ON GEOMETRY OF CONFORMALLY CO-SYMPLECTIC MANIFOLD

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ABSTRACT. *In this article, we introduced the concept of holomorphic sectional conharmonic tensor of a normal local conformally almost co-symplectic manifold ($\mathcal{NLCACS}$-manifold). In particular, we established some of its properties and analytical expressions. Consequently, analytical conditions for the $\mathcal{NLCACS}$-manifold to be a kind of point wise constancy conharmoniclly holomorphic sectional are obtained.*

Keywords: Normal local conformally almost co-symplectic manifold, conharmoniclly Φ-holomorphic sectional tensor.

1. INTRODUCTION

The concept of local conformally almost co-symplectic ($\mathcal{LCACS}$) structures introduced by Vaisman [1]. In particular, he proved that an almost contact metric manifold M is $\mathcal{LCACS}$ if, and only if, there is a differential 1-form α satisfies certain conditions.

Furthermore, in [2], the mentioned manifold was studied regarding the φ-sectional curvature. Subsequently, it is shown that pointwise constancy φ-sectional curvature condition implies that the η-Einstein and the k-nullity are equivalent. On the other hand, Kharitonova [3], established the full family of the structural equations of a $\mathcal{LCACS}$-structure.

The normal $\mathcal{LCACS}$-structures($\mathcal{NLCACS}$-structures) are studied in [4]. A complete family of structural equations was established. Moreover, the components of the Rieman-Christoffel and the Ricci tensors were determined. Moreover, The conditions for the constant curvature are given. In particular, it is proved that a special form of $\mathcal{NLCACS}$-manifolds has non-positive curvature. Finally, Abood and Al-Hussaini [5], employed the adjoined $\mathbb{G}$-structure space to study the geometry of the mentioned manifold of Φ-holomorphic sectional conharmonic curvature tensor.

2. Preliminaries

This section contains some concepts and facts related to the content of this paper. In particular, the structural equations and the components of the Rieman-Christoffel tensor of the $\mathcal{NLCACS}$-manifolds have been established.

Definition 2.1([6]). Suppose that M is an odd-dimensional manifold, a quadruple (γ,α,Φ,g) of tensors is called an almost contact metric structure ($\mathcal{AC}$-structure) on a manifold M, such that γ is a contact differential form of rank 1, α is a characteristic vector field, Φ is endomorphism tensor of type (1,1) of a module $\mathcal{X}(M)$, and $g=\langle\cdot,\cdot\rangle$ is a Riemannian metric tensor. Moreover, the following conditions hold:

$$(1)\gamma(\alpha)=1;\ (2)\Phi(\alpha)=0;\ (3)\gamma\circ\Phi=0;\ (4)\Phi^2-\mathrm{id}+\gamma\otimes\alpha;\ (5)\langle\Phi X,\Phi Y\rangle=\langle X,Y\rangle-\gamma(X)\gamma(Y), X,Y\in\mathcal{X}(M). \tag{2.1}$$

A manifold M endowed with the above structure is known as almost contact metric manifold ($\mathcal{AC}$-manifold).

It is easy to verify that $\Omega(X,Y)=\langle X,\Phi Y\rangle$ is anti-symmetric, that means is a differential 2-form on M. It is called a fundamental differential form of the $\mathcal{AC}$-structure [7].

Let (γ,α,Φ,g) be an $\mathcal{AC}$-structure on the manifold M^{2n+1}. In the module $\mathcal{X}(M)$ there are two mutually complement projections m,ℓ, where $m=\gamma\otimes\alpha$ and $\ell=-\Phi^2$. Thus $\mathcal{X}(M)=L\oplus\mathcal{M}$, where $L=Im\Phi=ker\gamma$, $dimL=2n$ and $\mathcal{M}=Im\,m=ker\Phi$, $dim\mathcal{M}=1$. L is called a contact distribution of the first kind, while $\mathcal{M}$ is called a distribution of the second type. Obviously, the distributions $\mathcal{M}$ and L are Φ-invariant. Moreover, we have $\tilde{\Phi}^2=-id,\langle\tilde{\Phi}X,\tilde{\Phi}Y\rangle=\langle X,Y\rangle, X,Y\in\mathcal{X}(M)$, where $\tilde{\Phi}=\Phi|_L$. Therefore, in the tangent space $T_m(M)$ at $m\in M$, one can construct an orthonormal frame $(m,e_0,e_1,\dots,e_n,\Phi e_1,\dots,\Phi e_n)$, where $e_0=\alpha_p$. Such frame is called a real adapted [8]. On the other hand, let $L^C=L\otimes C$ be the complexification of the distribution L. Moreover, there are two complement projectors $\rho=\frac{1}{2}\left(id-\sqrt{-1}\Phi\right)$ and $\bar{\rho}=\frac{1}{2}\left(id+\sqrt{-1}\Phi\right)$ which are intrinsically defined on the proper submodules $\mathcal{D}_\Phi^{\sqrt{-1}}$ and $\mathcal{D}_\Phi^{-\sqrt{-1}}$ of the endomorphism Φ corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. That is, the module L^C decomposes into a direct sum of proper submodules $L^C=\mathcal{D}_\Phi^{\sqrt{-1}}\oplus\mathcal{D}_\Phi^{-\sqrt{-1}}$.

Thus, the complexification of the module $\mathcal{X}(M)$ will be written as a direct sum of the proper submodules of the endomorphism Φ, that is $\mathcal{X}^C(M)=\mathcal{D}_\Phi^{\sqrt{-1}}\oplus\mathcal{D}_\Phi^{-\sqrt{-1}}\oplus\mathcal{D}_\Phi^0$, where $\mathcal{D}_\Phi^0=\mathcal{M}^C$. The projections on this direct sum are respectively, endomorphisms $\Pi=-\frac{1}{2}\left(\Phi^2+\sqrt{-1}\Phi\right),\overline{\Pi}=\frac{1}{2}\left(-\Phi^2+\sqrt{-1}\Phi\right)$ and $m=\mathrm{id}+\Phi^2$.

Consequently, we can construct a frame $(m,\sigma_0,\sigma_1,\dots,\sigma_n,\sigma_{\hat{1}},\dots,\sigma_{\hat{n}})$ of the complexification of the space $T_m(M)$, where $\sigma_0=\alpha_p,\sigma_a=\sqrt{2}\rho(e_a),\sigma_{\hat{a}}=\sqrt{2}\bar{\rho}(e_a)$ consisting of the eigen vectors of the operator Φ. Such a frame is called an A-frame [9]. One, easily possible to observe that the components matrices of the tensors Φ_m and g_m in the A-frame are represented as the following formulas:

$$\left(\Phi_j^i\right)=\begin{pmatrix}0&0&0\\0&\sqrt{-1}\mathrm{I_n}&0\\0&0&-\sqrt{-1}\mathrm{I_n}\end{pmatrix},\left(g_{ij}\right)=\begin{pmatrix}1&0&0\\0&0&\mathrm{I_n}\\0&\mathrm{I_n}&0\end{pmatrix}, \tag{2.2}$$

It is well known [8], that the collection of previous frames gives a $\mathbb{G}$-structural on M with the structural group $\{1\}\times U(n)$, formulated by the matrix $\begin{pmatrix}1&0&0\\0&\mathrm{A}&0\\0&0&\bar{\mathrm{A}}\end{pmatrix}$, where $A\in U(n)$. The mentioned $\mathbb{G}$-structural is called an associated.

Throughout this paper, we will assume that the range of the indexes $i,j,k,\dots$ are from 0 to 2n, while the range of the indexes $a,b,c,d,f,\dots$ are from 1 to n, further set $\hat{a}=a+n,\hat{\hat{a}}=a,\hat{0}=0$.

The structural endomorphism Φ and the metric structure g are respectively tensors of type (1,1) and (2,0) on the manifold M^{2n+1}. Therefore, regarding the basic theorem of tensor analysis and the covariant of the metric tensor of the Riemannian connection, their components as a family of functions on the space of the principal bundle of frames $\mathcal{B}(M^{2n+1})$, satisfy the relations:

1) $d\Phi^i_j - \Phi^i_k\omega^k_j + \Phi^k_j\omega^i_k = \Phi^i_{j,k}\omega^k$; 2) $dg_{ij} - g_{ik}\omega^k_j - g_{kj}\omega^k_i = 0$, (2.3)

where $\{\omega^i\}, \{\omega^i_j\}$ are the components of the displacement forms and the Riemannian connection ∇ respectively, and $\Phi^i_{j,k}$ are the components of the endomorphism tensor Φ in this connection.

Taking (2.2) into account, so the relations (2.3) on the associated $\mathbb{G}$-structural space can be rewritten in the forms [8]

$$\Phi^a_{b,k} = \Phi^{\hat{a}}_{\hat{b},k} = \Phi^0_{0,k} = 0;\ \omega^a_{\hat{b}} = \frac{\sqrt{-1}}{2}\Phi^a_{\hat{b},k}\omega^k;$$
$$\omega^{\hat{a}}_b = -\frac{\sqrt{-1}}{2}\Phi^{\hat{a}}_{b,k};\qquad \omega^a_0 = \sqrt{-1}\Phi^a_{0,k}\omega^k;$$
$$\omega^{\hat{a}}_0 = -\sqrt{-1}\Phi^{\hat{a}}_{0,k}\omega^k;\qquad \omega^0_a = -\sqrt{-1}\Phi^0_{a,k}\omega^k;$$
$$\omega^0_{\hat{a}} = \sqrt{-1}\Phi^0_{\hat{a},k}\omega^k;\qquad \omega^i_j + \omega^{\hat{j}}_{\hat{i}} = 0,\quad \omega^0_0 = 0. \quad (2.4)$$

In addition, we note that, because of the fact that the corresponding forms and tensors are real, $\overline{\omega^i} = \omega^{\hat{i}}, \overline{\omega^i_j} = \omega^{\hat{i}}_{\hat{j}}, \overline{\nabla\Phi^i_{j,k}} = \nabla\Phi^{\hat{i}}_{\hat{j},\hat{k}}$, where $t \to \bar{t}$ is the complex conjugation operator. And also, from (2.4) it follows that the system of functions $\{\Phi^i_{j,k}\}$ is anti-symmetric regarding the indexes i and j, that is $\Phi^i_{j,k} = \Phi^{\hat{j}}_{\hat{i},k}$.

Consequently, according to these relations, the first family of structural equations of the Riemannian connection $d\omega^i = -\omega^i_j \wedge \omega^j$ on the associated $\mathbb{G}$-structural space of an $\mathcal{AC}$-manifold can be written in the following form, called the first family of the structural equations of $\mathcal{AC}$-manifold [9]:

$$1)\ d\omega^a = -\omega^a_b\wedge\omega^b + \beta^{ab}{}_c\omega^c\wedge\omega_b + \beta^{abc}\omega_b\wedge\omega_c + \beta^a{}_b\omega\wedge\omega^b + \beta^{ab}\wedge\omega_b,$$
$$2)\ d\omega_a = \omega^b_a\wedge\omega_b + \beta_{ab}{}^c\omega_c\wedge\omega^b + \beta_{abc}\omega^b\wedge\omega^c + \beta_a{}^b\omega\wedge\omega_b + \beta_{ab}\wedge\omega^b,$$
$$3)\ d\omega = C_{bc}\omega^b\wedge\omega^c + C^{bc}\omega_b\wedge\omega_c + C^b_c\omega^c\wedge\omega_b + C_b\omega\wedge\omega^b + C^b\omega\wedge\omega_b, \quad (2.5)$$

where $\omega = \omega^0 = \omega_0 = \Pi^*(\eta)$, Π is a projection of the associated $\mathbb{G}$-structural space onto the manifold M,

$$\beta_{abc} = -\frac{\sqrt{-1}}{2}\Phi^{\hat{a}}_{[b,c]};\ \beta_{ab}{}^c = \frac{\sqrt{-1}}{2}\Phi^{\hat{a}}_{b,\hat{c}};$$
$$\beta^{abc} = \frac{\sqrt{-1}}{2}\Phi^a_{[\hat{b},\hat{c}]};\quad \beta^{ab}{}_c = -\frac{\sqrt{-1}}{2}\Phi^a_{\hat{b},c};$$
$$\beta^a{}_b = \sqrt{-1}\Phi^a_{0,b};\quad C^{ab} = \sqrt{-1}\Phi 0^0_{[\hat{a},\hat{b}]};$$
$$\beta_a{}^b = -\sqrt{-1}\Phi^{\hat{a}}_{0,\hat{b}};\ C_{ab} = -\sqrt{-1}\Phi^0_{[a,b]};$$
$$C_a = \sqrt{-1}\Phi^0_{a,0};\quad C^a = -\sqrt{-1}\Phi^0_{\hat{a},0};$$
$$C^a_b = -\sqrt{-1}\left(\Phi^0_{b,\hat{a}} + \Phi^0_{\hat{a},b}\right) = \beta^a{}_b - \beta_b{}^a;$$
$$\beta_{ab} = \sqrt{-1}\left(\frac{1}{2}\Phi^{\hat{a}}_{b,0} - \Phi^{\hat{a}}_{0,b}\right);$$
$$\beta^{ab} = -\sqrt{-1}\left(\frac{1}{2}\Phi^a_{\hat{b},0} - \Phi^a_{0,\hat{b}}\right). \quad (2.6)$$

Taking these relations into account and the anti-symmetry of the system of functions $\{\Phi^i_{j,k}\}$, we note that

$$\beta^{abc} = -\beta^{acb};\ \beta^{ab}{}_c = -\beta^{ba}{}_c;\ C^{ab} = -C^{ba};$$
$$\beta_{abc} = -\beta_{acb};\ \beta_{ab}{}^c = -\beta_{ba}{}^c;\ C_{ab} = -C_{ba};$$
$$\overline{\beta^{abc}} = \beta_{abc};\ \overline{\beta^{ab}{}_c} = \beta_{ab}{}^c;\ \overline{C^{ab}} = C_{ab};$$
$$\overline{\beta^{ab}} = \beta_{ab};\ \overline{C^a} = C_a;\ \overline{\omega^a_b} = -\omega^b_a. \quad (2.7)$$

The Nijenhuis tensor of an endomorphism Φ is a tensor $\mathcal{N}_\Phi$ of type (2.1), given by

$$\mathcal{N}_\Phi(X,Y) = \frac{1}{4}(\Phi^2[X,Y] + [\Phi X,\Phi Y] - \Phi[\Phi X,Y] - \Phi[X,\Phi Y]),\ X,Y \in \mathcal{X}(M).$$

Its vanishing is equivalent to the integrability of the structure [10]. A direct calculation, taking into account the identity $[X,Y] = \nabla_X Y - \nabla_Y X$, shows that [9]

$$\mathcal{N}_\Phi(X,Y) = \frac{1}{4}\{\nabla_{\Phi X}(\Phi)Y - \Phi\nabla_X(\Phi)Y - \nabla_{\Phi Y}(\Phi)X + \Phi\nabla_Y(\Phi)X\}.$$

Taking (2.4) into account, we obtain that, the components of the tensor N_Φ are determined by identities below:

$$1)\ \mathcal{N}^0_{ab} = -\frac{\sqrt{-1}}{2}\Phi^0_{[a,b]};$$
$$2)\ \mathcal{N}^0_{\hat{a}\hat{b}} = \frac{\sqrt{-1}}{2}\Phi^0_{[\hat{a},\hat{b}]};$$
$$3)\ \mathcal{N}^a_{\hat{b}\hat{c}} = \sqrt{-1}\Phi^a_{[\hat{b},\hat{c}]};$$
$$4)\ \mathcal{N}^{\hat{a}}_{bc} = -\sqrt{-1}\Phi^{\hat{a}}_{[b,c]};$$
$$5)\ \mathcal{N}^0_{\hat{a}b} = -\mathcal{N}^0_{b\hat{a}} = -\frac{\sqrt{-1}}{2}\Phi^0_{(\hat{a},b)};$$
$$6)\ \mathcal{N}^a_{\hat{b}0} = -\mathcal{N}^0_{0\hat{b}} = \frac{\sqrt{-1}}{4}\Phi^a_{\hat{b},0} - \frac{\sqrt{-1}}{2}\Phi^a_{0,\hat{b}}. \quad (2.8)$$

The remaining components of the Nijenhuis tensor are identically vanishes.

Definition 2.2[14]. An $\mathcal{AC}$-structure is said to be a normal if, $\mathcal{N}_\Phi + 2d\gamma\otimes\alpha = 0$.

The notion of a normality was introduced by Sasaki and Hatakeuyama in 1961 [14] which is one of the most significant concepts of contact geometry.

The following proposition immediately follows from (2.8).

Proposition 2.1[8]An $\mathcal{AC}$-structure is normal if ,and only if, the equalities below hold,

$$\Phi^{\hat{a}}_{b,c} = \Phi^a_{\hat{b},\hat{c}} = \Phi^{\hat{a}}_{b,0} = \Phi^a_{\hat{b},0} = \Phi^0_{a,b} = \Phi^0_{\hat{a},\hat{b}} = \Phi^0_{a,0} = \Phi^0_{\hat{a},0} = 0.$$

the first family of the structural equations of the normal structure are given by the formulas below:

$$d\omega^a = -\omega^a_b\wedge\omega^b + \beta^{ab}{}_c\omega^c\wedge\omega_b + \beta^a{}_b\omega\wedge\omega^b;$$
$$d\omega_a = \omega^b_a\wedge\omega_b + \beta_{ab}{}^c\omega_c\wedge\omega^b + \beta_a{}^b\omega\wedge\omega_b;$$
$$d\omega = C^b_c\omega^c\wedge\omega_b.$$

Definition 2.3 [12]. An $\mathcal{AC}$-structure $S = (\gamma,\alpha,\Phi,g)$ is said to be an almost co-symplectic structure ($\mathcal{ACS}$-structure), if the conditions below hold:

1) $d\gamma = 0$; 2) $d\Omega = 0$.

Proposition 2.2 [8]. Let Ω be the fundamental form of an AC-structure. Then the following equalities hold:

1) $\Pi^*\Omega = -\sqrt{-1}\omega^a\wedge\omega_a$;

2) $\Pi^* d\Omega = \sqrt{-1}\{\beta^{ab}{}_c\omega_a\wedge\omega_b\wedge\omega^c - \beta^{abc}\omega_a\wedge\omega_b\wedge\omega_c + \beta^a{}_b\omega_a\wedge\omega^b\wedge\omega + \beta^{ab}\omega_a\wedge\omega_b\wedge\omega - \beta_{ab}{}^c\omega^a\wedge\omega^b\wedge\omega_c + \beta_{abc}\omega^a\wedge\omega^a\wedge\omega^c - \beta_a{}^b\omega^a\wedge\omega_b\wedge\omega + \beta_{ab}\omega^a\wedge\omega^b\wedge\omega\}$.

Writing the conditions of the Definition 2.3 and taking Proposition 2.2 into account, the first family of the equations of the $\mathcal{ACS}$-structure takes the formula:

$$1)\ d\omega = 0;\ 2)\ d\omega^a = -\theta^a_b\wedge\omega^b + \beta^{abc}\omega_b\wedge\omega_c + F^{ab}\omega_b\wedge\omega;\ 3)\ d\omega_a = \theta^b_a\wedge\omega_b + \beta_{abc}\omega^b\wedge\omega^c + F_{ab}\omega^b\wedge\omega, \quad (2.9)$$

where,

$$\beta^{abc} = \frac{\sqrt{-1}}{2}\Phi^a_{[\hat{b},\hat{c}]};\quad \beta_{abc} = -\frac{\sqrt{-1}}{2}\Phi^{\hat{a}}_{[b,c]};$$
$$F^{ab} = \sqrt{-1}\Phi^0_{\hat{a},\hat{b}};\quad F_{ab} = -\sqrt{-1}\Phi^0_{a,b};$$
$$\beta^{[abc]} = \beta_{[abc]} = 0;\ \overline{\beta^{abc}} = \beta_{abc};$$
$$F^{[ab]} = F_{[ab]} = 0;\ \overline{F^{ab}} = F_{ab}. \quad (2.10)$$

Definition 2.4 [7]. A normal almost co-symplectic structure is called a co-symplectic structure.

The first family of the equations of the co-symplectic structure on the associated $\mathbb{G}$-structural space takes the form

1) $d\omega = 0$; 2) $d\omega^a = -\theta^a_b \wedge \omega^b$; 3) $d\omega_a = \theta^b_a \wedge \omega_b$. (2.11)

A conformally transformation of an $\mathcal{AC}$-structure $S = (\gamma, \alpha, \Phi, g)$ on a manifold M is a transition from S to an $\mathcal{AC}$-structure $\tilde{S} = (\tilde{\gamma}, \tilde{\alpha}, \tilde{\Phi}, \tilde{g})$. In this case $\tilde{\gamma} = e^{-\sigma}\gamma, \tilde{\alpha} = e^{\sigma}\alpha, \tilde{\Phi} = \Phi, \tilde{g} = e^{-2\sigma}g$, where σ is an arbitrary smooth function on M, called a defining function [13]. If $\sigma = const$ant, so a conformally transformation is called a trivial.

Definition 2.5 [13]. The $\mathcal{AC}$-structure $S = (\gamma, \alpha, \Phi, g)$ on M is called a local conformally almost co-symplectic ($\mathcal{LCACS}$) structure, if its restriction to some neighborhood U of a point $m \in M$ admits a conformal transformation into an $\mathcal{ACS}$-structure. We call this transformation a local conformally. A manifold M equipped with a $\mathcal{LCACS}$-structure is called a $\mathcal{LCACS}$-manifold.

Note that if $\sigma = constant$, then we obtain an $\mathcal{ACS}$-manifold.

Suppose that M is a normal $\mathcal{LCACS}$-manifold. Recall [4] that for normal $\mathcal{LCACS}$-manifolds, the non-zero components of the Rieman-Christoffel tensor $\mathcal{R}$ have the forms

1) $\mathcal{R}^a_{bc\hat{d}} = A^{ad}_{bc} - \delta^a_c\delta^d_b\sigma^2_0$; 2) $\mathcal{R}^a_{\hat{b}cd}$; 3) $\mathcal{R}^a_{0b0} = -(\sigma_{00} + \sigma^2_0)\delta^a_b$, (2.12)

where $\delta^{ab}_{cd} = \delta^a_c\delta^b_d - \delta^b_c\delta^a_d$. The remaining components obtained from those above which are given with allowance for the symmetrical properties and reality of the *Rieman-Christoffel* tensor.

The non-zero components of the Ricci tensor are given by the following relations [4]:

$r_{00} = -2n(\sigma_{00} + \sigma^2_0)$;

$r_{\hat{a}b} = A^{ac}_{bc} - 2n\delta^a_b\sigma^2_0 - \delta^a_b\sigma_{00}$. (2.13)

3. $\mathcal{NLCACS}$-Manifolds of Conharmonically Holomorphic Sectional Curvature

Let M^{2n+1} be an $\mathcal{AC}$-manifold. One of the subclasses of the conformal transformations is the conharmonic transformations, these transformations preserve the harmonicity of functions.

A conharmonic tensor [14] is invariant under conharmonic transformations which has the form

$C(X,Y,Z,W) = \mathcal{R}(X,Y,Z,W) - \frac{1}{2n-1}\{g(X,W)r(Y,Z) - g(X,Z)r(Y,W) + g(Y,Z)r(X,W) - g(Y,W)r(X,Z)\}$, $X, Y, Z \in \mathcal{X}(M)$,

where $\mathcal{R}, r$ and g are respectively denote the Rieman-Christoffel tensor, Ricci tensor and Riemannian metric.

The conharmonic curvature tensor of an $\mathcal{AC}$-structure on the space of all complex frames is calculated by the formula below [14]

$C^i_{jkl} = \mathcal{R}^i_{jkl} + \frac{1}{2n-1}\left(\delta^i_k r_{jl} - \delta^i_l r_{jk} + g_{jl}r^i_k - g_{jk}r^i_l\right)$, (3.1)

where $\mathcal{R}^i_{jkl}, r_{ij}$ and g_{ij} are respectively denote the components of the Rieman-Christoffel tensor, Ricci tensor and metric tensor.

It is easy to show from (3.1), that the conharmonic tensor possesses all the symmetrical properties of the Rieman-Christoffel tensor.

Definition 3.1. An $\mathcal{AC}$-manifold M is called a kind of point wise constant conharmonicly holomorphic sectional ($\mathcal{PCHS}$)-curvature c, if

$\langle C(X,\Phi X)X, \Phi X\rangle = c\|X\|^4, \forall X \in L$. (3.2)

If, in addition, $c = constant$, then the manifold is called a manifold of global conharmonicly holomorphic sectional ($\mathcal{CHS}$)-curvature.

Let M be an $\mathcal{AC}$-manifold of $\mathcal{PCHS}$-curvature c. The relation (3.2) regarding the associated $\mathbb{G}$-structural is written as the form $4C_{\hat{a}bc\hat{d}}X^{\hat{a}}X^bX^cX^{\hat{d}} = -4cg_{\hat{a}b}g_{c\hat{d}}X^{\hat{a}}X^bX^cX^{\hat{d}}$. Taking (2.2) into account, the previous relation can be rewritten as $(C_{\hat{a}bc\hat{d}} + c\delta^a_b\delta^d_c)X^{\hat{a}}X^bX^cX^{\hat{d}} = 0$. The polarization of this relation leads us to the following theorem.

Theorem 3.1. An $\mathcal{AC}$-manifold is a one of $\mathcal{PCHS}$-curvature c if, and only if, on the associated $\mathbb{G}$-structural space, we have

$C^{(ad)}_{(bc)} = -\frac{c}{2}\tilde{\delta}^{ad}_{bc}$, (3.3)

where $\tilde{\delta}^{ad}_{bc} = \delta^a_b\delta^d_c + \delta^d_b\delta^a_c$.

Suppose, in particular, that M is an $\mathcal{NLCACS}$-manifold. Regarding the associated $\mathbb{G}$-structural space, the conharmonic curvature tensor of a mentioned manifold has the following non-zero components [15]:

1) $C^a_{0b0} = \mathcal{R}^a_{0b0} - \frac{1}{2n-1}(\delta^a_b r_{00} + g_{00}r^a_b) = \frac{1}{2n-1}\left[2\delta^a_b\left\{\left(n + \frac{1}{2}\right)\sigma^2_0 + \sigma_{00}\right\} - A^{ac}_{bc}\right]$;

2) $C^a_{bc\hat{d}} = \mathcal{R}^a_{bc\hat{d}} - \frac{1}{2n-1}(\delta^a_c r_{b\hat{d}} + g_{b\hat{d}}r^a_c) = \frac{1}{2n-1}\left[2\delta^a_c\delta^d_b\left\{\left(n + \frac{1}{2}\right)\sigma^2_0 + \sigma_{00}\right\} + A^{ad}_{bc}(2n-1) - \delta^d_b A^{ah}_{ch} - \delta^a_c A^{ah}_{bh}\right]$;

3) $C^a_{\hat{b}cd} = \mathcal{R}^a_{\hat{b}cd} - \frac{1}{2n-1}(\delta^a_c r_{\hat{b}d} - \delta^a_d r_{\hat{b}c} + g_{\hat{b}d}r^a_c - g_{\hat{b}c}r^a_d) = \frac{1}{2n-1}\left[2\delta^{ab}_{cd}\left\{\left(n + \frac{1}{2}\right)\sigma^2_0 + \sigma_{00}\right\} + \delta^b_c A^{ah}_{dh} + \delta^a_d A^{bh}_{ch} - \delta^b_d A^{ah}_{ch} - \delta^a_c A^{bh}_{dh}\right]$, (3.4)

The remaining components obtained with allowance for the realness and symmetrical properties of this tensor as an algebraic curvature tensor, or are identical equal to zero.

Definition 3.2. An $\mathcal{NLCACS}$-manifold of a $\mathcal{PCHS}$-curvature is called an $\mathcal{NLCACS}$-conharmonic space form.

Theorem 3.2. An $\mathcal{NLCACS}$-manifold is a one of $\mathcal{PCHS}$-curvature c if, and only if, on the associated $\mathbb{G}$-structural space, the following equality holds:

$$\tilde{A}^{ad}_{bc} = -\frac{c}{2}\tilde{\delta}^{ad}_{bc}$$

where $\tilde{\delta}^{ad}_{bc} = \delta^a_b\delta^d_c + \delta^d_b\delta^a_c$.

Proof. Suppose that M is a $\mathcal{NLCACS}$-manifold of $\mathcal{PCHS}$-curvature c. So taking into account (3.3) and (3.4:2), we have $C^{(ad)}_{(bc)} = A^{(ad)}_{(bc)} - \delta^{(a}_{(c}\delta^{d)}_{b)}\sigma^2_0 - \frac{1}{2n-1}\left[\left(A^{(d|h|}_{(b|h|} - 2n\delta^{(d}_{(b}\sigma^2_0 - \delta^{(d}_{(b}\sigma_{00}\right)\delta^{a)}_{c)} + \left(A^{(a|h|}_{(c|h|} - 2n\delta^{(a}_{(c}\sigma^2_0 - \delta^{(a}_{(c}\sigma_{00}\right)\delta^{d)}_{b)}\right] = -\frac{c}{2}\tilde{\delta}^{ad}_{bc}$, that means $A^{ad}_{bc} - \frac{1}{2}\tilde{\delta}^{ad}_{bc}\sigma^2_0 - \frac{2}{(2n-1)}\left(A^{(d|h|}_{(b|h|}\delta^{a)}_{c)} - (n\sigma^2_0 + \sigma_{00})\tilde{\delta}^{ad}_{bc}\right) = -\frac{c}{2}\tilde{\delta}^{ad}_{bc}$. (3.5)

We introduce a pure tensor $\tilde{A}$ of type $\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$ with components $\tilde{A}^{ad}_{bc} = A^{ad}_{bc} - \frac{1}{2}\tilde{\delta}^{ad}_{bc}\sigma^2_0 - \frac{2}{(2n-1)}\left(A^{(d|h|}_{(b|h|}\delta^{a)}_{c)} - (n\sigma^2_0 + \sigma_{00})\tilde{\delta}^{ad}_{bc}\right)$, (3.6)

symmetrizing over any pair of superscripts. Then (3.6) is written in the form

$\tilde{A}^{ad}_{bc} = -\frac{c}{2}\tilde{\delta}^{ad}_{bc}$. (3.7)

Consider a 4-form $H(X,Y,Z,W) = \left(\tilde{A}^{ad}_{bc} + \frac{c}{2}\tilde{\delta}^{ad}_{bc}\right) X^bY^cZ_aW_d = \left\{A^{ad}_{bc} - \frac{1}{2}\tilde{\delta}^{ad}_{bc}\sigma_0^2 - \frac{2}{(2n-1)}\left(A^{(d|h|}_{(b|h|}\delta^{a)}_{c)} - (n\sigma_0^2+\sigma_{00})\tilde{\delta}^{ad}_{bc}\right) + \frac{c}{2}\tilde{\delta}^{ad}_{bc}\right\} X^bY^cZ_aW_d;\ \forall X,Y,Z,W \in L.$

The form $H(X,Y,Z,W)$ has the properties listed below

1) $H(X_1+X_2,Y,Z,W) = \left\{A^{ad}_{bc} - \frac{1}{2}\tilde{\delta}^{ad}_{bc}\sigma_0^2 - \frac{2}{(2n-1)}\left(A^{(d|h|}_{(b|h|}\delta^{a)}_{c)} - (n\sigma_0^2+\sigma_{00})\tilde{\delta}^{ad}_{bc}\right) + \frac{c}{2}\tilde{\delta}^{ad}_{bc}\right\}(X_1+X_2)^bY^cZ_aW_d = \left\{A^{ad}_{bc} - \frac{1}{2}\tilde{\delta}^{ad}_{bc}\sigma_0^2 - \frac{2}{(2n-1)}\left(A^{(d|h|}_{(b|h|}\delta^{a)}_{c)} - (n\sigma_0^2+\sigma_{00})\tilde{\delta}^{ad}_{bc}\right) + \frac{c}{2}\tilde{\delta}^{ad}_{bc}\right\}{X_1}^bY^cZ_aW_d + \left\{A^{ad}_{bc} - \frac{1}{2}\tilde{\delta}^{ad}_{bc}\sigma_0^2 - \frac{2}{(2n-1)}\left(A^{(d|h|}_{(b|h|}\delta^{a)}_{c)} - (n\sigma_0^2+\sigma_{00})\tilde{\delta}^{ad}_{bc}\right) + \frac{c}{2}\tilde{\delta}^{ad}_{bc}\right\}{X_2}^bY^cZ_aW_d = H(X_1,Y,Z,W) + H(X_2,Y,Z,W);$

2) $H(X,Y,Z,W) = H(Y,X,Z,W) = H(X,Y,W,Z);$

3) $H(X,Y,Z_1+Z_2,W) = H(X,Y,Z_1,W) + H(X,Y,Z_2,W);$

4) $H(\Phi X,Y,Z,W) = \sqrt{-1}H(X,Y,Z,W);$

5) $H(X,Y,\Phi Z,W) = -\sqrt{-1}H(X,Y,Z,W);$ $\forall X,Y,Z,W \in L.$

Based on these properties, we prove that $H(X,Y,Z,W) = 0$. By (3.7), we have that $H(X,X,X,X) = 0$. Make a replacement $X \to X+Y$, then we get $H(X+Y,X+Y,X+Y,X+Y) = 0$, which means that

$$2H(X,X,X,Y) + H(X,X,Y,Y) + 2H(X,Y,X,X) + 4H(X,Y,X,Y) + 2H(X,Y,Y,Y) + H(Y,Y,X,X) + 2H(Y,Y,X,Y) = 0. \quad (3.8)$$

Moreover, one can make in (3.8) the changing $X \to -X$ and then adding the result obtained by term with (3.8), we obtain $H(X,X,Y,Y) + 4H(X,Y,X,Y) + H(Y,Y,X,X) = 0$. (3.9) In the last equality, we make the change $X \to \Phi X$, the result is added term by term with (3.9), then we get $H(X,Y,X,Y) = 0$. Given this equality, the equality (3.9) takes the form $H(X,X,Y,Y) + H(Y,Y,X,X) = 0$, where we made the replacement $X \to X+Y$. As a result, we get

$$H(X,Z,Y,Y) + H(Y,Y,X,Z) = 0. \quad (3.10)$$

We make in (3.10) replacement $X \to \Phi X$, then $\sqrt{-1}H(X,Z,Y,Y) - \sqrt{-1} - H(Y,Y,X,Z),$

Consequently, $H(X,Z,Y,Y) - H(Y,Y,X,Z) = 0$. Adding the last equality to (3.10), we get $H(X,Z,Y,Y) = 0$. In the resulting equality, we make the replacement $Y = Y + W$, then $H(X,Z,Y,Y) + 2H(X,Z,Y,W) + H(X,Z,W,W) = 0$, therefore we have $H(X,Z,Y,W) = 0$.

By replacing $Z \leftrightarrow Y$, we obtain the desired, which means $H(X,Y,Z,W) = 0, \forall X,Y,Z,W \in L$. Thus, the theorem is proved. □

Now, we introduce the tensor H with components $H^{ad}_{bc} = \tilde{A}^{ad}_{bc} = A^{ad}_{bc} - \frac{1}{2}\tilde{\delta}^{ad}_{bc}\sigma_0^2 - \frac{2}{(2n-1)}\left(A^{(d|h|}_{(b|h|}\delta^{a)}_{c)} - (n\sigma_0^2+\sigma_{00})\tilde{\delta}^{ad}_{bc}\right)$ and call it a tensor of holomorphic conharmonic curvature of $\mathcal{LCACS}$-manifolds.

It is easy to show that the holomorphic conharmonic curvature tensor H has the following properties:

1) $\gamma \circ H(X,Y,Z) = 0;$

2) $\Phi \circ H(X,Y,Z) = H(\Phi X,Y,Z) = -H(X,Y,\Phi Z);$

3) $H(\alpha,Y,Z) = H(X,Y,\alpha) = 0; \forall X,Y,Z \in \mathcal{X}(M).$ (3.11)

Let M be an $\mathcal{NLCACS}$-manifold of $\mathcal{PCHS}$-curvature c. Then, by virtue of Theorem 3.2, we have $H^{ad}_{bc} = -\frac{c}{2}\tilde{\delta}^{ad}_{bc}$. We write this equality in the form $H^{ad}_{bc} = -\frac{c}{2}\delta^a_b\delta^d_c - \frac{c}{2}\delta^d_b\delta^a_c$. So we have

$$H(\sigma_b,\sigma_c,\sigma_{\hat{d}})^a = -\frac{c}{2}\langle\sigma_c,\sigma_{\hat{d}}\rangle(\sigma_b)^a - \frac{c}{2}\langle\sigma_b,\sigma_{\hat{d}}\rangle(\sigma_c)^a,$$

since $\{\sigma_a\}$ and $\{\sigma_{\hat{a}}\}$ are the basis vectors of the submodules $D_\Phi^{\sqrt{-1}}$ and D_Φ^{-1} and the projectors on these submodules are endomorphisms $\Pi = \rho \circ l = -\frac{1}{2}\left(\Phi^2 + \sqrt{-1}\Phi\right)$, $\bar{\Pi} = \bar{\rho} \circ l = \frac{1}{2}\left(-\Phi^2 + \sqrt{-1}\Phi\right)$ respectively, then

$$\left(\Phi^2+\sqrt{-1}\Phi\right) \circ H\left(\Phi^2X+\sqrt{-1}\Phi X, \Phi^2Y+\sqrt{-1}\Phi Y - \Phi^2Z+\sqrt{-1}\Phi Z\right) = -\frac{c}{2}\langle\Phi^2Y+\sqrt{-1}\Phi Y, -\Phi^2Z+\sqrt{-1}\Phi Z\rangle\left(\Phi^2X+\sqrt{-1}\Phi X\right) - \frac{c}{2}\langle\Phi^2X+\sqrt{-1}\Phi X, -\Phi^2Z+\sqrt{-1}\Phi Z\rangle\left(\Phi^2Y+\sqrt{-1}\Phi Y\right); X,Y,Z \in \mathcal{X}(M).$$

Expanding the brackets, we get

$$-\Phi^2 \circ H(\Phi^2X,\Phi^2Y,\Phi^2Z) - \Phi^2 \circ H(\Phi^2X,\Phi Y,\Phi Z) - \Phi^2 \circ H(\Phi X,\Phi^2Y,\Phi Z) + \Phi^2 \circ H(\Phi X,\Phi Y,\Phi^2Z) + \Phi \circ H(\Phi X,\Phi Y,\Phi Z) - \Phi \circ H(\Phi^2X,\Phi^2Y,\Phi Z) + \Phi \circ H(\Phi^2X,\Phi Y,\Phi^2Z) + \Phi \circ H(\Phi X,\Phi^2Y,\Phi^2Z) = -\frac{c}{2}(-\langle\Phi^2Y,\Phi^2Z\rangle\Phi^2X - \langle\Phi Y,\Phi Z\rangle\Phi^2X + \langle\Phi Y,\Phi^2Z\rangle\Phi X - \langle\Phi^2Y,\Phi Z\rangle\Phi X) - \frac{c}{2}(-\langle\Phi^2Y,\Phi^2Z\rangle\Phi^2X - \langle\Phi Y,\Phi Z\rangle\Phi^2X + \langle\Phi Y,\Phi^2Z\rangle\Phi X - \langle\Phi^2Y,\Phi Z\rangle\Phi X);\ X,Y,Z \in \mathcal{X}(M). \quad (3.12)$$

In view of (3.11), the identity (3.12) will take the form

$$H(X,Y,Z) = -\frac{c}{8}(-\langle\Phi Y,\Phi Z\rangle\Phi^2X + \langle Y,\Phi Z\rangle\Phi X - \langle\Phi Y,\Phi Z\rangle\Phi^2X\langle Y,\Phi Z\rangle\Phi X);\ X,Y,Z \in \mathcal{X}(M). \quad (3.13)$$

Thus, the following theorem is concluded:

Theorem 3.3. An $\mathcal{NLCACS}$-manifold is a manifold of $\mathcal{PCHS}$-curvature c if, and only if, the holomorphic conharmonic curvature tensor has the following formula

$$H(X,Y,Z) = -\frac{c}{8}(-\langle\Phi Y,\Phi Z\rangle\Phi^2X + \langle Y,\Phi Z\rangle\Phi X - \langle\Phi Y,\Phi Z\rangle\Phi^2X + \langle Y,\Phi Z\rangle\Phi X);\ X,Y,Z \in \mathcal{X}(M).$$

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