
\sum_k \frac{\partial g_{nj}}{\partial \xi_k}(b_i) & \text{in the pointwise case} \\
\sum_k < \frac{\partial g_{nj}}{\partial \xi_k}, f_i > \mathcal{L}(0,1) & \text{in the zonal case}
\end{cases}
\]

**Proof:** We will discuss the case where the sensors are of pointwise type and located inside the domain \( \Omega \). The suite of sensors \((b_i, \delta_{b_i})_{1 \leq i \leq q}\) is \(\omega^\alpha\)-strategic if and only if

\[
\{ x^* \in (L^2(\omega))^n \mid < H y, x^* >_{(L^2(\omega))^n} = 0, \forall y \in \mathcal{G} \} \Rightarrow x^* = 0.
\]

Suppose that the suite of sensors \((b_i, \delta_{b_i})_{1 \leq i \leq q}\) is \(\omega^\alpha\)-strategic but for a certain \( \eta \in \mathcal{G} \), then there exists a vector \( x_n = (x_{n1}, x_{n2}, \ldots, x_{nr_n})^T \neq 0 \), such that \( G_n x_n = 0 \). So, we can construct a nonzero \( x_0 \in L^2(\omega) \) considering \( < x_0, \phi_p >_{L^2(\omega)} = 0 \) if \( p \neq n \) and

\[
< x_0, \phi_n >_{L^2(\omega)} = x_n, 1 \leq j \leq r_n.
\]

Let \( x_0 = \sum_{j=1}^{r_n} x_j \phi_j \), \( x_0 = (x_0, x_0, \ldots, x_0) \), then

\[
< H y, x_0 >_{L^2(\omega)^n} = \sum_{j=1}^{r_n} < x_0, \frac{\partial}{\partial \xi_k}(K' y), x_0 >_{L^2(\omega)}
\]

\[
= \sum_{k=1}^{q} \frac{\partial}{\partial \xi_k} (x(T)), \; x_0 >_{L^2(\omega)}
\]

where \( \tilde{x} \) is the solution of the following system:

\[
\begin{cases}
\frac{d}{dt} \tilde{x}(t) = A\tilde{x}(t) + \sum_{i=1}^{q} \delta_{b_i} \gamma_i (T-t) & \text{in } Q \\
\tilde{x}(0) = 0 & \text{in } \Omega \\
\tilde{x}(t) = 0 & \text{in } \Sigma
\end{cases}
\]

(18)

Consider the system:

\[
\begin{cases}
\frac{\partial \xi}{\partial t}(t) = -A\xi(t) & \text{in } Q \\
\xi(0) = x_0 & \text{in } \Omega \\
\xi(t) = 0 & \text{in } \Sigma
\end{cases}
\]

(19)

Multiply (18) by \( \frac{\partial \xi}{\partial t} \) and integrate on \( Q \), we obtain

\[
\int_Q \frac{\partial \xi}{\partial t}(t) \frac{\partial}{\partial \xi_k} (\xi(t), t) d\xi dt + \int_Q (\sum_{i=1}^{q} \delta_{b_i} \gamma_i (T-t)) \frac{\partial \xi}{\partial \xi_k} (\xi(t), t) d\xi dt.
\]

But we have

\[
\int_Q \frac{\partial \xi}{\partial t}(t) \frac{\partial}{\partial \xi_k} (\xi(t), t) d\xi dt = \int_Q A^* \tilde{x}(t) \frac{\partial}{\partial \xi_k} (\xi(t), t) d\xi dt + \int_Q A (\xi(t), t) \tilde{x}(t) d\xi dt
\]

then:

\[
\int_Q \frac{\partial \xi}{\partial t}(t) \tilde{x}(t) d\xi = -\int_Q A (\xi(t), t) \tilde{x}(t) d\xi + \int_Q A^* \tilde{x}(t) \frac{\partial}{\partial \xi_k} (\xi(t), t) d\xi dt
\]

integrating by parts we obtain
Consequently, the observability can be reduced to one [19].

**Remark 3.8:** We can deduced the following result:

**Corollary 3.7:** In the one dimension case, a sensor is $\omega_G$-strategic if and only if $q \geq r = \text{sup} r_n$ and $\text{rank} G_n = r_n, \forall n \geq 1$, where $G_n$ is given in theorem 3.6.

**Remark 3.8:** From the previous results, we can get

1. The Theorem 3.6 implies that the required number of sensors is greater than or equal to the largest multiplicity of the eigenvalues.
2. By infinitesimally deforming the domain, the multiplicity can be reduced to one [19]. Consequently, $\omega_G$-strategic sensors can be achieved using only one sensor.

Now, we can deduced that various sensors which are not strategic in usual sense for systems, but may be $\omega_G$-strategic and achieve the $\omega_G$-observability. This is illustrated in the following counter-example.

### 3.2 A Counter-Example

Consider the system described by the parabolic equation

\[
\begin{aligned}
\frac{\partial x(x,t)}{\partial t} &= \frac{\partial^2 x}{\partial t^2} (x, t) \quad \text{in } ]0,1[ \times [0,T[ \\
x(0,1) &= x(1, t) = 0 \quad \text{in } ]0,T[ \\
x(x,0) &= x_0(x) \quad \text{in } ]0,1[ 
\end{aligned}
\]

(21)

Suppose that the measurement is given by pointwise sensor located in $b \in ]0,1[$ which is given by the following output function: 

\[
\int_0^T \frac{\partial y}{\partial t}(\xi, t) \frac{\partial y}{\partial t}(\eta, t) \, d\eta dt = -\int_0^T \frac{\partial y}{\partial t}(\xi, t) \frac{\partial y}{\partial t}(\eta, t) \, d\eta dt + \int_0^T (\sum_{i=1}^n \delta_{b_i} y_i(T-t)) \frac{\partial y}{\partial t}(\xi, t) \, d\eta dt.
\]

The boundary conditions give

\[
\int_0^T \frac{\partial y}{\partial t}(\xi, t) \frac{\partial y}{\partial t}(\eta, t) \, d\eta dt = -\sum_{k=1}^n \int_0^T \frac{\partial y}{\partial t}(b_i, t) y_i(T-t) \, dt.
\]

Then

\[
\lambda_j = (\omega_j^*(\Omega) \sum_{i=1}^n \delta_{b_i}(G_i x_i))_i
\]

for all $y \in L^2 (0, T; R^n)$, then $x_0 \in \ker H^*$ which contradicts the assumption that the suite of sensors is $\omega_G$-strategic.

We can deduced the following result:

\[
\frac{\partial y}{\partial t}(\xi, t) \frac{\partial y}{\partial t}(\eta, t) \, d\eta dt = -\sum_{k=1}^n \int_0^T \frac{\partial y}{\partial t}(b_i, t) y_i(T-t) \, dt = 0
\]

This is true for all $y \in L^2 (0, T; R^n)$, then $x_0 \in \ker H^*$ which contradicts the assumption that the suite of sensors is $\omega_G$-strategic.

### Example

Consider the system described by the parabolic equation

\[
\begin{aligned}
\frac{\partial x(x,t)}{\partial t} &= \frac{\partial^2 x}{\partial t^2} (x, t) \quad \text{in } ]0,1[ \times [0,T[ \\
x(0,1) &= x(1, t) = 0 \quad \text{in } ]0,T[ \\
x(x,0) &= x_0(x) \quad \text{in } ]0,1[ 
\end{aligned}
\]

(21)

Suppose that the measurement is given by pointwise sensor located in $b \in ]0,1[$ which is given by the following output function: 

\[
\int_0^T \frac{\partial y}{\partial t}(\xi, t) \frac{\partial y}{\partial t}(\eta, t) \, d\eta dt = -\sum_{k=1}^n \int_0^T \frac{\partial y}{\partial t}(b_i, t) y_i(T-t) \, dt
\]

and we have

\[
\varphi(\xi, t) = \sum_{p=1}^n e^{-\lambda_p(T-t)} \sum_{i=1}^n \delta_{b_i}(G_i x_i)_i
\]

Therefore

\[
\varphi(\xi, t) = \sum_{p=1}^n e^{-\lambda_p(T-t)} \sum_{i=1}^n \delta_{b_i}(G_i x_i)_i
\]

Thus

\[
\varphi(\xi, t) = \sum_{p=1}^n e^{-\lambda_p(T-t)} (G_p x_p)_i
\]

This is true for all $y \in L^2 (0, T; R^n)$, then $x_0 \in \ker H^*$ which contradicts the assumption that the suite of sensors is $\omega_G$-strategic.
y(., t) = \int_\Omega x(\xi, t) \delta(\xi - t)d\xi = x(b, t), \; t \in (0, T)

(22)

Where \( \varphi_n = \sqrt{2} \sin(n\pi \xi) \) and \( \lambda_n = -n^2\pi^2 \). First, we must prove that the system (21)-(22) is not weakly observable in \( \Omega \), that means the sensors \((\delta_b, b)\) is not strategic. For this purpose, we can write the system (21) as a state space one dimensional system

\[
x(\xi, t) = Ax(\xi, t) \\
x(\xi, 0) = x_0(\xi)
\]

Where \( A = \frac{\partial^2}{\partial \xi^2} \) generate the continuous semigroup \((S(t))_{t \geq 0}\) given by [17].

\[
S(t)x_0 = \sum_{i=1}^{n} e^{\lambda_i t} < x_0, \varphi_i >_{L^2(\Omega)} \varphi_i
\]

Where, \( \varphi_n = \sqrt{2} \sin(n\pi \xi), \lambda_n = -n^2\pi^2 \) are the eigenfunctions associated with the eigenvalues of \( A \). Then from solution of (21), we have

\[
y(\xi, t) = \sum_{i=1}^{n} e^{\lambda_i t} < x_0, \varphi_i >_{L^2(\Omega)} \varphi_i(b) = CS(t)x_0 = K(t)x_0
\]

The system (21)-(22) is weakly observable if \( ker \; K(t) = \{0\} \).

As proved in [27], if \( b \in Q \) then system (21)-(22) is not weakly observable on \( \Omega = (0,1) \) and a sensor \((\delta_b, b)\) is not strategic.

A sensor is \( \omega \)-strategic on \((0,1) \) \iff \( b \in S = \bigcup_{n=1}^{\infty} \left\{ \frac{1}{n}k \in [1, n - 1] \cap \mathbb{N} \right\}. \) Since \( \sin(n\pi b) = 0 \iff nb = k \Rightarrow b = \frac{k}{n} \).

Consequently, the system is weakly observable on \((0,1) \). Then, it is G-strategic on \((0,1) \) \iff \( b \in S_0 = \bigcup_{n=1}^{\infty} \left\{ \frac{2k+1}{2n} \right\}k \in [0,n-1]\backslash \mathbb{N}. \) Since \( \cos(n\pi b) = 0 \iff nb = 2k+1 \Rightarrow b = 2k+1/2n. \) Consequently, the system is weakly G-observable on \((0,1) \).□

**Corollary 3.9:** If the system (21)-(22) is exactly \( \omega_G \)-observable, rank condition in theorem (3.6) is satisfied and a sensor is \( \omega_G \)-strategic.

Now, assume that a sensor is not gradient strategic in whole the domain \( \Omega \) and let \((\bar{\varphi}_i)_{i \in \mathbb{N}^n}\) be a basis in \((L^2(\Omega))^n\). Let \( I \subset \mathbb{N}^n \) be such that \( ker \; \mathcal{K} = span \left\{ (\bar{\varphi}_i)_{i \in I} \right\} \) and \( I = \mathbb{N}^n \backslash J. \)

**Proposition 3.10:** The following properties are equivalent:

1. A sensor is \( \omega_G \)-strategic.
2. \( span \left\{ (x, \omega \bar{\varphi}_i)_{i \in J} \right\} = (L^2(\omega))^n \)
3. If \( x \in (L^2(\omega))^n \) is such that \( < x, x, \omega \bar{\varphi}_i >_{(L^2(\omega))^n} = 0 \) for all \( i \in J, \) then \( x = 0. \)
4. If \( \sum_{i \in J} a_i \bar{\varphi}_i = 0 \) in \( \Omega \omega, \) then \( a_i = 0 \) for all \( i \in J. \)

**Proof:** 1\( \Rightarrow 2 \) Since sensors are \( \omega_G \)-strategic this mean that the system is weakly \( \omega_G \)-observable.

Let \( x \in (L^2(\omega))^n \). Then for \( \epsilon > 0 \) \exists \; \gamma \in \mathcal{O} \) such that

\[
\|x - x, \omega \mathcal{K}^* \|_{(L^2(\omega))^n} \leq \epsilon, \quad \text{but}
\]

\[
\mathcal{K}^* y = \sum_{i \in \mathbb{N}^n} < \mathcal{K}^* y, \bar{\varphi}_i >_{(L^2(\omega))^n} \bar{\varphi}_i = \sum_{i \in J} < y, \mathcal{K}^* \bar{\varphi}_i >_{\Omega} \mathcal{K} \bar{\varphi}_i, \quad \text{and thus}
\]

\[
x, \omega \mathcal{K}^* y = \sum_{i \in J} < x, \omega \mathcal{K}^* \bar{\varphi}_i >_{\Omega} \mathcal{K} \bar{\varphi}_i, \quad \text{Then}
\]

\[
\|x - x, \omega \mathcal{K}^* \|_{(L^2(\omega))^n} < \epsilon
\]

and hence \( x \in \left\{ x, \omega \mathcal{K}^* \right\}_{i \in J}. \)

2\( \Rightarrow 3 \) Let \( x \in (L^2(\omega))^n. \) For any \( \epsilon > 0 \) \exists \; a_i (j \in J) \) such that

\[
\|x - \sum_{i \in J} a_j \omega \bar{\varphi}_j \|_{(L^2(\omega))^n} < \epsilon, \quad \text{with}
\]

\[
<x, \omega \mathcal{K}^* \bar{\varphi}_j >_{(L^2(\omega))^n} = 0, \forall j \in J
\]

we deduced that

\[
\|x\|_{(L^2(\omega))^n} < \epsilon . \quad \text{Thus, } x = 0.
\]

3\( \Rightarrow 4 \) Let \( \sum_{i \in J} a_j \bar{\varphi}_j = 0 \) in \( \Omega \omega. \)
Now consider \( x = \xi_0 (\Sigma_{i \in \mathcal{E}} a_i \phi_i) \). For \( j \in \mathcal{F} \), we have

\[
< x, x_0 \phi_i >_{(L^2(\Omega))} = \xi_0 (\Sigma_{i \in \mathcal{E}} a_i \langle \phi_i, \phi_i >_{(L^2(\Omega))} = 0 .
\]

Since \( x = 0 \), we get

\[
\xi_0 (\Sigma_{i \in \mathcal{E}} a_i \phi_i) = 0 \text{ in } \Omega \text{ and } a_i = 0, \forall i \in \mathcal{I}.
\]

4.1 Consider \( x \in (L^2(\omega))^n \) such that

\[
KV^* x = 0 . \text{ We have } \chi_0^* x \in (L^2(\Omega))^n \text{ then}
\]

\[
KV^* \chi_0^* x = K^* V (\Sigma_{i \in \mathcal{E}} < x, \chi_0 \phi_i >_{(L^2(\omega))} \phi_i) = K^* V (\Sigma_{i \in \mathcal{E}} < x, \chi_0 \phi_i >_{(L^2(\omega))} \phi_i) = 0 . \text{ Therefore,}
\]

\[
\Sigma_{i \in \mathcal{E}} < x, \chi_0 \phi_i >_{(L^2(\omega))} \phi_i \in \text{span} \{ (\phi_i)_{i \in \mathcal{I}} \}
\]

and then

\[
\Sigma_{i \in \mathcal{E}} < x, \chi_0 \phi_i >_{(L^2(\omega))} = 0, \forall j \in \mathcal{I} .
\]

Therefore

\[
\chi_0^* x = \Sigma_{i \in \mathcal{E}} < x, \chi_0 \phi_i >_{(L^2(\omega))} \phi_i = 0 \text{ in } \Omega .
\]

From the assumption we have \( < x, \chi_0 \phi_i >_{(L^2(\omega))} = 0, \forall i \in \mathcal{I} \). Hence \( x = 0 . \) □

We can deduced the following result:

**Corollary 3.11:** Under the hypotheses of Proposition 3.10, a sensors is \( \omega^a \)-strategic in all \( \omega \subset \Omega \) such that \( < \phi_i, \phi_j >_{(L^2(\omega))} = 0, \forall i, j \in \mathcal{I}, i \neq j \).

**Proof:** To deduce the result from previous Proposition 3.10, we take \( \Sigma_{i \in \mathcal{E}} a_i \phi_i = 0 \) in \( \Omega (\omega) \). Then we only need to show that \( a_i = 0, \forall i \in \mathcal{I} \). Let \( x = \Sigma_{i \in \mathcal{E}} a_i \phi_i \) in \( \Omega (\omega) \) and \( i_0 \in \mathcal{I} \). Then

\[
< x, \phi_{i_0} >_{(L^2(\omega))} = < \Sigma_{i \in \mathcal{E}} a_i \phi_i, \phi_{i_0} >_{(L^2(\omega))} = \Sigma_{i \in \mathcal{E}} a_i < \phi_i, \phi_{i_0} >_{(L^2(\omega))} = a_{i_0} .
\]

(23)

Since \( x = 0 \) in \( \Omega (\omega) \), under the assumption of Corollary 3.11 we have

\[
< x, \phi_{i_0} >_{(L^2(\omega))} = \Sigma_{i \in \mathcal{E}} a_i < \phi_i, \phi_{i_0} >_{(L^2(\omega))} = a_{i_0} \| \phi_{i_0} \|_{(L^2(\omega))}^2 .
\]

(24)

From (23)-(24), we obtain \( a_i = 0, \forall i \in \mathcal{I} \).

**4. APPLICATION TO SENSORS LOCATIONS**

In this section, we give specific results related to the different case presented in the above section. First we consider internal sensors (zonal, pointwise, filament in rectangular and disk domain) the presented result give information on the structure of \( \omega \). Consider the system

\[
\begin{align*}
\frac{\partial u}{\partial t}(\xi_1, \xi_2, t) &= \Delta x(\xi_1, \xi_2, t) & \text{in } \mathcal{Q}, \\
x(\xi_1, \xi_2, 0) &= x_0(\xi_1, \xi_2) & \text{in } \mathcal{Q}, \\
x(\eta_1, \eta_2, t) &= 0 & \text{in } \Sigma
\end{align*}
\]

(25)

Let \( \Omega = (0,1) \times (0,1) \) and let \( \omega = (a_1, b_1) \times (a_2, b_2) \) be the considered region is subset of \( \Omega \), the eigenfunctions and the eigenvalue of the system (25) are given by:

\[
\phi_{ij}(\xi_1, \xi_2) = \frac{2}{\sqrt{(b_1 - a_1)(b_2 - a_2)}} \sin \pi j (\xi_1) \sin \pi i (\xi_2)
\]

(26)

Associated with eigenvalue

\[
\lambda_{ij} = -\frac{i^2}{(b_1 - a_1)^2} + \frac{j^2}{(b_2 - a_2)^2}
\]

(27)

**4.1 Internal Zone Sensor**

Consider the system (25) together with output function (2) where the sensor supports \( D \) are located in \( \Omega \). The output (2) can be written by the form

\[
y(t) = \int_D x(\xi_1, \xi_2, t) f(\xi_1, \xi_2) d\xi_1 d\xi_2
\]

(28)
Where \( D \subset \Omega \) is location of zone sensor and \( f \in L^2(D) \). In this case of (see Figure 3), the eigenfunctions and the eigenvalues (26) and (27).

Fig. 3: Domain \( \Omega \), sub-region \( \omega \) and location \( D \) of internal zone sensor

However, if we suppose that

\[
\frac{(\beta_1 - \alpha_1)^2}{(\beta_2 - \alpha_2)^2} \notin Q
\]

Then multiplicity of \( \lambda_{ij} \) is \( r_{ij} = 1 \) and then one sensor \( (D, f) \) may be sufficient to achieve \( \omega_0 \)-observable of the systems (25) and (28) [19]. Let the measurement support is rectangular with

\[
D = [\xi_{01} - l_1, \xi_{01} + l_1] \times [\xi_{02} - l_2, \xi_{02} + l_2] \in \Omega
\]

Then, we have the following result

**Corollary 4.1:** If \( f_1 \) is symmetric about \( \xi_1 = \xi_{01} \) and \( f_2 \) is symmetric about \( \xi_2 = \xi_{02} \), then the sensor \( (D, f) \) is \( \omega_0 \)-strategic if

\[
\frac{j(\xi_{01} - \xi_1)}{(\beta_1 - \alpha_1)} \text{ and } \frac{j(\xi_{02} - \xi_2)}{(\beta_2 - \alpha_2)} \notin N \text{ for some } i, j.
\]

**4.2 Internal Pointwise Sensor**

In this case the output function is given by:

\[
y(t) = \int_D x(\xi_1, \xi_2, t) \delta(\xi_1 - b_1, \xi_2 - b_2) d\xi_1 d\xi_2
\]

(29)

With \( b = (b_1, b_2) \) is location of pointwise sensor as defined in (see Figure 4)

Fig. 4: Rectangular domain, and location \( b \) of internal pointwise sensor

If \( \frac{(\beta_1 - \alpha_1)}{(\beta_2 - \alpha_2)} \notin Q \), then \( r_{ij} = 1 \) and one sensor \( (b, \delta_b) \) may be sufficient for \( \omega_0 \)-observability of the systems (25)-(29)

**Corollary 4.2:** The sensor \( (b, \delta_b) \) is \( \omega_0 \)-strategic if

\[
\frac{j(\xi_{01} - \xi_1)}{(\beta_1 - \alpha_1)} \text{ and } \frac{j(\xi_{02} - \xi_2)}{(\beta_2 - \alpha_2)} \notin N \text{, for some } i, j.
\]

**4.3 Internal Filament Sensor**

Consider the case where the observation is given on the curve \( \sigma = \text{Im}(\gamma) \) with \( \gamma \in C^1(0,1) \) (see Figure 5)
Fig. 5: Rectangular domain, and location $\sigma$ of internal filament sensors

**Corollary 4.3:** If the measurements recovered by filament sensor $(\sigma, \delta_\sigma)$ such that is symmetric with respect to the line $\xi = \xi_0$. Then the sensor $(\sigma, \delta_\sigma)$ is $\omega_\gamma$-strategic if

$$\frac{i(\xi_0-\alpha_1)}{(\beta_1-\alpha_1)} \text{ and } \frac{j(\xi_0-\alpha_2)}{(\beta_2-\alpha_2)} \notin N, \text{ for } i, j = 1, \ldots, J.$$  

**Remark 4.4:** These results can be extended to the following:

1. Case of Neumann or mixed boundary conditions [4-5].
2. Case of disc domain $\Omega=(D,1)$ and $\omega=(0, r_\omega)$ where $\omega \subset \Omega$ and $0 < r_\omega < 1$ [1-3].
3. Case of boundary sensors where $C \notin L(X, R^4)$, we refer to see [13-14].
4. We can show that the observation error decreases when the number and support of sensors increases [23, 25].

5. **CONCLUSION**

We have been introduced a sufficient condition of regional gradient strategic sensors in order to achieve regional gradient observability. Many interesting results concerning the choice of sensors structure are given and illustrated in specific situations. Various questions still opened under consideration. For example, these result can be extended to the boundary case with parabolic and hyperbolic systems [8].

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