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Sufficiency and Duality for E-differentiable Multiobjective Programming Problems Involving Generalized V-E-invex Functions

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Abstract

In this paper, a class of E-differentiable multiobjective programming problems with both inequality and equality constraints is considered. For E-differentiable functions, the concepts of V-E-pseudo-invexity, strictly V-E-pseudo-invexity and V-E-quasi-invexity are introduced. Based upon these notions of generalized V-E-invex functions, the sufficiency of the so-called E-Karush-Kuhn-Tucker optimality conditions are established for the considered E-differentiable vector optimization problems with both inequality and equality constraints. Furthermore, the so-called vector Mond-Weir E-dual problem is defined for the considered E-differentiable multiobjective programming problem and several E-duality theorems in the sense of Mond-Weir are derived also under appropriate generalized V-E-invexity assumptions.

Key words: V-E-invex function, Generalized V-E-invexity, E-differentiable function, E-optimality conditions, E-Mond-Weir duality.

AMS Subject Classification: 90C26, 90C30, 90C46, 26B25

1 Introduction

Convexity notion plays an important role to derive the optimality conditions and duality results for various scalar and vector optimization problems. However, many operational research problems that are modeled by various optimization problems are not convex. During the past decades, therefore, generalized convex functions received much attention. Various classes of differentiable and nondifferentiable generalized convex functions have appeared in literature, not only for scalar optimization problems, but also for multiobjective programming problems. Optimality conditions and duality theorems for differentiable and nondifferentiable optimization problems have been studied extensively in the literature (see, for example, [1], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [18], [19], [21], [22], [23], [24], [25], [26], and others).

One of such important generalizations of the convexity notion is the concept of invexity introduced by Hanson [15] for scalar optimization problems. Jeyakumar and Mond [17] introduced a new class of nonconvex differentiable vector-valued functions, namely V-invex functions, in order to resolve the difficulty of demanding the same function η for objective and constraint functions in extremum problems dealing with the concept of invexity introduced by Hanson [15] for scalar optimization problems. They established sufficient optimality criteria and duality results in the multiobjective static case for weak Pareto solutions under V-invexity hypotheses. Megahed et al. [20] presented the concept of an E-differentiable convex function which transforms a (not necessarily) differentiable convex function to a differentiable function based on the effect of an operator $E : R^n \to R^n$. Antczak and Abdulaleem [5] proved the so-called E-optimality conditions and Wolfe E-duality for E-differentiable vector optimization problems with both inequality and equality constraints. Abdulaleem [1] introduced a new concept of generalized convexity as a generalization of the notion of E-differentiable E-convexity. Namely, he defined the concept of E-differentiable E-invexity in the case of (not necessarily) differentiable vector optimization problems with E-differentiable functions. Recently, Abdulaleem [2] introduced a new concept of generalized convexity as a generalization of the E-differentiable E-invexity notion and the concept of V-invexity. Namely, he defined the concept of E-differentiable V-E-invexity in the case of (not necessarily) differentiable vector optimization problems with E-differentiable functions.

In this paper, a new class of nonconvex E-differentiable vector optimization problems with both inequality and equality constraints is considered in which the involved functions are generalized V-E-invex. Therefore, the concepts of V-E-pseudo-invex, strictly V-E-pseudo-invex and V-E-quasi-invex functions for E-differentiable vector optimization problems are introduced. Further, the sufficiency of the so-called E-Karush-Kuhn-Tucker optimality conditions are derived for the considered E-differentiable vector optimization problem under appropriate generalized V-E-invexity hypotheses. Furthermore, the so-called vector E-dual problems in the sense of Mond-Weir is defined for E-differentiable vector dual problems. Then, several Mond-Weir E-duality results are established between the considered E-differentiable multicriteria optimization problem and its Mond-Weir vector dual problem also under appropriate generalized V-E-invexity hypotheses.

2 Preliminaries

Let R^n be the n-dimensional Euclidean space and R^n_+ be its nonnegative orthant. The following convention for equalities and inequalities will be used in the paper. For any vectors $x = (x_1, x_2, ..., x_n)^T$ and $y = (y_1, y_2, ..., y_n)^T$ in R^n, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, \cdots, n$;

(ii) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, \cdots, n$;

(iii) $x \geqq y$ if and only if $x_i \geqq y_i$ for all $i = 1, 2, \cdots, n$;

(iv) $x \geq y$ if and only if $x_i \geqq y_i$ for all $i = 1, 2, \cdots, n$ but $x \neq y$;

(v) $x \ngtr y$ is the negation of $x > y$.

Definition 2.1 *[1] Let $E : R^n \to R^n$. A set $M \subseteq R^n$ is said to be an E-invex set if and only if there exists a vector-valued function $\eta : M \times M \to R^n$ such that the relation*

$$E(u) + \lambda\eta(E(x), E(u)) \in M$$

holds for all $x, u \in M$ and any $\lambda \in [0, 1]$.

Let M be a nonempty E-invex subset of R^n.

Definition 2.2 *[20] Let $E : R^n \to R^n$ and $f : M \to R$ be a (not necessarily) differentiable function at a given point $u \in M$. It is said that f is an E-differentiable function at u if and only if $f \circ E$ is a differentiable function at u (in the usual sense), that is,*

$$(f \circ E)(x) = (f \circ E)(u) + \nabla(f \circ E)(u)(x - u) + \theta(u, x - u)\|x - u\|, \tag{2.1}$$

where $\theta(u, x - u) \to 0$ as $x \to u$.

Definition 2.3 *[2] Let $E : R^n \to R^n$ and $f : M \to R^k$ be an E-differentiable function on M. It is said that f is a V-E-invex function (a strictly V-E-invex function) with respect to η at $u \in M$ on M if, there exist functions $\eta : M \times M \to R^n$ and $\alpha_i : M \times M \to R_+ \setminus \{0\}$, $i = 1, 2, ..., k$, such that, for each $x \in M$ $(E(x) \neq E(u))$, the inequalities*

$$f_i(E(x)) - f_i(E(u)) \geqq \alpha_i(E(x), E(u))\nabla f_i(E(u))\eta(E(x), E(u)) \quad (>) \tag{2.2}$$

hold. If inequalities (2.2) are fulfilled for any $u \in M$ $(E(x) \neq E(u))$, then f is V-E-invex (strictly V-E-invex) with respect to η on M. Each function f_i, $i = 1, ..., k$, for which (2.2) is fulfilled is said to be α_i-E-invex (strictly α_i-E-invex) with respect to η at u on M.

Remark 2.4 *Note that the Definition 2.3 generalizes and extends several generalized convexity notions, previously introduced in the literature. Indeed, there are the following special cases:*

a) *In the case when* $\alpha_i(x,u) = 1$, $i = 1, ..., k$, *then the definition of a* V*-*E*-invex function reduces to the definition of an* E*-invex function introduced by Abdulaleem [1].*

b) *If* f *is differentiable and* $E(x) \equiv x$ *(*E *is an identity map), then the definition of a* V*-*E*-invex function reduces to the definition of a* V*-invex function introduced by Jeyakumar and Mond [17].*

c) *If* f *is differentiable,* $E(x) \equiv x$ *(*E *is an identity map) and* $\alpha_i(x,u) = 1$, $k = 1$, *then the definition of a* V*-*E*-invex function reduces to the definition of an invex function introduced by Hanson [13].*

d) *If* $\eta : R^n \times R^n \to R^n$ *is defined by* $\eta(x,u) = x - u$ *and* $\alpha_i(x,u) = 1$, $i = 1, ..., k$, *then we obtain the definition of an* E*-differentiable* E*-convex vector-valued function introduced by Megahed et al. [20].*

e) *If* f *is differentiable,* $E(x) = x$ *and* $\eta(x,u) = x - u$ *and* $\alpha_i(x,u) = 1$, $i = 1, ..., k$, *then the definition of a* V*-*E*-invex function reduces to the definition of a differentiable convex vector-valued function.*

f) *If* f *is a differentiable scalar function,* $\eta(x,u) = x - u$ *and* $\alpha_i(x,u) = 1$, *then we obtain the definition of a differentiable* E*-convex function introduced by Youness [26].*

Now, we introduce various classes of generalized E-differentiable V-E-invex functions.

Definition 2.5 *Let* $E : R^n \to R^n$ *and* $f : M \to R^k$ *be an* E*-differentiable function on* M. *It is said that* f *is a* V*-*E*-pseudo-invex function with respect to* η *at* $u \in M$ *on* M *if, there exist functions* $\eta : M \times M \to R^n$ *and* $\alpha_i : M \times M \to R_+ \setminus \{0\}$, $i = 1, 2, ..., k$, *such that, for each* $x \in M$, *the relations*

$$\sum_{i=1}^{k} \alpha_i(E(x), E(u)) f_i(E(x)) < \sum_{i=1}^{k} \alpha_i(E(x), E(u)) f_i(E(u))$$

$$\Rightarrow \sum_{i=1}^{k} \nabla f_i(E(u)) \eta(E(x), E(u)) < 0 \tag{2.3}$$

hold. If (2.3) *are fulfilled for any* $u \in M$, *then* f *is* V*-*E*-pseudo-invex with respect to* η *on* M. *Each function* f_i, $i = 1, ..., k$, *satisfying* (2.3) *is said to be* α_i*-*E*-pseudo-invex with respect to* η *at* u *on* M.

Definition 2.6 *Let* $E : R^n \to R^n$ *and* $f : M \to R^k$ *be an* E*-differentiable function on* M. *It is said that* f *is a strictly* V*-*E*-pseudo-invex function with respect to* η *at* $u \in M$ *on* M *if, there exist functions* $\eta : M \times M \to R^n$ *and* $\alpha_i : M \times M \to R_+ \setminus \{0\}$, $i = 1, 2, ..., k$, *such that, for each* $x \in M$, $E(x) \neq E(u)$, *the relations*

$$\sum_{i=1}^{k} \alpha_i(E(x), E(u)) f_i(E(x)) \leqq \sum_{i=1}^{k} \alpha_i(E(x), E(u)) f_i(E(u))$$

$$\Rightarrow \sum_{i=1}^{k} \nabla f_i(E(u))\eta(E(x), E(u)) < 0 \tag{2.4}$$

hold. If (2.4) *are fulfilled for any* $u \in M$, $E(x) \neq E(u)$, *then* f *is strictly* V*-*E*-pseudo-invex with respect to* η *on* M. *Each function* f_i, $i = 1, ..., k$, *satisfying* (2.4) *is said to be strictly* α_i*-*E*-pseudo-invex with respect to* η *at* u *on* M.

Note that each strictly V-E-pseudo-invex function is V-E-pseudo-invex and each E-differentiable E-pseudo-invex function is V-E-pseudo-invex. Also, each V-pseudo-invex function is V-E-invex and each V-E-invex function is V-E-pseudo-invex, but the converse is not true.

Now, we present an example of such a V-E-invex function which is not an E-invex function with respect to η.

Example 2.7 *Let* $f : R \to R^2$ *be defined by* $f(x) = (2e^{\sqrt[3]{x}},\ 4e^{\sqrt[3]{x}})$ *and* $E : R \to R$ *be an operator defined by* $E(x) = x^3$ *and* η *be defined by*

$$\eta(E(x), E(u)) = \begin{cases} \frac{-1}{4} & \text{if } x < u, \\ e^u - e^x & \text{if } x \geqq u. \end{cases}$$

where $\alpha_i(E(x), E(u)) : R \times R \to R_+ \setminus \{0\}$ *defined by*

$$\alpha_i(E(x), E(u)) = \begin{cases} \frac{4e^u - 4e^x}{e^u} & \text{if } x < u, \\ \frac{1}{e^u} & \text{if } x \geqq u. \end{cases}$$

Then f *is* V*-*E*-invex on* R, *but* f *is not* E*-invex with respect to* η *defined above as can be seen by taking* $x = \ln 4$, $u = \ln 10$, *since the inequalities*

$$f_i(E(x)) - f_i(E(u)) < \nabla f_i(E(u))\eta(E(x), E(u))$$

hold. Hence, by the definition of an E*-invex function [1], it follows that* f *is not* E*-invex with respect to* η *given above.*

Definition 2.8 *Let* $E : R^n \to R^n$ *and* $f : M \to R^k$ *be an* E*-differentiable function on* M. *It is said that* f *is a* V*-*E*-quasi-invex function with respect to* η *at* $u \in M$ *on* M *if there exist functions* $\eta : M \times M \to R^n$ *and* $\alpha_i : M \times M \to R_+ \setminus \{0\}$, $i = 1, 2, ..., k$, *such that for each* $x \in M$, *the relations*

$$\sum_{i=1}^{k} \alpha_i(E(x), E(u)) f_i(E(x)) \leqq \sum_{i=1}^{k} \alpha_i(E(x), E(u)) f_i(E(u))$$

$$\Rightarrow \sum_{i=1}^{k} \nabla f_i(E(u))\eta(E(x), E(u)) \leqq 0 \tag{2.5}$$

hold. If (2.5) *are fulfilled for any* $u \in M$, *then* f *is* V*-*E*-quasi-invex on* M. *Each function* f_i, $i = 1, ..., k$, *satisfying* (2.5) *is said to be* α_i*-*E*-quasi-invex with respect to* η *at* u *on* M.

Note that every E-differentiable V-E-invex is V-E-quasi-invex and every V-E-pseudo-invex function is V-E-quasi-invex.

Now, we present an example of such an E-differentiable V-E-pseudo-invex function which is not V-E-invex or E-invex.

Example 2.9 *Let $f : R \to R^2$ be defined by $f(x) = (2e^{\sqrt[3]{x}}, e^{\sqrt[3]{x}})$, $E : R \to R$ be an operator defined by $E(x) = x^3$ and $\eta : R \times R \to R$ be defined by*

$$\eta(x,u) = \begin{cases} \frac{1}{(\sqrt[3]{x}-\sqrt[3]{u})e^{\sqrt[3]{u}}} & \text{if } x > u, \\ -1 & \text{if } x \leqq u. \end{cases}$$

where $\alpha : R \times R \to R_+ \setminus \{0\}$ defined by

$$\alpha_i(x,u) = \begin{cases} \sqrt[3]{x} - \sqrt[3]{u} & \text{if } x > u, \\ 2 & \text{if } x \leqq u. \end{cases}$$

Let $x \leqq u$. Then, we have

$$\sum_{i=1}^{k} \alpha_i(E(x), E(u)) f_i(E(x)) = 6e^x \leqq 6e^u = \sum_{i=1}^{k} \alpha_i(E(x), E(u)) f_i(E(u)).$$

Moreover, we have $\sum_{i=1}^{k} \nabla f_i(E(u))\eta(E(x), E(u)) = -3e^u < 0$. Let $x > u$. Then, we have $\sum_{i=1}^{k} \nabla f_i(E(u))\eta(E(x), E(u)) = \frac{3}{x-u} > 0$. Moreover, we have

$$\sum_{i=1}^{k} \alpha_i(E(x), E(u)) f_i(E(x)) = 3(x-u)e^x > 3(x-u)e^u = \sum_{i=1}^{k} \alpha_i(E(x), E(u)) f_i(E(u)).$$

Therefore, by Definition 2.5, f is an E-differentiable V-E-pseudo-invex function on R. However, it is not V-E-invex on R with respect to η and α given above. Indeed, if we set $x = ln(1.5)$, $u = ln(1)$, then we have

$$f_i(E(x)) - f_i(E(u)) < \alpha_i(E(x), E(u)) \nabla f_i(E(u)) \eta(E(x), E(u)).$$

Hence, by Definition 2.3, it follows that f is not V-E-invex with respect to η and α given above (see Abdulaleem [2]). Also, f is not E-invex on R with respect to η given above. Indeed, if we set $x = ln(1.5)$, $u = ln(1)$, then we have

$$f(E(x)) - f(E(u)) < \nabla f(E(u)) \eta(E(x), E(u)).$$

Hence, by the definition of an E-invex function (see Abdulaleem [1]), it follows that f is not E-invex with respect to η given above.

3 E-differentiable multiobjective programming

In this paper, we consider the following (not necessarily differentiable) multiobjective programming problem (MOP) with both inequality and equality constraints:

$$\begin{aligned} &\text{minimize } f(x) = (f_1(x), ..., f_q(x)) \\ &\text{subject to } g_j(x) \leqq 0, \ \ j \in J = \{1, ..., p\}, \\ &\qquad h_k(x) = 0, \ \ k \in K = \{1, ..., s\}, \\ &\qquad x \in R^n, \end{aligned} \qquad \text{(MOP)}$$

where $f_i : R^n \to R$, $i \in I = \{1, ..., q\}$, $g_j : R^n \to R$, $j \in J$, $h_k : R^n \to R$, $k \in K$, are real-valued functions defined on R^n. We shall write $g := (g_1, ..., g_p) : R^n \to R^p$ and $h := (h_1, ..., h_s) : R^n \to R^s$ for convenience. Let

$$\Omega := \{x \in R^n : g_j(x) \leqq 0, \ \ j \in J, \, h_k(x) = 0, \ \ k \in K\}$$

be the set of all feasible solutions of (MOP). Further, we denote by $J(x)$ the set of inequality constraint indices that are active at a feasible solution x, that is, $J(x) = \{j \in J : g_j(x) = 0\}$.

Definition 3.1 *A feasible point $\overline{x}$ is said to be a weak Pareto (weakly efficient) solution of (MOP) if and only if there is no other feasible solution x such that*

$$f(x) < f(\overline{x}).$$

Definition 3.2 *A feasible point $\overline{x}$ is said to be a Pareto (efficient) solution of (MOP) if and only if there is no other feasible solution x such that*

$$f(x) \leq f(\overline{x}).$$

Let $E : R^n \to R^n$ be a given one-to-one and onto operator. Now, for the considered E-differentiable multiobjective programming problem (MOP), we define its associated differentiable vector optimization problem as follows:

$$\begin{aligned} &\text{minimize } f(E(x)) = (f_1(E(x)), ..., f_q(E(x))) \\ &\text{subject to } g_j(E(x)) \leqq 0, \ \ j \in J = \{1, ..., p\}, \\ &\qquad h_k(E(x)) = 0, \ \ k \in K = \{1, ..., s\}, \\ &\qquad x \in R^n. \end{aligned} \qquad (\text{VP}_E)$$

Let

$$\Omega_E := \{x \in R^n : g_j(E(x)) \leqq 0, \ \ j \in J, \, h_k(E(x)) = 0, \ \ k \in K\}$$

be the set of all feasible solutions of (VP_E).

Definition 3.3 *A feasible point $E(\overline{x})$ is said to be a weak E-Pareto (weakly E-efficient) solution of (MOP) if and only if there is no other feasible solution $E(x)$ such that*

$$f(E(x)) < f(E(\overline{x})).$$

Definition 3.4 *A feasible point $E(\overline{x})$ is said to be an E-Pareto (E-efficient) solution of (MOP) if and only if there is no other feasible solution $E(x)$ such that*

$$f(E(x)) \leq f(E(\overline{x})).$$

Lemma 3.5 *[5] Let $E : R^n \to R^n$ be a one-to-one and onto. Then $E(\Omega_E) = \Omega$.*

Lemma 3.6 *[5] Let $\overline{x} \in \Omega$ be a weak Pareto solution (Pareto solution) of the considered multiobjective programming problem (MOP). Then, there exists $\overline{z} \in \Omega_E$ such that $\overline{x} = E(\overline{z})$ and $\overline{z}$ is a weak Pareto (Pareto) solution of the E-vector optimization problem (VP_E).*

Lemma 3.7 *[5] Let $\overline{z} \in \Omega_E$ be a weak Pareto (Pareto) solution of the E-vector optimization problem (VP_E). Then $E(\overline{z})$ is a weak Pareto solution (Pareto solution) of the considered multiobjective programming problem (MOP).*

Theorem 3.8 *[1] (E-Karush-Kuhn-Tucker necessary optimality conditions). Let $\overline{x} \in \Omega_E$ be a weak Pareto solution of the problem (VP_E) (and, thus, $E(\overline{x})$ be a weak E-Pareto solution of the problem (MOP)). Further, let f, g, h be E-differentiable at $\overline{x}$ and the E-Guignard constraint qualification [1] be satisfied at $\overline{x}$. Then there exist Lagrange multipliers $\overline{\xi} \in R^q$, $\overline{\nu} \in R^p$, $\overline{\mu} \in R^s$ such that*

$$\sum_{i=1}^{q} \overline{\xi}_i \nabla (f_i \circ E)(\overline{x}) + \sum_{j=1}^{p} \overline{\nu}_j \nabla (g_j \circ E)(\overline{x}) + \sum_{k=1}^{s} \overline{\mu}_k \nabla (h_k \circ E)(\overline{x}) = 0, \tag{3.1}$$

$$\overline{\nu}_j (g_j \circ E)(\overline{x}) = 0, \;\; j \in J(E(\overline{x})), \tag{3.2}$$

$$\overline{\xi} \geq 0, \, \overline{\nu} \geqq 0. \tag{3.3}$$

Definition 3.9 *$(\overline{x}, \overline{\xi}, \overline{\nu}, \overline{\mu}) \in \Omega_E \times R^q \times R^p \times R^s$ is said to be a Karush-Kuhn-Tucker point for the considered constrained E-vector optimization problem (VP_E) if the Karush-Kuhn-Tucker necessary optimality conditions (3.1)-(3.3) are satisfied at $\overline{x}$ with Lagrange multiplier $\overline{\xi}$, $\overline{\nu}$, $\overline{\mu}$.*

Now, we prove the sufficient optimality conditions for a feasible solution to be a weak Pareto solution (a Pareto solution) in the considered multiobjective programming problem (MOP) under the concepts of generalized V-E-invexity hypotheses, that is, under assumption that constituting it are E-differentiable generalized V-E-invex at a feasible point satisfying the E-Karush-Kuhn-Tucker necessary optimality conditions (3.1)-(3.3).

Theorem 3.10 *Let* $(\overline{x}, \overline{\xi}, \overline{\nu}, \overline{\mu}) \in \Omega_E \times R^q \times R^p \times R^s$ *be a Karush-Kuhn-Tucker point of the constrained E-vector optimization problem (VP$_E$). Furthermore, assume that* $\overline{\xi}_i f_i(E(\cdot))$, $i \in I$, *is* α_i*-E-pseudo-invex with respect to* η *at* $\overline{x}$ *on* Ω_E, $\overline{\nu}_j g_j(E(\cdot))$, $j \in J(E(\overline{x}))$, *is* β_j*-E-quasi-invex with respect to* η *at* $\overline{x}$ *on* Ω_E, $\overline{\mu}_k h_k(E(\cdot))$, $k = 1, ..., s$, *is* γ_k*-E-quasi-invex with respect to* η *at* $\overline{x}$ *on* Ω_E. *Then* $\overline{x}$ *is a weak Pareto solution of the problem (VP$_E$) and, thus,* $E(\overline{x})$ *is a weak E-Pareto solution of the problem (MOP).*

Proof: By assumption, $(\overline{x}, \overline{\xi}, \overline{\nu}, \overline{\mu}) \in \Omega_E \times R^q \times R^p \times R^s$ is a Karush-Kuhn-Tucker point in the considered constrained E-vector optimization problem (VP$_E$). Then, by Definition 3.9, the Karush-Kuhn-Tucker necessary optimality conditions (3.1)-(3.3) are satisfied at $\overline{x}$ with Lagrange multipliers $\overline{\xi} \in R^q$, $\overline{\nu} \in R^p$ and $\overline{\mu} \in R^s$. We proceed by contradiction. Suppose, contrary to the result, that $\overline{x}$ is not a weak Pareto solution in (VP$_E$). Hence, by Definition 3.3, there exists $E(x') \in \Omega$ such that

$$f_i(E(x')) < f_i(E(\overline{x})), \; i \in I. \tag{3.4}$$

Using $\alpha_i(E(x'), E(\overline{x})) > 0$, $i \in I$, we get

$$\sum_{i=1}^{q} \alpha_i(E(x'), E(\overline{x}))\overline{\xi}_i f_i(E(x')) < \sum_{i=1}^{q} \alpha_i(E(x'), E(\overline{x}))\overline{\xi}_i f_i(E(\overline{x})). \tag{3.5}$$

Since $\overline{\xi} f(E(\cdot))$, is V-E-pseudo-invex with respect to η at $\overline{x}$ on Ω_E, by Definition 2.5, the inequality

$$\sum_{i=1}^{q} \overline{\xi}_i \nabla f_i(E(\overline{x}))\eta(E(x'), E(\overline{x})) < 0 \tag{3.6}$$

holds. Since $E(x') \in \Omega$, therefore, the E-Karush-Kuhn-Tucker necessary optimality conditions (3.2) and (3.3) imply

$$g_j(E(x')) - g_j(E(\overline{x})) \leqq 0, \;\; j \in J(E(\overline{x})).$$

Using $\beta_j(E(x'), E(\overline{x})) > 0$, $j \in J(E(\overline{x}))$, we get

$$\sum_{j=1}^{p} \beta_j(E(x'), E(\overline{x}))\overline{\nu}_j g_j(E(x')) - \sum_{j=1}^{p} \beta_j(E(x'), E(\overline{x}))\overline{\nu}_j g_j(E(\overline{x})) \leqq 0.$$

Since $\overline{\nu}_j g_j(E(\cdot))$, $j \in J(E(\overline{x}))$, is V-E-quasi-invex with respect to η at $\overline{x}$ on Ω_E, by Definition 2.8, we get

$$\sum_{j=1}^{p} \overline{\nu}_j \nabla g_j(E(\overline{x}))\eta(E(x'), E(\overline{x})) \leqq 0. \tag{3.7}$$

From $x' \in \Omega_E$, $\overline{x} \in \Omega_E$, it follows that

$$h_k(E(x')) - h_k(E(\overline{x})) = 0. \tag{3.8}$$

Using $\gamma_k(E(x'), E(\overline{x})) > 0$, $k = 1, ..., s$, we obtain

$$\sum_{k=1}^{s} \gamma_k(E(x'), E(\overline{x}))\overline{\mu}_k h_k(E\,(x')) - \sum_{k=1}^{s} \gamma_k(E(x'), E(\overline{x}))\overline{\mu}_k h_k(E\,(\overline{x})) = 0. \tag{3.9}$$

Since $\overline{\mu}_k h_k(E(\cdot))$, $k = 1, ..., s$, is V-E-quasi-invex with respect to η at $\overline{x}$ on Ω_E, by Definition 2.8, we have

$$\sum_{k=1}^{s} \overline{\mu}_k \nabla h_k\,(E\,(\overline{x}))\,\eta\,(E\,(x')\,, E\,(\overline{x})) \leqq 0. \tag{3.10}$$

Combining (3.6), (3.7) and (3.10), we get that the following inequality

$$\left[\sum_{i=1}^{q} \overline{\xi}_i \nabla f_i(E\,(\overline{x})) + \sum_{j=1}^{p} \overline{\nu}_j \nabla g_j\,(E\,(\overline{x})) + \sum_{k=1}^{s} \overline{\mu}_k \nabla h_k\,(E\,(\overline{x}))\right] \eta\,(E\,(x')\,, E\,(\overline{x})) < 0,$$

holds, which is a contradiction to the E-Karush-Kuhn-Tucker necessary optimality condition (3.1). By assumption, $E : R^n \to R^n$ is an one-to-one and onto operator. Since $\overline{x}$ is a weak Pareto solution of the problem (VP$_E$), by Lemma 3.7, $E\,(\overline{x})$ is an E-Pareto solution of the problem (MOP). Thus, the proof of this theorem is completed. □

Theorem 3.11 *Let $\left(\overline{x}, \overline{\xi}, \overline{\nu}, \overline{\mu}\right) \in \Omega_E \times R^q \times R^p \times R^s$ be a Karush-Kuhn-Tucker point of the constrained E-vector optimization problem (VP$_E$). Furthermore, assume that $\overline{\xi}_i f_i(E(\cdot))$, $i \in I$, is strictly α_i-E-pseudo-invex with respect to η at $\overline{x}$ on Ω_E, $\overline{\nu}_j g_j(E(\cdot))$, $j \in J\,(E\,(\overline{x}))$, is β_j-E-quasi-invex with respect to η at $\overline{x}$ on Ω_E, $\overline{\mu}_k h_k(E(\cdot))$, $k = 1, ..., s$, is γ_k-E-quasi-invex with respect to η at $\overline{x}$ on Ω_E. Then $\overline{x}$ is a Pareto solution of the problem (VP$_E$) and, thus, $E\,(\overline{x})$ is an E-Pareto solution of the original multiobjective programming problem (MOP).*

Proof: By assumption, $\left(\overline{x}, \overline{\xi}, \overline{\nu}, \overline{\mu}\right) \in \Omega_E \times R^q \times R^p \times R^s$ is a Karush-Kuhn-Tucker point in the considered constrained E-vector optimization problem (VP$_E$). Then, by Definition 3.9, the Karush-Kuhn-Tucker necessary optimality conditions (3.1)-(3.3) are satisfied at $\overline{x}$ with Lagrange multipliers $\overline{\xi} \in R^q$, $\overline{\nu} \in R^p$ and $\overline{\mu} \in R^s$. We proceed by contradiction. Suppose, contrary to the result, that $\overline{x}$ is not a Pareto solution of problem (VP$_E$). Hence, by Definition 3.4, there exists $E(x') \in \Omega$ such that

$$f_i(E(x')) \leq f_i\,(E\,(\overline{x}))\,, \; i \in I. \tag{3.11}$$

Using $\alpha_i(E(x'), E(\overline{x})) > 0$, $i \in I$, we get

$$\sum_{i=1}^{q} \alpha_i(E(x'), E(\overline{x}))\overline{\xi}_i f_i(E(x')) \leqq \sum_{i=1}^{q} \alpha_i(E(x'), E(\overline{x}))\overline{\xi}_i f_i\,(E\,(\overline{x}))\,. \tag{3.12}$$

Since $\overline{\xi} f(E(\cdot))$ is strictly V-E-pseudo-invex with respect to η at $\overline{x}$ on Ω_E, by Definition 2.6, the inequality

$$\sum_{i=1}^{q} \overline{\xi}_i \nabla f_i(E\,(\overline{x}))\eta\,(E\,(x')\,, E\,(\overline{x})) < 0 \tag{3.13}$$

holds. Since $\overline{\nu}_j g_j(E(\cdot))$, $j \in J(E(\overline{x}))$, $\overline{\mu}_k h_k(E(\cdot))$, $k = 1, ..., s$, are V-E-quasi-invex with respect to η at $\overline{x}$ on Ω_E, by Definition 2.8, the inequalities

$$\sum_{j=1}^{p} \overline{\nu}_j \nabla g_j(E(\overline{x}))\, \eta(E(x'), E(\overline{x})) \leqq 0, \tag{3.14}$$

$$\sum_{k=1}^{s} \overline{\mu}_k \nabla h_k(E(\overline{x}))\, \eta(E(x'), E(\overline{x})) \leqq 0 \tag{3.15}$$

hold, respectively. Combining (3.13), (3.14) and (3.15), we get that the following inequality

$$\left[\sum_{i=1}^{q} \overline{\xi}_i \nabla f_i(E(\overline{x})) + \sum_{j=1}^{p} \overline{\nu}_j \nabla g_j(E(\overline{x})) + \sum_{k=1}^{s} \overline{\mu}_k \nabla h_k(E(\overline{x}))\right] \eta(E(x'), E(\overline{x})) < 0$$

holds, which is a contradiction to the E-Karush-Kuhn-Tucker necessary optimality condition (3.1). By assumption, $E : R^n \to R^n$ is an one-to-one and onto operator. Since $\overline{x}$ is a Pareto solution of the problem (VP$_E$), by Lemma 3.7, $E(\overline{x})$ is an E-Pareto solution of the problem (MOP). Thus, the proof of this theorem is completed. □

Example 3.12 *Consider the following nondifferentiable vector optimization problem*

$$\begin{aligned} f(x) = (f_1(x), f_2(x)) = \left(\sqrt[3]{x_1} e^{-\sqrt[3]{x_1}},\ \sqrt[3]{x_2} e^{-\sqrt[3]{8x_2}}\right) &\to V\text{-}\min \\ s.t.\ g_1(x) = 1 - e^{-\sqrt[3]{8x_1}} &\leqq 0, \qquad \textit{(MOP1)} \\ g_2(x) = 1 - e^{-\sqrt[3]{8x_2}} &\leqq 0. \end{aligned}$$

Note that the set of all feasible solutions of the considered vector optimization problem (MOP1) is $\Omega = \{(x_1, x_2) \in R^2 : 1 - e^{-\sqrt[3]{8x_1}} \leqq 0 \wedge 1 - e^{-\sqrt[3]{8x_2}} \leqq 0\}$. *Further, note that the functions constituting (MOP1) are nondifferentiable at* $(0, 0)$. *Let* $E : R^2 \to R^2$ *be an one-to-one and onto mapping defined as follows* $E(x_1, x_2) = (x_1^3, x_2^3)$. *Now, for the considered multiobjective programming problem (MOP1), we define its associated constrained* E*-vector optimization problem (VP$_E$1) as follows*

$$\begin{aligned} f(E(x)) = (f_1(E(x)), f_2(E(x))) = \left(x_1 e^{-x_1},\ x_2 e^{-2x_2}\right) &\to V\text{-}\min \\ s.t.\ g_1(E(x)) = 1 - e^{-2x_1} &\leqq 0, \qquad \textit{(VP}_E\textit{1)} \\ g_2(E(x)) = 1 - e^{-2x_2} &\leqq 0. \end{aligned}$$

Note that the set of all feasible solutions of the considered E*-vector optimization problem (VP$_E$1) is* $\Omega_E = \{(x_1, x_2) \in R^2 : x_1 \geqq 0 \ \wedge \ x_2 \geqq 0\}$ *and* $\overline{x} = (0, 0)$ *is a feasible solution of the problem (VP$_E$1). Let* η *be defined by* $\eta(E(x), E(u)) = (-x_1 e^{x_1}, -x_2 - 1)$ *and, moreover,*

$\alpha_1(E(x),E(u)) = \begin{cases} e^{x_1+u_1} & \text{if } x \leqq u, \\ 0 & \text{if } x > u. \end{cases}$, $\alpha_2(E(x),E(u)) = \begin{cases} e^{2x_2+2u_2} & \text{if } x \leqq u, \\ 0 & \text{if } x > u. \end{cases}$,

$\beta_1(E(x),E(u)) = \begin{cases} e^{-2x_1} & \text{if } x \geqq u, \\ 0 & \text{if } x < u. \end{cases}$, $\beta_2(E(x),E(u)) = \begin{cases} e^{-2x_2} & \text{if } x \geqq u, \\ 0 & \text{if } x < u. \end{cases}$.

Further, note that all functions constituting the considered vector optimization problem (MOP1) are E-differentiable at $(0,0)$. *Then, by Definition 2.5, it can be shown that the objective function* f *is V-E-pseudo-invex at* $\overline{x}$ *on* Ω_E. *Moreover, by Definition 2.8, it can be shown that the constraint function* g_1, g_2 *are V-E-quasi-invex at* $\overline{x}$ *on* Ω_E. *Thus, all hypotheses of Theorem 3.10 are fulfilled and, therefore, we conclude that* $\overline{x} = (0,0)$ *is a weak Pareto solution of the E-vector optimization problem* ($VP_E 1$) *and, thus,* $E(\overline{x})$ *is a weak E-Pareto solution of the considered multiobjective programming problem (MOP1). Further, note that the functions constituting* ($VP_E 1$) *are not E-invex at* $\overline{x}$ *on* Ω_E *(see, Abdulaleem [1]). Therefore, it is not possible to prove that* $\overline{x}$ *is a weak E-Pareto solution of (MOP1) using the sufficient conditions under E-invexity hypotheses [1].*

4 Mond-Weir E-duality

In this section, for the considered E-differentiable vector optimization problem (MOP), we give the definition of its Mond-Weir vector E-dual problem (MWVD$_E$). Then, we prove several E-duality results between vector optimization problems (MOP) and (MWVD$_E$) under appropriate generalized V-E-invexity hypotheses. Let $E : R^n \to R^n$ be a given one-to-one and onto operator. We define the following vector dual problem in the sense of Mond-Weir related for the differentiable multicriteria E-optimization problem (VP$_E$) as follows:

$$(f \circ E)(y) = (f_1(E(y)), ..., f_q(E(y))) \to V - \max$$
$$\text{s.t. } \sum_{i=1}^{q} \xi_i \nabla (f_i \circ E)(y) + \sum_{j=1}^{p} \nu_j \nabla (g_j \circ E)(y) + \sum_{k=1}^{s} \mu_k \nabla (h_k \circ E)(y) = 0,$$
$$\sum_{j=1}^{p} \nu_j (g_j \circ E)(y) + \sum_{k=1}^{s} \mu_k (h_k \circ E)(y) \geqq 0, \qquad (\text{MWVD}_E)$$
$$\xi \in R^q, \xi \geq 0 \ , \ \nu \in R^p, \nu \geqq 0, \ \mu \in R^s,$$

where all functions are defined in the similar way as for the considered E-vector optimization problem (VP$_E$). Further, let

$$\Gamma_E = \Big\{ (y,\xi,\nu,\mu) \in R^n \times R^q \times R^p \times R^s :$$
$$\sum_{i=1}^{q} \xi_i \nabla (f_i \circ E)(y) + \sum_{j=1}^{p} \nu_j \nabla (g_j \circ E)(y) + \sum_{k=1}^{s} \mu_k \nabla (h_k \circ E)(y) = 0,$$
$$\sum_{j=1}^{p} \nu_j (g_j \circ E)(y) + \sum_{k=1}^{s} \mu_k (h_k \circ E)(y) \geqq 0, \ \xi \geq 0, \ \nu \geqq 0 \Big\}$$

be the set of all feasible solutions of the problem (MWVD$_E$). Let us denote, $Y_E = \{y \in R^n : (y,\xi,\nu,\mu) \in \Gamma_E\}$.

Theorem 4.1 *(Mond-Weir weak duality between (VP$_E$) and (MWVD$_E$)). Let x and (y, ξ, ν, μ) be any feasible solutions of the problems (VP$_E$) and (MWVD$_E$), respectively. Further, assume that the following hypotheses are fulfilled:*

a) each function $\xi_i (f_i \circ E)(\cdot)$, $i = 1, ..., q$, is α_i-E-pseudo-invex with respect to η at y on $\Omega_E \cup Y_E$,

b) $\nu_j (g_j \circ E)(\cdot)$, $j = 1, ..., p$, is β_j-E-quasi-invex with respect to η at y on $\Omega_E \cup Y_E$,

c) $\mu_k (h_k \circ E)(\cdot)$, $k = 1, ..., s$, is γ_k-E-quasi-invex with respect to η at y on $\Omega_E \cup Y_E$.

Then

$$(f \circ E)(x) \nless (f \circ E)(y). \tag{4.1}$$

Proof: Let x and (y, ξ, ν, μ) be any feasible solutions of the problems (VP$_E$) and (WMVD$_E$), respectively. We proceed by contradiction. Suppose, contrary to the result, that the inequality

$$(f \circ E)(x) < (f \circ E)(y) \tag{4.2}$$

holds. By the feasibility (y, ξ, ν, μ) in (WMVD$_E$), the above inequality yields

$$\sum_{i=1}^{q} \xi_i (f_i \circ E)(x) < \sum_{i=1}^{q} \xi_i (f_i \circ E)(y). \tag{4.3}$$

Using $\alpha_i(E(x), E(y)) > 0$, $i \in I$, we obtain that the inequality

$$\sum_{i=1}^{q} \alpha_i(E(x), E(y))\xi_i (f_i \circ E)(x) < \sum_{i=1}^{q} \alpha_i(E(x), E(y))\xi_i (f_i \circ E)(y) \tag{4.4}$$

holds. Since the function $\xi (f \circ E)(\cdot)$ is V-E-pseudo-invex at y on $\Omega_E \cup Y_E$, by (4.4) and Definition 2.5, the inequality

$$\sum_{i=1}^{q} \xi_i \nabla (f_i \circ E)(y) \eta (E(x), E(y)) < 0 \tag{4.5}$$

holds. Since $\nu_j (g_j \circ E)(\cdot)$, $\mu_k (h_k \circ E)(\cdot)$, are V-E-quasi-invex at y on $\Omega_E \cup Y_E$, by Definition 2.8 implies that the inequalities

$$\sum_{j=1}^{p} \nu_j \nabla (g_j \circ E)(y) \eta (E(x), E(y)) \leqq 0, \tag{4.6}$$

$$\sum_{k=1}^{s} \mu_k \nabla (h_k \circ E)(y) \eta (E(x), E(y)) \leqq 0 \tag{4.7}$$

hold, respectively. Combining (4.5), (4.6) and (4.7), we get that the inequality

$$\left[\sum_{i=1}^{q}\xi_i\nabla f_i(E(y))+\sum_{j=1}^{p}\nu_j\nabla g_j(E(y))+\sum_{k=1}^{s}\mu_k\nabla h_k(E(y))\right]\eta(E(x),E(y))<0 \quad (4.8)$$

holds, contradicting the first constraint of the vector Mond-Weir E-dual problem (MWVD$_E$). This means that the proof of the Mond-Weir weak duality theorem between the E-vector optimization problems (VP$_E$) and (MWVD$_E$) is completed. □

Theorem 4.2 *(Mond-Weir weak E-duality between (MOP) and (MWVD$_E$)). Let $E(x)$ and (y,ξ,ν,μ) be a feasible solutions of the problems (MOP) and (MWVD$_E$), respectively. Further, assume that all hypotheses of Theorem 4.1 are fulfilled. Then, Mond-Weir weak E-duality between (MOP) and (MWVD$_E$) holds, that is,*

$$(f\circ E)(x)\not< (f\circ E)(y).$$

Proof: Let $E(x)$ and (y,ξ,ν,μ) be any feasible solutions of the problems (MOP) and (MWVD$_E$), respectively. Then, by Lemma 3.5. it follows that x is any feasible solution of (VP$_E$). Since all hypotheses of Theorem 4.1 are fulfilled, the Mond-Weir weak E-duality theorem between the problems (MOP) and (MWVD$_E$) follows directly from Theorem 4.1. □

If some stronger V-E-invexity hypotheses are imposed on the functions constituting the considered E-differentiable multiobjective programming problem, then the following result is true.

Theorem 4.3 *(Mond-Weir weak duality between (VP$_E$) and (MWVD$_E$)). Let x and (y,ξ,ν,μ) be any feasible solutions of the problems (VP$_E$) and (MWVD$_E$), respectively. Further, assume that the following hypotheses are fulfilled:*

a) each function $\xi_i(f_i\circ E)(\cdot)$, $i=1,...,q$, is strictly α_i-E-pseudo-invex with respect to η at y on $\Omega_E\cup Y_E$,

b) $\nu_j(g_j\circ E)(\cdot)$, $j=1,...,p$, is β_j-E-quasi-invex with respect to η at y on $\Omega_E\cup Y_E$,

c) $\mu_k(h_k\circ E)(\cdot)$, $k=1,...,s$, is γ_k-E-quasi-invex with respect to η at y on $\Omega_E\cup Y_E$.

Then

$$(f\circ E)(x)\nleqslant (f\circ E)(y).$$

Proof: Let x and (y,ξ,ν,μ) be any feasible solutions of the problems (VP$_E$) and (WMVD$_E$), respectively. We proceed by contradiction. Suppose, contrary to the result, that the inequality

$$(f\circ E)(x)\leqslant (f\circ E)(y) \quad (4.9)$$

holds. By the feasibility (y, ξ, ν, μ) in (WMVD_E), the above inequality yields

$$\sum_{i=1}^{q} \xi_i \left(f_i \circ E\right)(x) \leqq \sum_{i=1}^{q} \xi_i \left(f_i \circ E\right)(y). \tag{4.10}$$

Using $\alpha_i(E(x), E(y)) > 0$, $i \in I$, we obtain

$$\sum_{i=1}^{q} \alpha_i(E(x), E(y)) \xi_i \left(f_i \circ E\right)(x) \leqq \sum_{i=1}^{q} \alpha_i(E(x), E(y)) \xi_i \left(f_i \circ E\right)(y) \tag{4.11}$$

holds. Since each function $\xi_i \left(f_i \circ E\right)(\cdot)$, $i = 1, ..., q$, is strictly α_i-E-pseudo-invex at y on $\Omega_E \cup Y_E$, by Definition 2.6 and (4.11), the inequality

$$\sum_{i=1}^{q} \xi_i \nabla \left(f_i \circ E\right)(y) \eta \left(E\left(x\right), E\left(y\right)\right) < 0 \tag{4.12}$$

holds. Since $\nu_j \left(g_j \circ E\right)(\cdot)$, $j = 1, ..., p$, $\mu_k \left(h_k \circ E\right)(\cdot)$, $k = 1, ..., p$, are V-E-quasi-invex at y on $\Omega_E \cup Y_E$, by Definition 2.8 and (4.12) implies that the inequalities

$$\sum_{j=1}^{p} \nu_j \nabla \left(g_j \circ E\right)(y) \eta \left(E\left(x\right), E\left(y\right)\right) \leqq 0, \tag{4.13}$$

$$\sum_{k=1}^{s} \mu_k \nabla \left(h_k \circ E\right)(y) \eta \left(E\left(x\right), E\left(y\right)\right) \leqq 0 \tag{4.14}$$

hold, respectively. Combining (4.12), (4.13) and (4.14), we get that the inequality

$$\left[\sum_{i=1}^{q} \xi_i \nabla f_i(E\left(y\right)) + \sum_{j=1}^{p} \nu_j \nabla g_j(E\left(y\right)) + \sum_{k=1}^{s} \mu_k \nabla h_k(E\left(y\right))\right] \eta \left(E\left(x\right), E\left(y\right)\right) < 0 \tag{4.15}$$

holds, contradicting the first constraint of the vector Mond-Weir E-dual problem (MWVD_E). This means that the proof of the Mond-Weir weak duality theorem between the E-vector optimization problems (VP_E) and (MWVD_E) is completed. □

Theorem 4.4 *(Mond-Weir weak E-duality between (MOP) and ($MWVD_E$)). Let $E\left(x\right)$ and (y, ξ, ν, μ) be any feasible solutions of the problems (MOP) and ($MWVD_E$), respectively. Further, assume that all hypotheses of Theorem 4.3 are fulfilled. Then, weak E-duality between (MOP) and ($VMVD_E$) holds, that is,*

$$\left(f \circ E\right)(x) \nleq \left(f \circ E\right)(y).$$

Proof: Let $E(x)$ and (y, ξ, ν, μ) be any feasible solutions of the problems (MOP) and (MWVD$_E$), respectively. Then, by Lemma 3.5. it follows that x is any feasible solution of (VP$_E$). Since all hypotheses of Theorem 4.3 are fulfilled, the Mond-Weir weak E-duality theorem between the problems (MOP) and (MWVD$_E$) follows directly from Theorem 4.3. □

Theorem 4.5 *(Mond-Weir strong duality between (VP$_E$) and (MWVD$_E$) and also Mond-Weir strong E-duality between (MOP) and (MWVD$_E$)). Let $\overline{x} \in \Omega_E$ be a weak Pareto solution (a Pareto solution) of the E-vector optimization problem (VP$_E$) (and, thus, $E(\overline{x})$ be a weak E-Pareto solution (an E-Pareto solution) of the E-vector optimization problem (MOP)). Further, assume that the E-constraint qualification [1] is satisfied at $\overline{x}$. Then, there exist $\overline{\xi} \in R^q$, $\overline{\nu} \in R^p$, $\overline{\nu} \geqq 0$, $\overline{\mu} \in R^s$ such that $\left(\overline{x}, \overline{\xi}, \overline{\nu}, \overline{\mu}\right)$ is feasible for (MWVD$_E$) and the objective functions of (VP$_E$) and (MWVD$_E$) are equal at these points. If also all hypotheses of the Mond-Weir weak duality (Theorem 4.1 (Theorem 4.3)) are satisfied, then $\left(\overline{x}, \overline{\xi}, \overline{\nu}, \overline{\mu}\right)$ is a weak efficient solution (an efficient solution) of a maximum type in (MWVD$_E$).*
In other words, if $E(\overline{x}) \in \Omega$ is a (weak) E-Pareto solution of the multiobjective programming problem (MOP), then $\left(\overline{x}, \overline{\xi}, \overline{\nu}, \overline{\mu}\right)$ is a (weak) efficient solution of a maximum type in the vector E-dual problem (MWVD$_E$) in the sense of Mond-Weir. This means that the Mond-Weir strong E-duality holds between the problems (MOP) and (MWVD$_E$).

Proof: Since $\overline{x} \in \Omega_E$ is a (weak) Pareto solution of the problem (VP$_E$) and the E-constraint qualification [1] is satisfied at $\overline{x}$, by Theorem 3.8, there exist $\overline{\xi} \in R^q$, $\overline{\nu} \in R^p$, $\overline{\nu} \geqq 0$, $\overline{\mu} \in R^s$ such that the following conditions are satisfied

$$\sum_{i=1}^{q} \overline{\xi}_i \nabla \left(f_i \circ E\right)(\overline{x}) + \sum_{j=1}^{p} \overline{\nu}_j \nabla \left(g_j \circ E\right)(\overline{x}) + \sum_{k=1}^{s} \overline{\mu}_k \nabla \left(h_k \circ E\right)(\overline{x}) = 0,$$

$$\overline{\nu}_j \left(g_j \circ E\right)(\overline{x}) = 0, \;\; j \in J\left(E\left(\overline{x}\right)\right),$$

$$\overline{\xi} \geq 0, \, \overline{\nu} \geqq 0.$$

Thus, $\left(\overline{x}, \overline{\xi}, \overline{\nu}, \overline{\mu}\right)$ is a feasible solution of the problem (MWVD$_E$). This means that the objective functions of (VP$_E$) and (MWVD$_E$) are equal. If we assume that all hypotheses of the Mond-Weir weak duality (Theorem 4.1 (Theorem 4.3)) are fulfilled, $\left(\overline{x}, \overline{\xi}, \overline{\nu}, \overline{\mu}\right)$ is a (weak) efficient solution of a maximum type for the vector E-dual problem (MWVD$_E$) in the sense of Mond-Weir.

Moreover, we have, by Lemma 3.5, that $E(\overline{x}) \in \Omega$. Since $\overline{x} \in \Omega_E$ is a weak Pareto solution of the problem (VP$_E$), by Lemma 3.7, it follows that $E(\overline{x})$ is a weak E-Pareto solution in the problem (MOP). Then, by the Mond-Weir strong duality between (VP$_E$) and (MWVD$_E$), we conclude that also the Mond-Weir strong E-duality holds between the problems (MOP) and (MWVD$_E$). This means that if $E(\overline{x}) \in \Omega$ is a weak E-Pareto solution of the problem (MOP), there exist $\overline{\xi} \in R^q$, $\overline{\nu} \in R^p$, $\overline{\nu} \geqq 0$, $\overline{\mu} \in R^s$ such that

$\left(\overline{x},\overline{\xi},\overline{\nu},\overline{\mu}\right)$ is a weakly efficient solution of a maximum type in the Mond-Weir dual problem (MWVD$_E$). □

Theorem 4.6 *(Mond-Weir converse duality between (VP$_E$) and (MWVD$_E$)). Let $\left(\overline{x},\overline{\xi},\overline{\nu},\overline{\mu}\right)$ be a (weakly) efficient solution of a maximum type in Mond-Weir dual problem (MWVD$_E$) such that $\overline{x} \in \Omega_E$. Moreover, assume that:*

a) each function $\overline{\xi}_i \left(f_i \circ E\right)(\cdot)$, $i = 1, ..., q$, is strictly α_i-E-pseudo-invex with respect to η at $\overline{x}$ on $\Omega_E \cup Y_E$,

b) $\overline{\nu}_j \left(g_j \circ E\right)(\cdot)$, $j = 1, ..., p$, is β_j-E-quasi-invex with respect to η at $\overline{x}$ on $\Omega_E \cup Y_E$,

c) $\overline{\mu}_k \left(h_k \circ E\right)(\cdot)$, $k = 1, ..., s$, is γ_k-E-quasi-invex with respect to η at $\overline{x}$ on $\Omega_E \cup Y_E$.

Then $\overline{x}$ is a (weak) Pareto solution of the problem (VP$_E$).

Proof: Let $\left(\overline{x},\overline{\xi},\overline{\nu},\overline{\mu}\right)$ be a (weakly) efficient solution of a maximum type in Mond-Weir dual problem (MWVD$_E$) such that $\overline{x} \in \Omega_E$. By means of contradiction, we suppose that there exists $x' \in \Omega_E$ such that the inequality

$$\left(f \circ E\right)(x') < \left(f \circ E\right)(\overline{x}) \tag{4.16}$$

holds. By the feasibility of $\left(\overline{x},\overline{\xi},\overline{\nu},\overline{\mu}\right)$ in the problem (MWVD$_E$), the above inequality yields

$$\sum_{i=1}^{q} \overline{\xi}_i f_i(E(x')) < \sum_{i=1}^{q} \overline{\xi}_i f_i(E((\overline{x})). \tag{4.17}$$

Thus,

$$\sum_{i=1}^{q} \alpha_i(E(x'), E(\overline{x}))\overline{\xi}_i f_i(E(x')) < \sum_{i=1}^{q} \alpha_i(E(x'), E(\overline{x}))\overline{\xi}_i f_i\left(E\left(\overline{x}\right)\right). \tag{4.18}$$

Since each function $\overline{\xi}_i \left(f_i \circ E\right)(\cdot)$, $i = 1, ..., q$, is α_i-E-pseudo-invex with respect to η at $\overline{x}$ on $\Omega_E \cup Y_E$, by Definition 2.5 and (4.18), the inequality

$$\left[\sum_{i=1}^{q} \overline{\xi}_i \nabla f_i(E\left(\overline{x}\right))\right] \eta\left(E\left(x'\right), E\left(\overline{x}\right)\right) < 0 \tag{4.19}$$

holds. Since $\overline{\nu}_j \left(g_j \circ E\right)(\cdot)$, $j = 1, ..., p$, $\overline{\mu}_k \left(h_k \circ E\right)(\cdot)$, $k = 1, ..., p$, are V-E-quasi-invex at $\overline{x}$ on $\Omega_E \cup Y_E$, by Definition 2.8 and (4.19) implies that the inequalities

$$\sum_{j=1}^{p} \overline{\nu}_j \nabla\left(g_j \circ E\right)(\overline{x}) \eta\left(E\left(x'\right), E\left(\overline{x}\right)\right) \leqq 0, \tag{4.20}$$

$$\sum_{k=1}^{s} \overline{\mu}_k \nabla\left(h_k \circ E\right)(\overline{x}) \eta\left(E\left(x'\right), E\left(\overline{x}\right)\right) \leqq 0 \tag{4.21}$$

hold, respectively. Combining (4.19), (4.20) and (4.21), we get that the inequality

$$\left[\sum_{i=1}^{q}\overline{\xi}_i\nabla\left(f_i\circ E\right)\left(\overline{x}\right)+\sum_{j=1}^{p}\overline{\nu}_j\nabla g_j\left(E\left(\overline{x}\right)\right)+\sum_{k=1}^{s}\overline{\mu}_k\nabla h_k\left(E\left(\overline{x}\right)\right)\right]\eta\left(E\left(x'\right),E\left(\overline{x}\right)\right)<0$$

holds, which is a contradiction to the first constraint of (MWVD$_E$). This means that the proof of the converse duality theorem between the E-vector optimization problems (VP$_E$) and (MWVD$_E$) is completed. □

Theorem 4.7 *(Mond-Weir converse E-duality between (MOP) and (MWVD$_E$)). Let $\left(\overline{x},\overline{\xi},\overline{\nu},\overline{\mu}\right)$ be a (weakly) efficient solution of a maximum type in Mond-Weir dual problem (MWVD$_E$). Further, assume that all hypotheses of Theorem 4.6 are fulfilled. Then $E(\overline{x})\in\Omega$ is a (weak) E-Pareto solution of the problem (MOP).*

Proof: Let $E(\overline{x})$ and $\left(\overline{x},\overline{\xi},\overline{\nu},\overline{\mu}\right)$ be any feasible solutions of the problems (MOP) and (MWVD$_E$), respectively. Then, by Lemma 3.5. it follows that $\overline{x}$ is any feasible solution of (VP$_E$). Since all hypotheses of Theorem 4.6 are fulfilled, the Mond-Weir converse E-duality between (MOP) and (MWVD$_E$) follows directly from Theorem 4.6. □

5 Concluding remarks

This paper represents E-duality results and optimality conditions for E-differentiable multiobjective programming problems with both inequality and equality constraints. We have established the sufficiency of the so-called E-Karush-Kuhn-Tucker optimality conditions for the considered E-differentiable vector optimization problems with both inequality and equality constraints under generalized V-E-invexity hypotheses. Further, the so-called vector Mond-Weir E-dual problems have been formulated for such E-differentiable multiobjective programming problems. Then, various E-duality theorems between the considered E-differentiable vector optimization problem and its Mond-Weir vector dual problem have been proved under generalized V-E-invexity hypotheses. It is pointed out that our sufficiency of the so-called E-Karush-Kuhn-Tucker optimality conditions and E-duality results of generalized V-E-invexity hypotheses are more general than the classical ones found for instance in [1, 15, 17, 20, 26].

However, some interesting topics for further research remain. It would be of interest to investigate whether it is possible to prove similar results for other classes of E-differentiable vector optimization problems. We shall investigate these questions in subsequent papers.

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