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Maximal Order of an NG-group

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Abstract

This study was aimed to consider the NG-group that consisting of transformations on a nonempty set A has no bijection as its element. In addition, it tried to find the maximal order of these groups. It found the order of NG-group not greater than n. Our results proved by showing that any kind of NG-group in the theorem be isomorphic to a permutation group on a quotient set of *A* with respect to an equivalence relation on *A*.

Keywords: NG-group; Permutation group; Equivalence relation; χ-subgroup

Introduction

This study considered the problem that the maximal order of a group consisting of transformations on a nonempty set *A* and the group has no bijection as its element. Recall a permutation group on *A* is a group consisting of bijections from *A* to *A* with respect to compositions of mappings. It is well known that any permutation group on a set *A* with cardinality n has an order not greater than *n*!.

In previous studies, there are some authors [1,2], problem 1.4 in [3] considering groups which consist of non-bijective transformations on *A* where the binary operation is the composition of mappings. Our first result is on the orders of such groups.

Theorem1.1. Let *A* be a set with cardinality *n*. Suppose *NG be* groups consisting of non-bijective transformations on *A*, where the binary operation on *NG* is the composition of transformation. Then the order of *NG* is not greater than $(n\text{-}1)!$ and there are such groups having order$(n\text{-}1)!$.

Then it was proven Theorem1.1 by showing that any kind of group in the theorem be isomorphic to a permutation group on a quotient set of *A* with respect to an equivalence relation on *A*.

Definition 1.1. A class of group χ is called an SHP-class if it is closed under taking subgroups, homomorphic images, and products of normal subgroups. The latter condition means that if U and V are normal in G and both U and V lie in χ, then $UV\in\chi$. If a group G belongs to χ, we will say G is an χ-group.

Remark 1.2. If χ is an SHP-class and $U,V \lhd G$ are such that G/U and G/V are χ- groups, then $G/(U\cap V)$ is isomorphic to a subgroup of the χ-group $(U/G)\times\{G/V)$, and thus $G/(U\cap V)$ is an χ-group. It follows that given a finite group G, there exists a unique smallest normal subgroup N such that $G/N\in\chi$, and we write $N= G\chi$. The following theorem was found by the author; see also lemma 2.32 in [4].

Theorem 1.2. Let χ be an SHP-class, and suppose $G=UV$, where U and V are subnormal in G. then $G\chi = U\chi\ V\chi$. It could take the SHP-class to the class of p-groups, the class of nilpotent groups, etc. Theorem1.2 will imply Lemma 9.15, problem 9B.5, Corollary 9.27, problem 9C.2, as corollaries.

Remark 1.3. It was noted in sec.4 of [5] that if it replaces the condition that χ is an SHP-class by some weaker condition that the class χ is such that whose composition factors all lie in some given set of simple groups then theorem 1.2 will fail in this case.

Definition 1.2. Let χ be an SHP-class and G be a finite group. The result was denoted the maximal normal χ-subgroup of G by $G\chi$. Then was considered the question that if $G=UV$ with U,V subnormal in G then it holds that $G\chi =U\chi V\chi$ or not. If p is a prime and take the SHP-class χ to be the class of all finite p-group, then for any finite group $G\chi$ will be $O_p(G)$ and results have the following theorem.

Theorem1.3. Let p and q be two primes such that $q\equiv 1 (\mathrm{mod}\ p)$. Let $N=G_q$ be a cyclic group of order q and $H=<x>\times<y>$ an elementary abelian group of order p^2. Let $<x>$ act on N faithfully and $<y>$ act on N trivially. Set $G= N\rtimes H$ to the semidirect product of N and H. Let $U=N<x>$ and $V=<xy>$. Then [1] U,V are both subnormal in G. [2] $O_p(G)=<y>$ and $O_p(U)=O_p(V)=1$. In particular, $Op(G)\neq O_p(U)O_p(V)$.

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Definition 2.1. A binary relation ~ in A is called an equivalence relation on A. If it satisfies the following three conditions:

(i) $a\sim a$ for any $a \in A$;
(ii) for any $a,b\in A$, if $a\sim b$ then $b\sim a$;
(iii) for any $a,b,c\in A$, if $a\sim b$ and $b\sim c$ then $a\sim c$.

For the set A, where use A^A to dented the set of all its transforms, for any $f\in A^A$, we use Im(f) to denote the image of f. Also, Z and $Z_>$ will respective dented the set of integers and positive integers.

Definition 2.2. Let ~ be an equivalence relation on A, for an element $a\in A$, it is call $\{x\in A|x \sim\}$ the equivalence class of a determined by ~, which is denoted by$[a]_\sim$. And $A/\sim=\{[a]_\sim|a\in A\}$ is called the quotient set of S relative to the equivalence relation ~.

Lemma2.3. Theorem 1 [1], for any $f\in G$ and the e the identity element of G,
$\sim_e=\sim_f$

Proof, for any $a\in A$, so result goal is to show that $[a]_f=[a]_e$.

On one hand, if $x\in[a]_f$, i.e. $f(x)=f(a)$. Since G is a group with identity element e, there is a transformation $f'\in G$ such that $f'f=e=f'f$. Therefore,

$$e(x)=f'(f(x))=f'(f(a))=e(a),$$

Which yields that $x\in[a]_e$.

On the other hand, if $y\in[a]_e$ i.e. $e(a)e(y)$. Hence,

$$f(a)=(fe)(a)=f(e(y))=(fe)(y)=f(y),$$

Which implies $y\in\in[a]_f$.. It follows that $[a]_e.=[a]_f$ for any $a\in A$, as wanted.

Remark 2.1. For Lemma 2.4, the current result see that $\sim_f=\sim_g$ for any element $f,g\in G$

The following Theorem is the revised version of Theorem 2, [1].

Theorem 2.5. Let f be an element in A^A and $\hat{}f$ be the induced transformation of f on $A/\sim_f$, i.e

$$\hat{}f\textbf{:}\ A/\sim_f \longrightarrow A/\sim_f,\ [x]_f \mapsto [f(x)]_f.$$

Then the following hold:

(i) The exists a groups $G\subseteq A^A$ containing f as the identity element *iff* $f^2=f$.

(ii) There is a groups $G\subseteq A^A$ containing f as the identity element *iff* $\hat{}f$ is abijective on $A/\sim_f$.

The following two corollaries are from [1], and we make some corrections to the original proofs. Actually, this adopt the restriction of finiteness on A in the first corollary from the original one. And then used the finiteness on A in the second corollary; the original one did not use it.

Corollary 2.6. Let f be an element in A^A. Then $f^2=f$ *iff* the induced mapping

$\hat{}f$ on $A/\sim_f$ is the identity element.

Proof. On one hand, suppose that $f^2=f$. Then for any $[x]_f\in A/\sim_f$, as$f(x)=f(f(x))$, then see that $[x]_f=[f(x)]_f$. It follows that

$$\hat{}f([x]_f)=[f(x)]_f=[x]_f\,;$$

This implies that $\hat{}f$ is the identity mapping on $A/\sim_f$.

On the other hand, assume that $\hat{}f$ is the identity mapping on $A/\sim_f$. Then for any $[x]_f\in A/\sim_f$, the condition that $\hat{}f([x]_f)=[x]_f$ will imply that $[f(x)]_f=[x]_f$ and hence $f(f(x))=f(x)$. It follows that $f^2=f$ as required.

Corollary 2.7. Suppose that A is a finite set and f is an element in A^A. Then there is a group $G\subseteq A^A$ containing f as an element *iff* $\mathrm{Im}(f)=\mathrm{Im}(f^2)$.

Proof. On one hand, suppose that there is a group $G\subseteq A^A$ containing f as an element. Let e be the identity element of G. Then by Theorem 2.5, the induced mapping $\hat{}f$ is a bijection on $A/\sim_f$. In particular, $\hat{}f$ is surjective and thus for any $x\in A$, there is *a* $[y]_f\in A/\sim_f$ such that $\hat{}f([y]_f)=[x]_f=[f(y)]_f\,;$

Which yields that $f(x)=f(f(y))=(f^2)(y)$. As a result, $\mathrm{Im}(f)\subseteq\mathrm{Im}(f^2)$ and thus $\mathrm{Im}(f)=\mathrm{Im}(f^2)$.

On the other hand, suppose that $\mathrm{Im}(f)=\mathrm{Im}(f^2)$. Thus, for any $f(x)\in\mathrm{Im}(f)$ there is a $y\in A$ such that $f(x)=f(f(y))$ and hence $\hat{}f([y]f)=[x]f\,;$ which implies that $\hat{}f$ is surjective on $A/\sim_f$. Note that results are assuming that A is finite and so is $A/\sim_f$. this study has that the induced mapping $\hat{}f$ is bijective. By Theorem 2. 5, the assertion follows.

Remark 2. 2. Let $G \subseteq A^A$ be a group. That has seen, in Remark 2.1, that $\sim_f = \sim_g$ for any elements in G and we will denote the common equivalence relation by ~. Also, by Theorem 2.5, each element $f \in G$ will induce a bijection $\hat{f}$ on $A/\sim$**.**

The following theorem is crucial since it turns a group $G \subseteq A^A$ into a permutation group.

Theorem 2.8. Let $G \subseteq A^A$ be a group. Set $\hat{G} = \{\hat{f} \mid f \in G\}$; then $\hat{G}$ is a permutation group on $A/\sim$ and $\rho : G \longrightarrow \hat{G}$, $f \mapsto \hat{f}$, is an isomorphism. *Proof*. For any $f, g \in G$ and any $[a] \in A/\sim$, results have $\rho(fg)([a]) = [(fg)(a)] = [f(g(a))] = \rho(f)([g(a)]) = (\rho(f)\,\rho(g))([a])$; which implies that $\rho(fg) = \rho(f)\,\rho(g)$ and thus ρ is a homomorphism.

By the definition of $\hat{G}$, it is obvious that ρ is surjective.

Now suppose that $\rho(f) = \rho(g)$ for two elements $f, g \in G$, i.e.$[f(a)] = [g(a)]$, $\forall a \in A$: Let e be the identity element of G, then we have $[f(a)]_e = [g(a)]_e$; $\forall a \in A$. It follows that $e(f(a)) = e(g(a))$; $\forall a \in A$ .Hence,$f(a) = (ef)(a) = e(f(a)) = e(g(a)) = g(a)$, $\forall a \in A$, and therefore $f = g$. so it conclude that ρ is injective. As a consequence, ρ is an isomorphism.

Definition 2.3. A subgroup H of a group G is called characteristic in G, denoted H char G, if every automorphism of G maps H to itself, that is$\rho(H) = H$ for all $\rho \in \mathrm{Aut}(G)$.

Remark 2.3. If H is characteristic in G in K and K is characteristic in G, then H is characteristic in G.

Let G be a finite group. It has the following two lemmas. They are from Section 2 of (*5*).

Lemma 2.9. Suppose that χ is an SHP-class.

(a) Let ≤ G be a subgroup. Then $H^\chi \leq G^\chi$.
(b) Let $N \lhd G$ be a normal subgroup of G and write $\bar{G} = G/N$. then $\bar{G}^\chi = \bar{G}^{\chi}$.
(c) G^χ is characteristic in G.
(d) $O_\chi(G)$ is characteristic in G.

The following lemma is a generalization of Problem 2A.1 in (*8*).

Lemma 2.10. Let A and B be two subnormal χ-subgroups of G. Then the subgroup $\langle A,B \rangle$ generated by A and B are χ-subgroup of G.

Proof. Let A be a subnormal χ-subgroup of G. The resulting use induction on the subnormal depth r, $A \subseteq O_\chi$ of A in G to show that if $r = 1$, then $A \lhd$ and thus $A \subseteq O_\chi(G)$ since $O_\chi(G)$ is the largest normal χ-subgroup of G.

Suppose $r > 1$ and the containment holds for r-1. Let $A_1 = A \lhd \ldots \lhd H_{r-1} \lhd H_r = G$
be a subnormal series from A to G: Then $A \subseteq O_\chi(G)$ by inductive hypothesis.
Since $O_\chi(G)$ char H_{r-1} and $H_{r-1} \lhd G$; $O_\chi(G) \lhd G$ and then $O_\chi(G)\ (H_{r-1}) \subseteq O_\chi(G)$.
It was concluded that $A \subseteq O_\chi(G)$.
In general, for any two subnormal χ-subgroups A and B, $A,B \subseteq O_\chi(G)$ and thus
$\langle A,B \rangle \subseteq O_\chi(G)$ as wanted.

Proofs of Main Results

Now let A be a set having n letters written as $\{1, 2,\ldots, n\}$. The results have the following theorem, which is Theorem 1.1.

Theorem 3.1. Let A be a set with cardinality n with $n \geq 3$. Suppose NG is a group consisting of non-bijective transformations on A, where the binary operation on NG is the composition of transformations. Then the order of NG is not greater than $(n-1)!$ and there are such groups having order $(n-1)!$:

Proof. Let NG be a group consisting of non-bijective transformations on A. By Remark 2.1, it is known that $\sim_f = \sim_g$ for any element $f, g \in NG$ and it denote the common equivalence relation by ~. Note that NG is a group consisting of non-bijective transformations, then we see that the equivalence relation is not the equality relation =on A. Thus, these results have that the quotient set $A/\sim$ has an order less than n-1.

Additionally, NG is isomorphic to a permutation group on $A/\sim$ by Theorem 2.8. It follows that the order of NG is less than $(n-1)!$ as any permutation group on $A/\sim$ has order less than $(n-1)!$.
Note that In defining a permutation s on the set $\{1, 3, \ldots, n\}$, there are n-1 choices for $\rho(1)$, n-2 choices of $\rho(3) \neq \rho(1)$, n- 2 choices of $\rho(4)$ $(\neq \rho(1), \rho(3))$, etc., i.e. totally $(n-1)(n-2)1 = (n-1)!$.

Theorem 3.2. Let χ be an SHP-class, and suppose $G = UV$; where U and V are subnormal in G. Then $G^{\chi} = U^{\chi}V^{\chi}$.

Proof. This work use induction of the subnormal depth of U in G to prove the result.
First, if the subnormal depth of U in G is one, i.e. $U \lhd G$. Since U^{χ} is characteristic in U and U is normal in G we see that $U^{\chi}$is normal in G.
Let $\overline{G} = G/U^{\chi}$. By the hypothesis, $\overline{G} = \overline{U}\,\overline{V}$ *where* $\overline{U} = U/U^{\chi}$, $\overline{V} = VU^{\chi}/U^{\chi}$.
Thus, $\overline{U}$ is a normal χ-group of $\overline{G}$ and $\overline{V}$ is subnormal in $\overline{G}$. By Lemma 2.10, we have $\overline{G}^{\chi} = \overline{V}^{\chi}$. By Lemma 2.9 (b), $\overline{G^{\chi}} = \overline{G}^{\chi}$, $\overline{V^{\chi}} = \overline{V}^{\chi} = \overline{U}^{\chi}\,\overline{V}^{\chi}$.
By correspondence theorem, it has $G^{\chi} = U^{\chi}V^{\chi}$; as required.

Now suppose that the subnormal depth of U in G is r with $r > 1$: Let
$U_1 = U \lhd \ldots \lhd U_r \lhd G$
be a subnormal series from U to G with length r. By Dedekind's lemma,
$U_r = U(V \cap U_r)$. As both U and $V \cap U_r$ are subnormal in U_r and U has subnormal depth r- 1 in U_r, then obtain that
$(U_r)^{\chi} = U^{\chi} (V \cap U_r)^{\chi}$
by inductive hypothesis. Also, $G = U_rV$ with U_r normal in G and V subnormal in G, and hence
$G^{\chi} = (U_r)^{\chi}V^{\chi}$ by the first paragraph of the proof. It follows that
$G^{\chi} = (U_r)^{\chi}V^{\chi} = U^{\chi} (V \cap U_r)^{\chi}V^{\chi} = U^{\chi}V^{\chi}$,
because $(V \cap U_r)^{\chi} \subseteq V^{\chi}$ by Lemma 2.9 (a).

Theorem 3.3. Let p and q be two primes such that $q \equiv 1 (\mathrm{mod}\, p)$. Let $N = C_q$ be a cyclic group of order q and $H = \langle x \rangle$ an elementary abelian group of order p^2. Let $\langle x \rangle$ act on N faithfully and $\langle y \rangle$ act on N trivially. Set $G = N \rtimes H$ to be the semidirect product of N and H. Let $U = N\langle x \rangle$ and $V = N\langle xy \rangle$. Then

(i) U, V are both subnormal in G and $G = UV$.

(ii) $O_p(G) = \langle y\rangle$ and $O_p(U) = O_p(V) = 1$. In particular, $Op(G) \neq O_p(U)O_p(V)$.

Proof. Since N is normal in G and the quotient group $G=N/H$ is abelian, it deduced that the derived subgroup G' is contained in N. It follows that both U and V contain G' as a subgroup, which implies that U and V are normal in G. Obviously, $G = UV$. Assertion (i) holds.

Note that the Sylow p-subgroup of G is not normal since $\langle x\rangle$ act on N faithfully and hence $O_p(G)$ has an order less than p^2. However, as $\langle y\rangle$ act on N trivially, N normalizes $\langle y\rangle$ which yields that $\langle y\rangle$ is a normal p-subgroup of G. It is easy to see that $O_p(G) = \langle y\rangle$. Both $\langle x\rangle$ and $\langle xy\rangle$ act faithfully on N, which yields that $O_p(U) = Op(V) = 1$; as wanted.

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