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Generalization of injective S-acts

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ABSTRACT
The injective topic is well known for its influential relevance in module theory and scholars have worked hard to identify generalizations for it. One of these generalizations is M-Mininjective and mininjective, so we extend these notions to S-act theory as well, since act theory represents the generalization of module theory. If every S-homomorphism from a simple M-cyclic subact of M_S into N_S can be extended to M_S, an S-act N_S is called M-mininjective. An S-act M_S is referred to as mininjective, if for each simple right ideal A of S and every S-homomorphism from A into M_S can be extended to S-homomorphism from S into M_S. We looked at the properties and characterizations of S-act where all subacts are M-cyclic and simple and all subacts are merely simple. These topics are shown using examples. With the provided concepts, we were able to accomplish improved results, obtaining novel characterizations of mininjective acts in terms of duality conditions. Additionally, the conditions under which subacts inherit the mininjective and M-mininjective properties are studied. The connection between the act of maximal right ideals of S and the act of minimal subact of T_M is explicated. Finally, the conditions under which the classes of M-mininjective acts and the classes of mininjective S-acts will coincide are defined. Our work's conclusions have been explained.

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1. Introduction

Theoretical computer science, the theory of differential equations and functional analysis are all examples of direct applications of the theory of acts. The action of a semigroup in semigroup theory is a generalization of group action in group theory, with the semigroup's elements acting as transformations of the set. S-systems, S-sets, S-operands, S- polygons, transition systems, and S-automata are only a few of the names for S-acts [11]. We advise the reader to the references [2, 3, 5–7, 12, 13, 17–20] for further information on injective acts generalizations.

Throughout this paper, unless otherwise stated, we assume that S is a monoid with zero element and that every S-act is unitary right S-act with zero element Θ which is denoted by M_S. A **right S-act** M_S with zero signifies a nonempty act with a function $f: M\times S \longrightarrow M$ $f(m,s)=ms$ such that the following properties hold: **(1)** m1= m, **(2)** m(st)=(ms) t, for all m∈M and s, t∈S, 1 refers to the identity element of S. A **subact** N of an S-act MS, is a nonempty subact of M such that xs∈N for all x∈N and s∈S. An S-act A_S is called a **cyclic (or principal)** act if it is generated by one element. It is denoted by $A_S=< u >$ where u $\in A_S$, then A_S =uS ([11], P. 63). Let g refer to a function from an S-act A_S into an S-act B_S; then, g will be called an **S-homomorphism**; if for any a $\in A_S$ and s ∈ S following which we have g(as)=g(a)s [6]. An S-act B_S is **a retract subact** of S-act A_S if and only if there exists a subact W of A_S and S-epimorphism $f: A_S \longrightarrow W$ such that $B_S \cong W$ and f(w)=w for every w∈W ([11], P. 84). A subact N of a right S-act M_S is referred to as **fully invariant** if f(N)⊆N for every endomorphism f

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of M_S and M_S is called a **duo act** if every subact of M_S is fully invariant [16]. An S-act M_S is called **simple** if it contains no subact other than M_S itself; meanwhile, it is called **Θ-simple** if it contains no subact other than M_S and one element subact Θ_S [11]. An S-act M_S is referred to as **principally self-generator** if every $x \in M_S$, there is an S-homomorphism $f : M_S \longrightarrow xS$ such that $x = f(x_1)$ for $x_1 \in M_S$ [1]. An **S-congruence** ρ on a right S-act M_S represents an equivalence relation on M_S such that whenever $(a, b) \in \rho$, then $(as, bs) \in \rho$ for all $s \in S$ [8]. The **identity S- congruence** on M_S will be denoted by I_M such that $(a, b) \in I_M$ if and only if $a = b$ [4]. **A right annihilator** of an S-act M_S is denoted by $\gamma_S(T)$ where T refers to a subact of M_S and is equal to the act $\{(a, b) \in S \times S | ta = tb, \text{ for all } t \in T\}$. If K is a subact of $M \times M$, then $\gamma_S(K) = \{s \in S | as = bs, \text{ for all } (a, b) \in K\}$ is **a right ideal** of S and a left annihilator of an S-act M_S is denoted by $\ell_M(H)$, where H denotes a subact of S and it is equal to the act $\{(m, n) \in M \times M | mx = nx \text{ for all } x \in H\}$ but if J is a subact of $S \times S$, then, $\ell_M(J) = \{a \in M | am = an \text{ for all } (m, n) \in J\}$ and, if $\ell_M(J) \neq \Theta$, then, it is a subact of M_S [10].

In 1966, P. Berthiaume introduced the concept of injective S-act. An S-act M_S is said to be **injective** if, for any S-monomorphism h from S-act A_S into B_S and S-homomorphism f from A_S into M_S, there is S-homomorphism g from B_S into M_S such that $g \circ h = f$ [6].

This paper is subdivided into three parts. Section two is focused on the mininjective (M-simple injective) S-act. We obtained novel characterizations of this class. Certain classes of subacts that inherit this property are considered. Examples are given to elucidate this concept. In section three characterizations of M-mininjective S-act over monoids are investigated. The properties of M-mininjective S-act over monoids are examined. The relationship between the M-mininjective S-acts and the classes of mininjective S-acts is exhibited. As a result, the conditions for these classes to coincide have been specified. In section four, we present the conclusions of our work.

2. M-Simple injective S-acts

Definition 2.1. Let M_S and N_S are S-acts. N_S is called a **minimal M-injective act** (or **M-simple injective**), if every S-homomorphism from simple subact A of M_S to N_S can be extended to S-homomorphism from M_S to N_S. An S-act M_S is called a **minimal quasi-injective** act if it is a minimal M-injective.

The following Propositions 2.2 and 2.3 demonstrate the conditions under which subacts inherit the property of the minimal M-injective:

Proposition 2.2. *Retract of minimal M-injective (M-simple) S-act is minimal M-injective.*

Proof. Let A be a retract subact of minimal M-injective S-act N_S and B be a simple subact of M_S. Let $f : B \longrightarrow A$ be an S-homomorphism. As N_S is a minimal M-injective S-act, there exists an S-homomorphism $g : M_S \longrightarrow N_S$ and $g \circ i = j_A \circ f$, where i is the inclusion map of B into M_S and j_A is the injection map of A into N_S. Put $h = \pi_A \circ g$. Thus, $h \circ i = \pi_A \circ g \circ i = \pi_A \circ j_A \circ f = I_A \circ f = f$. Therefore, A is minimal M-injective. □

Proposition 2.3. *Every fully invariant subact of minimal quasi-injective act is a minimal M- injective act.*

Proof. Let M_S be minimal quasi-injective act and N be fully invariant subact of M_S and α be S-homomorphism from simple subact A of M_S to N. Since M_S is a minimal M-injective, then there exists S-homomorphism $\beta : M_S \longrightarrow M_S$ such that $\beta \circ i_N \circ i_A = i_N \circ \alpha$ where $i_A; i_N$ are the inclusion maps of A into N and N into M_S respectively. We get $\beta(i_N(N)) \subseteq i_N(N)$, thus $\beta(N) \subseteq N$. Therefore, N is minimal M-injective. □

Remarks and Examples 2.4. In [1], Abbas M.S. and the author gave the definition of principally quasi injective as follows: an S-act M_S is referred to as **principally quasi injective** if every S-homomorphism from a principal subact of M_S to M_S extends to an S-endomorphism of M_S (in short, **PQ-injective act**).

(1) Every principal is simple, so every principally quasi injective is minimal M-injective but the converse is not true in general; for example, Z with multiplication is a Z-act over itself, then Z is; minimal Z-injective Z-act but not principally injective.
(2) Every semisimple S-act is minimal M-injective.
(3) Every simple S-act is minimal M-injective S-act.
(4) Every isomorphism of minimal M-injective S-act is minimal M-injective.

The next proposition is a generalization of Lemma 2.1.4, which exists in [17] (which also represents a generalization of lemma 1.1 in [14]):

Proposition 2.5. *Let M_S be an S-act with T=End (M_S). The following conditions are equivalent:*

(1) Every S-homomorphism from a simple subact of M_S to M_S can be extended to S-endomorphism;
(2) $\ell_M(\gamma_S(mS)) = Tm$, where mS is simple subact;
(3) If $\gamma_S(mS) \subseteq \gamma_S(nS)$, then $Tn \subseteq Tm$ where mS, nS are simple,
(4) If S-homomorphisms $\alpha, \beta : mS \longrightarrow M_S$ are given where mS is a simple subact with β being monomorphism, there exists $\sigma \in T$ such that $\sigma \circ \beta = \alpha$.

Proof. ($1 \longrightarrow 2$) Let $\alpha m \in Tm$ where $\alpha \in T$. For each $s, t \in S$ with ms=mt, we have $\alpha(ms) = \alpha(mt)$, so $\alpha m \in \ell_M(\gamma_S(m))$. Thus, we obtain $Tm \subseteq \ell_M(\gamma_S(m))$. Conversely, if $n \in \ell_M(\gamma_S(m)))$, then define $\sigma : mS \longrightarrow M_S$ by $\sigma(ms) = ns$; for $s \in S$. If ms=mt, for $s, t \in S$, then$(s, t) \in \gamma_S(mS) \subseteq \gamma_S(nS)$; hence ns=nt, so σ is well-defined. It is an easy matter to see that σ is an S- homomorphism. By (1), σ can extend to $\bar{\sigma} \in T$. So $n = \sigma(m) = \bar{\sigma}(m) \in Tm$. Thus, we have $\ell_M(\gamma_S(m)) \subseteq Tm$ and this implies to $\ell_M(\gamma_S(m)) = Tm$

($2 \longrightarrow 3$) If $\gamma_S(mS) \subseteq \gamma_S(nS)$, then $n \in Tn = \ell_M(\gamma_S(nS)) \subseteq \ell_M(\gamma_S(mS)) = Tm$, so $n \in Tm$ and hence$Tn \subseteq Tm$.

($3 \longrightarrow 4$) Let $(s, t) \in \gamma_S(\beta(mS))$ for $s, t \in S$. Then $\beta(ms) = \beta(mt)$. Since β is monomorphism, ms=mt and $\alpha(m)s = \alpha(m)t$; hence$(s, t) \in \gamma_S(\alpha(mS))$ Then, we have $\gamma_S(\beta(mS)) \subseteq \gamma_S(\alpha(mS))$ By using (3), we obtain $\alpha mS \in T\beta(mS)$. So there is $\sigma \in T$ such that $\alpha(ms) = \sigma\beta(ms)$ and hence $\alpha = \sigma\beta$.

($4 \longrightarrow 1$) Take $\beta : mS \longrightarrow M_S$ to be the inclusion homomorphism in (4). □

Proposition 2.6. *Let $M_1 \cong M_2$. If N_S is minimal M_1-injective (or M_1-simple injective) act, then N_S is minimal M_2-injective (or M_2-simple injective).*

Proof. Let A be a simple subact of S-act M_2 and f be S-homomorphism from A to N. Let g be S-isomorphism from M_2 to M_1. Put $B = g(A)$, it is clear that B is a simple subact of M_1. Define $\alpha : B \longrightarrow N$ by $\alpha(g(a)) = f(a)$, where $a \in A$. It is obvious that α is well-defined. Since N_S is a minimal M_1-injective ac t, there exists $\beta : M_1 \longrightarrow N_S$ such that $\beta \circ i = \alpha$, then $f(a) = \alpha(g(a)) = (\beta \circ i)(g(a)) = \beta(i(g(a))) = \beta(g(a)) = (\beta \circ g)(a)$. Hence, N_S is minimal M_2-injective. □

Proposition 2.7. *Let M_S and N_S be two S-acts and M_1 be subacts of M_S. If N_S is a minimal M-injective and then, N_S is a minimal M_1-injective.*

Proof. Let A be a simple subact of M_1 and $f : A \longrightarrow N_S$ be S-homomorphism. As N is a minimal M-injective act, there exists S-homomorphism $g : M_S \longrightarrow N_S$ such that $g \circ i_{M_1} \circ i_A = f$, where i_A, and i_{M_1} are the inclusion maps of A into M_1 and M_1 into M_S respectively. Define $g_1 : M_1 \longrightarrow N_S$ by $g_1 = g \circ i_{M_1}$. It is clear that g_1 is S-homomorphism, so $g_1 \circ i_A = f$ and then N_S is minimal M_1-injective. □

In [9], Harada defined a mininjective ring, which motivates us to generalize this concept to a monoid as follows:

Definition 2.8. An S-act M_S is called **mininjective**, if for each simple right ideal A of S and every S-homomorphism from A into M_S can be extended to S-homomorphism from S into M_S. A monoid S is **mininjective**; if S_S is mininjective as a right S-act.

Characterizations of the mininjective monoid are clarified in the next proposition:

Proposition 2.9. *The following conditions are equivalent for a commutative monoid S.*

(1) S is a right mininjective;
(2) If aS is simple, $a \in S$*; then,* $\ell_S\gamma_S(a) = Sa$*;*
(3) If bS is simple and $\gamma_S(b) \subseteq \gamma_S(a)$, $a, b \in S$ *then,* $Sa \subseteq Sb$*;*
(4) If aS is simple and $\alpha : aS \longrightarrow S$ *is S-linear then,* $\gamma_S(a) \in Sa$.

Proof. ($1\longrightarrow 2$) Let $\Theta \neq b \in \ell_S\gamma_S(a)$. Define $\alpha : aS \longrightarrow S$ by $\alpha(as) = bs$. If as=at, for $s, t \in S$, then $(s,t) \in \gamma_S(a) \subseteq \gamma_S(b)$; hence, bs=bt, which shows that α is well-defined. As S is mininjective by (1), then there exists $\beta : S \longrightarrow S$ such that $\beta \circ i = \alpha$, where i is the inclusion map of aS into S. Thus, $bs = \alpha(as) = (\beta \circ i)(as) = \beta(i(as)) = \beta(as) = \beta(s)a \in Sa$ and $\ell_S\gamma_S(a) \subseteq Sa$. For the other direction, let $\alpha a \in Sa$. Now, for each $s, t \in S$ with as=at, we have α (as) $= \alpha$ (at), so $\alpha a \in \ell_S\gamma_S(a)$

($2\longrightarrow 3$) Assume that $\gamma_s(b) \subseteq \gamma_s(a)$, then $\ell_S\gamma_S(a) \subseteq \ell_S\gamma_S(b)$, since $a \in Sa$, so $a \in \ell_S\gamma_S(a) = Sa$ by (2), then $a \in Sb$. Thus, we obtain $Sa \subseteq Sb$.

($3\longrightarrow 4$) Let $(s, t) \in \gamma_S(\alpha(a))$ for $s, t \in S$. Then $\alpha(as) = \alpha(at)$. Since α is monomorphism, then as=at and $\beta(a)s = \beta(a)t$ for $\beta \in S$, hence $(s, t) \in \gamma_S(\beta(a))$. Then, $\gamma_S(\alpha(a)) \subseteq \gamma_S(\beta(a)$. By using (3), we have $\beta a \in S\alpha(a)$. So there is $\sigma \in S$ such that $\beta(a) = (\sigma \circ \alpha)(a)$ and hence $\beta = \sigma \circ \alpha$.

($4\longrightarrow 1$) Take $\alpha : aS \longrightarrow S_S$ to be the inclusion homomorphism in (4). □

Definition 2.10. Let M_S be a right S-act with T=End (M_S). The **dual hom (M_S, S)** of M_S **is a left S-act**. If $s \in S$ and $\alpha \in$ hom (M_S, S), then the map $s\alpha$ is defined by $(s\alpha)(m) = s(\alpha(m))$ for all $m \in M_S$.

Lemma 2.11. *If M=mS be a cyclic S-act and* $A = \gamma_S(m)$*, then hom* $(M_S, S) \cong l_M(A) \cong l_M\gamma_S(m)$.

Proof. For any $x \in \ell_M(A)$, let α_x: $M_S \longrightarrow S$ by $\alpha_x(ms) = xs$. Then, we obtain that α_x is well-defined S-homomorphism. Then, define $\beta : \ell_M(A) \longrightarrow$ hom (M_S, S) by $\beta(x) = \alpha_x$. Thus, β is a left S-isomorphism. □

The next theorem affords a significant characterization of the mininjective monoid in terms of duality (also it is considered a generalization of Proposition 2.2 in [15]):

Theorem 2.12. *The following conditions are equivalent for a monoid S*

(1) S is right mininjective monoid;
(2) Hom (M_S, S) is a simple left S-act for all simple right S-act M_S;
(3) $\ell_M(A)$ *is a simple left S-act for all maximal right ideals A of S.*

Proof. ($1\longrightarrow 2$) Let $\alpha\beta \in$ Hom(M_S, S), where M_S is simple, and assume that $\alpha \neq \Theta$. Then, $\beta \circ \alpha^{-1}$: $\alpha(M) \longrightarrow S$ is homomorphism. Since α(M) is simple, $\beta \circ \alpha^{-1}$ can be extended to an endomorphism σ of S by (1). Thus $\beta = \sigma \circ \alpha$.

($2\longrightarrow 3$) Let M_S =mS ($m \in M$) be cyclic S-act and then take $A = \gamma_S(m)$. Thus $\ell_M(A) \cong$ Hom (M_S, S) by Lemma 2.11, which implies that $\ell_M(A)$ is simple by (2). ($3\longrightarrow 1$) Let $\alpha : mS \longrightarrow S$ be an S-homomorphism, where mS is simple and let $i : mS \longrightarrow S$ be the inclusion map. Put $A = \gamma_S(m)$. Then, A is the maximal right ideal of S, so $\ell_M(A) \cong$ Hom (mS, S) by Lemma 2.11. Hence (mS,S) is simple whence $\alpha = \beta \circ i$ for some $\beta \in S$. □

Definition 2.13. An S-act M_S is called **Kasch** if $\ell_M(A \times A) \neq \Theta$ for any maximal right ideal A of S.

Important properties of a minimal injective Kasch act are given in the next theorem:

Theorem 2.14. *Let M_S be a right mininjective S-act that is right Kasch with T=End (M_S). Then, map $\alpha : A \longrightarrow l_M(A \times A)$ from the act of all maximal right ideals A of S to the act of all minimal subacts of T_M. Then:*

(1) α is one-to-one.
(2) α is bijection if and only if $\ell_M \gamma_S(B) = B$ for all minimal subacts B of T_M. In this case α^{-1}: $B \longrightarrow \gamma_S(B)$.

Proof. (1) Let A be maximal right ideal. Then, we have $\ell_M(A \times A) \neq \Theta$ (since M_S is Kasch). Thus $\ell_M(A \times A)$ is simple by Theorem 2.12. Since $A \times A \subseteq \gamma_S \ell_M(A \times A) \neq S \times S$, so $A \times A = \gamma_S \ell_M (A \times A)$, because A is maximal.

(2) If α is onto and B is a minimal subact of T_M, where $B = \ell_M(A \times A)$, where A is the maximal right ideal of S and $A \times A \subseteq S \times S$, then, $\ell_M \gamma_S(B) = \ell_M \gamma_S \ell_M(A \times A) = \ell_M(A \times A) = B$. Conversely, assume that $\ell_M \gamma_S (B) = B$ for all minimal subacts B of T_M, so the proof is complete when we establish the following claims: □

Claim (1): γ_S (B) is a maximal right ideal of S for all minimal subacts B of T_M.

Proof. Let A be maximal right ideal of S and $\gamma_S(B) \subseteq A \times A$. Then, $\Theta \neq \ell_M(A \times A) \subseteq \ell_M \gamma_S(B) = B$ (since M_S is Kasch). As B is minimal, so $B = \ell_M(A \times A)$. Thus, $\gamma_S(B) = \gamma_S \ell_M(A \times A) \supseteq A \times A$ and then $\gamma_S(B) = A \times A$. □

Claim (2): $\ell_M(A \times A)$ is a minimal subact of $T_M$for all maximal ideals A of S.

Proof. Let A be maximal ideal of S. As M_S is Kasch S-act, so $\ell_M(A \times A) \neq \Theta$. Thus, there exists $m \in \ell_M(A \times A)$ which implies that $A = \gamma_S(m)$ and hence $\ell_M(A \times A) = \ell_M \gamma_S(m) = Tm$ by Proposition 2.5. By Theorem 2.12 Tm is a minimal subact of T_M. It implies that $\ell_M(A \times A)$ is minimal. □

Proposition 2.15. *If M_S is an S-act that contains a simple S-subact essential in M_S, then M_S is mininjective.*

Proof. It is obvious so it is omitted. □

The next proposition illustrates under which condition on minimal M-injective act to be principally quasi injective:

Lemma 2.16. [16] *Over a monoid S, the following statement holds: a right S-system M_S is duo if and only if for each endomorphism f of M_S and for each element a of M_S, $f(a) = as$ for some $s \in S$. In particular, if S is commutative and M_S is duo right S-system, then End (M_S) is a commutative monoid.*

Proposition 2.17. *Let M_S be a multiplication S-act. If M_S is minimal M-injective S-act, then M_S is PQ-injective.*

Proof. Assume that M_S is minimal M-injective and multiplication S-act. Let N be S-subact of M_S and f be S-homomorphism from N into M_S. Since M_S is multiplication system, so N=MI for some right ideal I of S. Since, every multiplication act is duo, so by using Lemma 2.16 and since every principal is simple, then by the minimalist property of the injective, we have for each endomorphism g of M_S and each element a of M_S, g (a)=as for some $s \in S$. Now, for each $n \in N$ and $s \in S$, we have $ns \in N$

(since N=MI); thus, ns =f(n) =g(n), which means that g is an extension of f and M_S is principally quasi injective. □

3. M-mininjective S-acts

Definition 3.1. Let M_S and N_S be S-act, N_S is called **M-mininjective**, if for every S- homomorphism from a simple M-cyclic subact of M_S into N_S can be extended to M_S. A monoid S is a right mininjective if and only S_S is minimal injective as S-act.

Proposition 3.2. *Let A be a simple M-cyclic subact of M_S. If A is an M-mininjective act, then A is a retract subact of M_S.*

Proof. Let I_A: A $\longrightarrow$ A be the identity map. Since A is M-mininjective, there exists S-homomorphism $f : M_S \longrightarrow A$ such that fi $=I_A$, where i is the inclusion map of A into M_S. This means that i has left inverse, so A is retract of M_S. □

Proposition 3.3. *Let $N = \oplus_{i\in I} N_i$ where $N_i | i \in I$ and I is finite index act be a family of S-acts. Then, N is M-mininjective if and only if N_i is M-mininjective.*

Proof. Assume that $N_S = \oplus_{i\in I} N_i$ is M-mininjective. Let X be a simple M-cyclic subact of M_S and f be S-homomorphism from X into N_i. Since N_S is M-mininjective S-act, there exists S-homomorphism $g : M_S \longrightarrow N_S$ such that $g \circ i_X = j \circ f$, where i_X is the inclusion map of X into M_S, and j is the injection map of N_i into N_S. Define $h : M_S \longrightarrow N_i$ such that $h = \pi_i \circ g$, where π_i is the projection map of N_S onto N_i, then $h \circ i_X = \pi_i \circ g \circ i_X = \pi_i \circ j \circ f = f$. That is for all $a \in X$, $h(a) = h(i_X(a)) = \pi_i(g(a)) = \pi_i(g(i_X(a))) = \pi_i(j(f(a))) = (\pi_i \circ j)(f(a)) = f(a)$. Figure 1 explains that:

Hence N_i is M-mininjective S-act. Conversely, assume that N_i is M-mininjective for each $i \in I$ and f is S-homomorphism from a simple M-cyclic subact X of M_S into N_S. Since N_i is M-mininjective, S-homomorphism $\beta_i : M_S \longrightarrow N_i$ exists such that $\beta_i \circ i_X = \pi_i \circ f$, where π_i is the natural projection of N_S onto N_i. So there exists S-homomorphism $\beta : M_S \longrightarrow N_S$ such that $\beta_i = \pi_i \circ \beta$. We claim that $\beta \circ i_X = f$. For this since $\beta_i \circ i_X = \pi_i \circ \beta \circ i_X$, then $\pi_i \circ f = \pi_i \circ \beta \circ i_X$, so we obtain $f = \beta \circ i_X$. Figure 2 illustrates that:

Therefore, N is M-mininjective. □

Corollary 3.4. *The retraction subact of M-mininjective S-act is M-mininjective.*

The following theorem elucidates characterizations of the M-mininjective act:

Theorem 3.5. *Let M_S be an S-act with T= End (M_S). The following conditions are equivalent.*

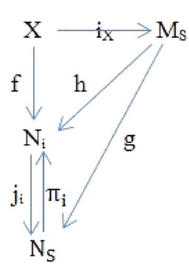

Figure 1. Illustrates that N_S is M-mininjective S-act.

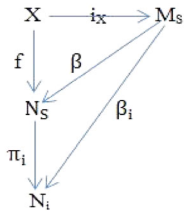

Figure 2. Explains that N_i is M-mininjective act.

(1) M_S is M-mininjective;
(2) For all $\alpha \in T$ if α (M) is simple then, $\ell_T(\ker\alpha) = T\alpha$;
(3) $\mathrm{Ker}\alpha \subseteq \mathrm{Ker}\,\beta$ implies that $T\beta \subseteq T\alpha$ for any $\alpha, \beta \in T$ and $\beta \neq \Theta$, where $\alpha(M)$ is simple;
(4) For all $\alpha \in T$, if α (M) is simple and $\beta : \alpha(M) \longrightarrow M_S$ is an S-homomorphism, then $\beta\alpha \in T\alpha$.
(5) $\ell_T((\beta(M) \times \beta(M)) \bigcap \ker\alpha) = \ell_T(\beta(M) \times \beta(M)) \bigcup T\alpha$, for each $\alpha, \beta \in T$ with $\alpha(M)$ is simple.

Proof. (1$\longrightarrow$2) Let $\beta \in T\alpha$, then $\beta = \sigma\alpha$ for some $\sigma \in T$. For each $s, t \in S$ with ms=mt, we have $\beta(ms) = \beta(mt)$, so $\beta \in \ell_T(\ker\alpha)$. Conversely, let $\beta \in \ell_T(\ker\alpha)$, then define $\sigma : \alpha(M_S) \longrightarrow M_S$ by $\sigma(\alpha(m)) = \beta(m)$ for some $m \in M_S$. It is clear that σ is a well-defined S-homomorphism with $\mathrm{Ker}\alpha \subseteq \mathrm{Ker}\beta$. In fact, if $\alpha(m_1) = \alpha(m_2)$, then $(m_1, m_2) \in \ker\alpha \subseteq \ker\beta$. So $(m_1, m_2) \in \ker\beta$ and then $\beta(m_1) = \beta(m_2)$. Therefore $\sigma(\alpha(m_1)) = \sigma(\alpha(m_2))$. Since M_S is M-mininjective, there exists an S-homomorphism $f : M_S \longrightarrow M_S$ such that $f \circ i = \sigma$, where i is the inclusion map of $\alpha(M_S)$ into M_S. Thus, $\beta(m) = \sigma(\alpha(m)) = fi(\alpha(m)) = f(\alpha(m)) = (f\alpha)(m)$, where $m \in M_S$. Hence, $\beta \in T\alpha$ and $\ell_T(\ker\alpha) \subseteq T\alpha$. Then, we have $T\alpha = \ell_T(\ker\alpha)$.

(2$\longrightarrow$3) Assume that $\mathrm{Ker}\alpha \subseteq \mathrm{Ker}\beta$, then $\ell_T(\mathrm{Ker}\beta) \subseteq \ell_T(\mathrm{Ker}\alpha)$. Hence, we have $T\beta \subseteq T\alpha$ for any $\alpha, \beta \in T$.

(3$\longrightarrow$4) Define $\beta : \alpha(M) \longrightarrow M_S$ by $\beta(\alpha(m)) = \sigma(m)$, for each $m \in M_S$, then, it is clear that β is well-defined since $\mathrm{Ker}\alpha \subseteq \mathrm{Ker}\sigma$. Since, $\sigma \in T\sigma$, by (3) we have $\sigma \in T\alpha$ and hence $\beta\alpha \in T\alpha$.

(4$\longrightarrow$5) Let $\sigma \in \ell_T(\beta(M_S) \times \beta(M_S) \bigcap \ker\alpha)$. We claim that $\mathrm{Ker}\alpha\beta \subseteq \mathrm{Ker}\sigma\beta$, for this let $(m_1, m_2) \in \mathrm{Ker}\alpha\beta$, so $\alpha(\beta(m_1)) = \alpha(\beta(m_2))$. This implies that $(\beta(m_1), \beta(m_2)) \in (\beta(M_S) \times \beta(M_S) \bigcap Ker\alpha)$, then $\sigma(\beta(m_1)) = \sigma(\beta(m_2))$. Thus, we have $(m_1, m_2) \in \mathrm{Ker}\sigma\beta$. By (4), we have $T\sigma\beta \subseteq T\alpha\beta$, and $\sigma \circ \beta = u \circ \alpha \circ \beta$ for some $u \in T$, and therefore, there is $u \in T$ such that $\sigma \circ \beta = u \circ \alpha \circ \beta$ for each $\alpha, \beta \in T$, in particular $\sigma = u \circ \alpha$. Thus, $\sigma \in \ell_T(\beta(M_S) \times \beta(M_S))$. This means that $\sigma \in \ell_T(\beta(M_S) \times \beta(M_S)) \bigcup T$.

Conversely, let $\sigma \in \ell_T(\beta(M_S) \times \beta(M_S)) \bigcup T\alpha$, so this means $\sigma \in \ell_T(\beta(M_S) \times \beta(M_S))$ or $\sigma = u \circ \alpha$ for some $u \in T$. If $\sigma \in \ell_T(\beta(M_S) \times \beta(M_S))$, this means $\sigma(\beta(m_1)) = \sigma(\beta(m_2))$, $\forall\, m_1, m_2 \in M_S$. Now, for each $m_1, m_2 \in M_S$, we have $(m_1 m_2) \in (\ker\alpha \cap \beta(M_S) \times \beta(M_S))$, which implies that $\alpha(m_1) = \alpha(m_2)$ and $\beta(m_1) = \beta(m_2)$. Since u is well-defined, so $u \circ \alpha(m_1) = u \circ \alpha(m_2)$. If $\sigma = u \circ \alpha$, then this implies that $\sigma(m_1) = \sigma(m_2)$. Thus, we have $\sigma \in \ell_T(\beta(M_S) \times \beta(M_S) \bigcap \mathrm{Ker}\alpha)$. If $\sigma \in \ell_T(\beta(M_S) \times \beta(M_S))$, then we have $\sigma\beta(m_1) = \sigma\beta(m_2)$. Hence, we have $\sigma \in \ell_T(\beta(M_S) \times \beta(M_S) \bigcap Ker\alpha)$.

(5$\longrightarrow$2) By taking $\beta = I_M$, identity map of M_S.

(4$\longrightarrow$2) Let $\beta \in \ell_T(\ker\alpha)$. Then, we obtain $\mathrm{Ker}\alpha \subseteq \mathrm{Ker}\beta$. Define $\sigma : \alpha(M_S) \longrightarrow M_S$ by $\sigma(\alpha(m)) = \beta(m)$ for some $m \in M_S$. It is clear that σ is a well-defined S-homomorphism. In fact, if $\alpha(m_1) = \alpha(m_2)$, then $(m_1 m_2) \in \ker\alpha \subseteq \ker\beta$. So $(m_1 m_2) \in \ker\beta$ and then $\beta(m_1) = \beta(m_2)$. Therefore, $\sigma(\alpha(m_1)) = \sigma(\alpha(m_2))$. By using (4), we have $\sigma\alpha \in T\alpha$, again by using(4),$\beta \in T\alpha$. Therefore, we have $\ell_T(\ker\alpha) \subseteq T\alpha$. On the other hand $\alpha \in T\alpha$, then $\alpha \in \ell_T(\ker\alpha)$. Hence, $T\alpha \subseteq \ell_T(\ker\alpha)$and then $T\alpha = \ell_T(\ker\alpha)$.

(3$\longrightarrow$1) Let N be an M_S-cyclic subact of S-act M_S, so there exists S-epimorphism $\alpha : M_S \longrightarrow N$ such that $\alpha(M_S) = N$. Let φ be S-homomorphism from N into M_S and i be the inclusion map of N into M_S. It is clear that $\varphi\alpha$ is S-endomorphism of M_S. Since $\mathrm{Ker}\alpha \subseteq \mathrm{Ker}\varphi\alpha$, whence for each

$(x, y) \in \mathrm{Ker}\alpha$ implies $\alpha(x) = \alpha(y)$ and since φ is well-defined, so $\varphi(\alpha(x)) = \varphi(\alpha(y))$. Thus, we have $(x, y) \in \mathrm{Ker}(\varphi \circ \alpha)$. By (3), we have $T\varphi\alpha \subseteq T\alpha$. Thus, $\varphi \circ \alpha \in T\alpha$ and so $\varphi \circ \alpha = \sigma \circ \alpha$ for some $\sigma \in T$. This shows that M_S is M-mininjective S-act. □

Definition 3.6. An S-act M_S is called **uniserial**; if its subacts are linearly ordered by inclusion. A monoid S is called **a uniserial** monoid if it is uniserial as an S-act.

Proposition 3.7. *Let MS be a uniserial S-act and T=End (MS). If MS is M-mininjective, then T is left uniserial monoid.*

Proof. Let N and H be left ideals of T. Assume that $H \not\subseteq N$. To prove $N \subseteq H$. Let $\alpha \in N$, $\beta \in H$ and $\beta \notin N$. If $\ker\alpha \subseteq \ker\beta$, then $T\beta \subseteq T\alpha$, by Theorem 3.5 and hence $\beta \in T\alpha \subseteq N$ which is a contradiction. Since M_S is uniserial, it follows that $\mathrm{Ker}\beta \subseteq \mathrm{Ker}\alpha$ and therefore $T\alpha \subseteq T\beta$ by Theorem 3.5, this implies that $N \subseteq H$. Therefore, T is left uniserial monoid.

M-mininjective S-act coincided with minimal M-injective (or M-simple injective) S-act by using some conditions, and this will be clarified in the next proposition: □

Proposition 3.8. *Let M_S be principal self-generator S-act. Then, M_S is minimal M-injective S-act if and only if M_S is M-mininjective.*

For the future work, this article can be generalized for the notion of almost min-quasi-injective act and this is satisfied when for any simple subact mS of S-act M_S, there exists subact X_m of M_S such that $\ell_M\gamma_S(m) = Tm\dot{\bigcup}X_m$.

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The author declares that there is no conflict of interest.

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